Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Large-Scale Nonlinear Multivariable Systems (Decomposition, Modeling and Control Problems)

A.V. Akhmetzyanov

V.A. Trapeznikov Institute of Control Sciences, Russian Academy of Sciences,
Moscow, Russia (e-mail: atlas@ipu.rssi.ru)

Abstract: A class of multivariable systems is considered where modeling and control problems related to real physical processes can be solved only using approximate computational approach. The simulation processes are defined as solving mesh (finite-difference and finite-element) approximations of initial-boundary problems corresponding to original equations of mathematical physics for proper physical processes. The high dimensionality issues arising in the frame of such approach are overcome by means of decomposition and partitioning combined with multigrid spatial versions of approximating operator equations in function spaces. Multilevel computational methods for modeling and solving optimal control problems are oriented to using multiprocessor computer systems with parallel computing in message passing interface environment. The proposed results are actual both in theoretical and applied aspects. For instance, using the proposed approach to resolving problems of natural hydrocarbon deposit development simulation and optimal control opens wide capabilities for choosing efficient strategic decisions.

1. MODELING PROBLEMS

The basic process in a controlled plant for a multivariable system class under consideration is defined by a parabolic non-linear partial differential equation:

$$\chi_0(\phi)\partial\phi / \partial t = \operatorname{div} \gamma(\phi) \operatorname{grad} \phi\,, \qquad (1)$$

where $\chi_0(\phi)$ and $\gamma(\phi)$ are non-linear analytic functions,

$$\mathbf{x} = (x_1, x_2, x_3) \in \Omega \subset R^3,\ t \in [0,T],$$

$\phi(\mathbf{x},t)$ is a generalized solution obtained under specified initial and boundary conditions (e.g., of first, second and third kinds) on the outer and the inner boundaries

$$\Gamma_0 = \partial\Omega \text{ and } \Gamma_i = \partial\omega_i, i = \overline{1,N}$$

of the multiply-connected domain Ω. In a two-dimensional case, domain Ω outer and inner boundary location may be represented as in a Fig. 1. For simplicity just that two-dimensional controlled plant will be considered, as a generalization of results to 3D plants of any complexity creates no difficulties in principle. Traditionally, control actions are defined through the selection of boundary conditions on inner boundaries, but the cases of control actions on outer boundaries can exist. Let the initial and boundary conditions on boundaries

$$\Gamma_0 \text{ and } \Gamma_i, i = \overline{1,N}$$

are defined by expressions:

$$\phi(\mathbf{x},t)_{t=0} = \phi_0(\mathbf{x}), \quad \mathbf{x} \in \Omega;$$

$$a_0\phi(\mathbf{x},t)|_{\Gamma_0} + \gamma(\phi)\partial\phi(\mathbf{x},t)/\partial n|_{\Gamma_0} = u_0,$$

$$\forall \mathbf{x} \in \Gamma_0 = \partial\Omega, \forall t \in [0,T]; \qquad (2)$$

$$a_i\phi(\mathbf{x},t)|_{\Gamma_i} + \gamma(\phi)\partial\phi(\mathbf{x},t)/\partial n)|_{\Gamma_i} = u_i,$$

$$\forall \mathbf{x} \in \Gamma_i, i = \overline{1,N}, \forall t \in [0,T].$$

Problem (1)-(3) seems to be unsolvable even with functions $\chi(\phi)$ and $\gamma(\phi)$ invariant everywhere over the domain Ω. Because of this, we reduce the equation (1) to the form

$$\chi(\Phi)\partial\Phi / \partial t = \operatorname{divgrad} \Phi, \qquad (1^*)$$

where

$$\Phi(\mathbf{x},t) = \int_0^{\phi} \gamma(s)ds$$

is a Kirchhoff's transform, which linearizes both a right-hand side of equation (1) and non-linear members in boundary conditions (2), and converts linear members into non-linear. Let

$$\chi(\Phi) = \delta(\Phi) r(\Phi)$$

where

$$u(\Phi) = 1 - \beta\Phi, \beta \in R\,,$$

and after the proper change of variables transform (1) into the following form:

$$\delta(\Phi)r(\Phi)\frac{d\Phi}{dr}\frac{\partial r}{\partial t} - \mathrm{div}\left(\frac{d\Phi}{dr}\,\mathrm{grad}\, r\right) = 0 .$$

Further, putting

$$\psi(\Phi) = \delta(\Phi)r(\Phi)d\Phi / dr - 1/\alpha ,$$

$$\alpha \in R,\ \alpha/\beta > 0 ,$$

and replacing

$$\delta(\Phi)r(\Phi)d\Phi / dr$$

in (4) by

$$(1/\alpha) + \varepsilon\psi(\Phi)$$

where $0 \le \varepsilon \le 1$, construct a perturbed equation

$$\partial r / \partial t - \varepsilon F(r) = 0$$

where

$$F(r) = \left(\xi\chi\left((r-1)/\beta\right) - 1\right)\partial r / \partial t .$$

At $\varepsilon = 1$ or $\varepsilon = 0$ the perturbed equation transforms respectively into the original equation or into the unperturbed one

$$\partial r_0 / \partial t - \xi\,\mathrm{divgrad}\, r_0 = 0 .$$

Subtracting perturbed and unperturbed equation gives the following equation with respect to corrections $v = r - r_0$:

$$\partial v / \partial t - \xi\,\mathrm{divgrad}\, v + \varepsilon\varphi(v + r_0)\partial r_0 / \partial t = 0 ,$$

where

$$\varphi(r) = \xi\chi\left((r-1)/\beta\right) - 1 .$$

Given the sufficiently smooth source data, the initial-boundary problem for the unperturbed equation has a unique solution, while a sufficient condition for a uniqueness of the initial-boundary problem solution with respect to a correction r is defined by a relation

$$0 < \xi < 2/\chi_{\max} ,$$

where

$$\chi_{\max} = \max_{\phi} \chi_0(\phi) = \max_{\Phi} \chi(\Phi) .$$

Thus, given the unperturbed task solution r_0, we have the following sequence of equation sets for calculating corrections v_j, $j = 1, \ldots, J$:

$$\partial v_1 / \partial t - \xi\,\mathrm{divgrad}\, v_1 = f_1(r_0, t) =$$

$$= -\varphi(r_0)\partial r_0 / \partial t,$$

$$\partial v_2 / \partial t - \xi\,\mathrm{divgrad}\, v_2 = f_2(r_0, v_1, t) =$$

$$= -\varphi(r_0)\partial v_1 / \partial t - (\alpha\chi' v_1 / \beta)\partial r_0 / \partial t,$$

etc. Calculating J corrections allows find a proper approximation from the formula

$$r_{(J)} = r_0 + \sum_{i=0}^{i=J} \varepsilon^i v_i ,$$

$$\| r\text{-}r_{(J)} \| \le c\varepsilon^{(J+1)}$$

with $v_0 = 0$.

Solving initial-boundary problems for a sequence of constructed linear equations is performed by a decomposition of the domain Ω into subsets $\Omega_1, \Omega_2, \ldots, \Omega_N$, such that

$$\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_N = \Omega ,$$

and in addition, for any pair of indexes $i, j = \overline{1, M}$ ($i \ne j$) either

$$\Omega_i \cap \Omega_j \ne \varnothing ,$$

or

$$\dim\left(\mathrm{mes}(\Omega_i \cap \Omega_j)\right) = 1 \vee 0\ ^{1},$$

that is, into subsets that can have only common boundaries or vertices. The said partitioning can be constructed by Voronoi-Delaunay approach with nodes corresponding to coordinates of centers $\omega_i, i = \overline{1, N}$, and obtained Voronoi's polyhedrons or Delaunay's domains represent finite volumes. In particular, Fig. 1 illustrates such partitioning for a rectangular two-dimensional domain Ω, where first all hatched elements are numbered and then all non-hatched ones (two-color partitioning).

After discretization by spatial variables and performing transformations needed to account boundary conditions, in place of a sequence of boundary problems, we obtain an appropriate sequence of Cauchy problems for evolutionary equations expressed as

$$\begin{aligned} &dr_0 / dt + Ar_0 = 0, \quad r(x,0) = r^0(x); \\ &dv_s / dt + Av_s = f_s, \quad \nu_s(x,0) = 0,\ s \ge 1, \end{aligned} \tag{4}$$

where A is a linear positive definite operator. According to the two-color decomposition scheme, represent the original domain Ω by a union of subsets

$$D_1 = \Omega_1 \cup \ldots \cup \Omega_{M1} ,$$

$$D_2 = \Omega_{M1+1} \cup \ldots \cup \Omega_{M2} ,$$

which have only common vertices, that is,

[1] $\dim\left(\mathrm{mes}(\Omega_i \cap \Omega_j)\right)$ is a dimension of measure of a subset $\Omega_i \cap \Omega_j$.

$$D_1 \cup D_2 = \Omega$$

and

$$\dim\big(\mathrm{mes}(D_1 \cap D_2 \neq \varnothing)\big) = 0 .$$

For instance, the two-color partitioning in Fig. 1 yields the following sets:

$$D_1 = \Omega_1 \cup \Omega_1 \cup \cdots \cup \Omega_{10} ,$$

$$D_2 = \Omega_{11} \cup \Omega_{12} \cup \ldots \cup \Omega_{20} .$$

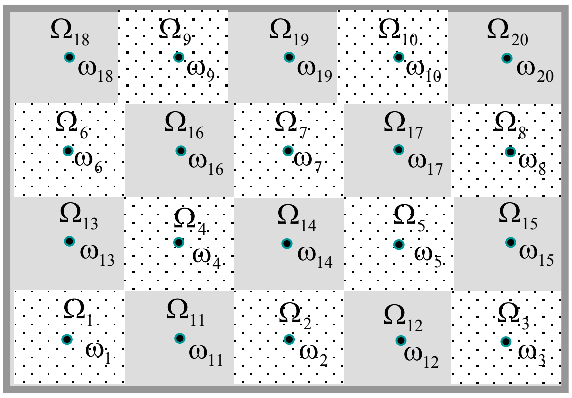

Fig. 1. Domain decomposition

These sets D_1 and D_2 each consists, in its turn, of non-overlapping subsets defining appropriate sets of independent subproblems. Decomposition of problems (1.4) must provide breakdown of the source problem solution into solutions of separate subproblems defined on non-overlapping sets D_1 and D_2. The specific choice of a decomposition scheme is defined by selection of a computation method in separate subsets $\Omega_j, j = \overline{1,N}$ under balanced boundary conditions on common boundaries of subdomains $\partial\Omega_j, j = \overline{1,N}$.

Now construct a decomposition scheme based on identity partitioning by defining a describing function

$$\delta_s(x) = \{> 0, x \in D_s ;\ 0, x \notin D_s\}, s = 1,2 ,$$

such that

$$\delta_1(x) + \delta_2(x) = 1, x \in \Omega = D_1 \cup D_2 .$$

The most natural choice is a decomposition scheme

$$A^{(s)} = \delta_s A ,\ s = 1,2 .$$

In this event,

$$A^{(s)} \neq (A^{(s)})^* ,\quad s = 1,2$$

that is, operators are not self-adjoint ones.

Let r_n be a solution of unperturbed task (index «0» is omitted) in a time point $t = n\tau$ where $\tau > 0$ is a time step, while transition to a new temporal layer is performed sequentially in two stages according to equations:

$$\begin{aligned}&(r_{n+1/2} - r_n)/\tau + \delta_1(x) A u_{n+1/2} + \\ &+ \delta_2(x) A^{(2)} r_n = 0, \\ &(r_{n+1} - r_{n+1/2})/\tau + \\ &+ \delta_2(x) A^{(2)} (r_{n+1} - r_n) = 0.\end{aligned} \tag{5}$$

Equations (5) can be reduced to a symmetrical form, if suppose

$$A^{1/2} r_{n+s/2} = w_{n+s/2}$$

and multiply these equations by a matrix $A^{1/2}$. In so doing, we obtain

$$\begin{aligned}&(w_{n+1/2} - w_n)\tau + \widetilde{A}^{(1)} w_{n+1/2} + \widetilde{A}^{(2)} w_n = 0, \\ &(w_{n+1} - w_{n+1/2})\tau + \widetilde{A}^{(2)} (w_{n+1} - w_n) = 0,\end{aligned} \tag{6}$$

where

$$\widetilde{A}^{(s)} = A^{1/2} \delta_s(x) A^{1/2} , \widetilde{A}^{(s)} = (\widetilde{A}^{(s)})^* \geq 0 ,\ s = 1,2 .$$

The scheme (6) is stabilized by initial data, as the kind of estimate

$$\| (E + \tau\widetilde{A}^{(2)}) r_{n+1} \|_A \leq \| (E + \tau\widetilde{A}^{(2)}) r_n \|_A$$

holds for it.

In a similar manner, partitioning schemes can be constructed for disturbed equations (4) stabilized by right-hand sides.

The computation scheme in each element (finite volume) $\Omega_j, j = \overline{1,N}$ is implemented using multigrid options of FLM or FEM on quasiuniform thickening grids under a boundary condition on the inner boundary (2), defining both the required and admissible control actions.

In the general case of deformable domain Ω a left-hand side of equation (1*) contains the more sophisticated function $\chi(\Phi)$, and in this event the proposed approach not only stays true, but also appears to be more efficient than other applicable methods.

1.1. Peculiarities of Nonuniform System Simulation.

The stratified (vertical) and zonal (horizontal) nonuniformities dominate generally in controlled plants under consideration. Taking into account these nonuniformities is put into practice by domain decomposition methods as well. To this end, elements of original nonuniform domain Ω partitioning into uniform subsets $\Omega_1, \Omega_2, \ldots, \Omega_M$ can be united in such sets D_1 and D_2, that

$$D_1 \cup D_2 = \Omega .$$

If

$$\dim\left(\operatorname{mes}(\Omega_i \cap \Omega_j)\right) = 0 ,$$

that is,

$$\Omega_i \cap \Omega_j, \forall i \neq j, i, j = \overline{1, M}$$

contain only common vertexes, then nonuniform plant model can be built using the symmetric analog (6) of two-component partitioning of the kind (5). If, however, for some $i \neq j$ the intersections $\Omega_i \cap \Omega_j$, $i, j = \overline{1,M}$ contain common boundaries of dimension

$$\dim\left(\operatorname{mes}(\Omega_i \cap \Omega_j)\right) \neq 0 ,$$

then it is necessary to use a two-component partitioning. In simple cases (e.g. for stratified or zonal nonuniformities in Fig. 2 and Fig. 3) nonuniform plant models can be built in a manner described below.

For a vertical nonuniformity (Fig. 2), initial top-level partitioning scheme is carried out by sets

$$D_1 = \Omega_1 \cup \Omega_2 \text{ and } D_2 = \Omega_3 ,$$

in accordance with the equation set of kind (6), obtained by reducing the equation set (5) to a symmetric form.

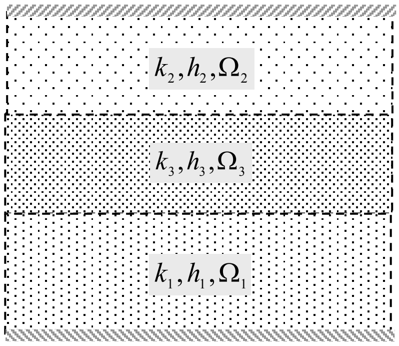

Fig. 2. Stratified reservoir nonunifofmity

For a horizontal nonuniformity (Fig. 3), where

$$\dim\left(\operatorname{mes}(\Omega_3 \cap \Omega_4)\right) = 1 ,$$

initial top-level partitioning scheme is carried out by subsets

$$D_1 = \Omega_1 \cup \Omega_2 ,\ D_2 = \Omega_3 \text{ and } D_3 = \Omega_4 .$$

Here we introduce a describing function

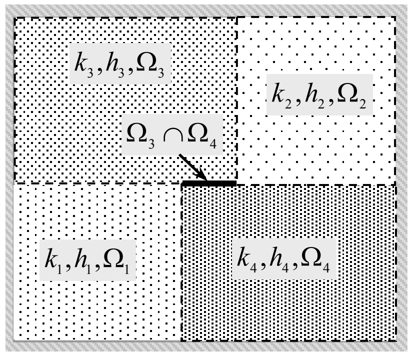

Fig. 3. Zonal nonuniformity of reservoir

$$\delta_s(x) = \{ > 0, (x) \in D_s ; 0, (x) \notin D_s \},\ s = \overline{1,3};$$

$$\delta_1(x) + \delta_2(x) + \delta_3(x) = 1 ,\ x \in \Omega ;$$

then $A^{(s)} = \delta_s A$, $A^{(s)} \neq (A^{(s)})^*, s = \overline{1,3}$ and

$$\begin{gathered}(r_{n+s/3} - r_{n+(s-1)/3}) / \tau + \theta\delta_s(x) A^{(s)} r_{n+s/3} + \\ + (1-\theta)\delta_s(x) A^{(s)} r_{n+(s-1)/3} = 0, \quad s = \overline{1,3}.\end{gathered} \tag{7}$$

Equations (7) reduce to a symmetric form, if we suppose

$$A^{1/2} r_{n+s/3} = w_{n+s/3}$$

and multiply them by the matrix $A^{1/2}$ that yields

$$(w_{n+s/3} - w_{n+(s-1)/3})/\tau + \theta\delta_s(x)\widetilde{A}^{(s)} w_{n+s/3} +$$
$$+(1-\theta)\delta_s(x)\widetilde{A}^{(s)} w_{n+(s-1)/3} = 0, \quad s = \overline{1,3}, \tag{8}$$

where

$$\widetilde{A}^{(s)} = A^{1/2}\delta_s(x)A^{1/2},\ \widetilde{A}^{(s)} = (\widetilde{A}^{(s)})^* \geq 0, \quad s = \overline{1,3}.$$

The scheme (8) is stabilized by initial data, since the estimate

$$\| w_{n+1} \| \leq \| w_0 \|$$

holds for it; whence it follows that

$$\| r_{n+1} \|_A \leq \| r_0 \|_A .$$

It is apparent that each subproblem of this (top or zero) level in its turn is subjected to domain decomposition and to the first-level partitioning by equations of type (8) after transforming appropriate equations (7) to a symmetric form. Subsequent calculations in each element (finite volume) $\Omega_j, j = \overline{1,N}$ are carried out as before by multigrid methods on thickening quasiuniform grids.

1.2. Spatial Nonuniformity.

In the general case, that is, at spatial nonuniformity of the controlled plant, 3D decomposition scheme requires four-colored partitioning of domain Ω (field reservoir), that is,

$$\Omega = D_1 \cup D_2 \cup D_3 \cup D_4,$$
$$D_i \cap D_j = \varnothing,\ i \neq j .$$

Then at each temporal layer, the scheme of multi-component partitioning can be implemented in four steps as follows. As before, define a describing function

$$\delta_s(x) = \{>0, x \in D_s; \quad 0, x \notin D_s\}, s = \overline{1,4};$$
$$\delta_1(x) + \delta_2(x) + \delta_3(x) + \delta_4(x) = 1, x \in \Omega .$$

The decomposition scheme with nonselfadjoint operators

$$A^{(s)} = \delta_s A,\ A^{(s)} \neq (A^{(s)})^*,\ s = \overline{1,4}$$

determines a transition to the other temporal layer for four subsequent steps according to

$$(r_{n+s/4} - u_{n+(s-1)/4})/\tau + \theta\delta_s(x)Ar_{n+s/4} +$$
$$+(1-\theta)\delta_s(x)Ar_{n+(s-1)/4} = 0, \quad s = \overline{1,4}; \tag{9}$$

The equations (9) reduce to symmetric form by assuming

$$A^{1/2}(\sigma_{n+(s-1)/4})\, r_{n+s/4} = w_{n+s/4}$$

with subsequent multiplying this equations by matrixes $A^{1/2}$, that yields

$$(w_{n+s/4} - w_{n+(s-1)/4})/\tau + \theta\,\delta_s(x)\widetilde{A}^{(s)} w_{n+s/4} +$$
$$+(1-\theta)\,\delta_s(x)\widetilde{A}^{(s)} w_{n+(s-1)/4} = 0, \quad s = \overline{1,4}. \tag{10}$$

For equations (10), where matrixes

$$\widetilde{A}^{(s)} = A^{1/2}\delta_s(x)A^{1/2},\ \widetilde{A}^{(s)} = (\widetilde{A}^{(s)})^* \geq 0,$$

the estimate

$$\| w_{n+1} \| \leq \| w_0 \|$$

holds, whence it follows that

$$\| r_{n+1} \|_A \leq \| r_0 \|_A ,$$

that is, the stability by initial data is available.

When the 3D analog of nonuniformities shown in Fig. 2 is the case, both domain decomposition and the corresponding multicomponent partitioning will contain more than four components. Since these type nonuniformities are rather infrequent for the multivariable system class under consideration and are irregular in their structure, the peculiarities of such controlled plants are not treated here.

As in the previous case, each subproblem of obtained top or zero level in its turn is subjected to domain decomposition and to the next first level partitioning in accordance with equations (10) after transforming corresponding equations (9) to a symmetric form. Subsequent calculations in each element $\Omega_j, j = \overline{1,N}$ are carried out as before by multigrid methods on thickening quasiuniform grids.

2. PROBLEMS OF OPTIMAL CONTROL

We take as efficiency criterion for selection of control actions

$$u = \{u_1, \ldots, u_N\} \in U$$

the following functional:

$$J = \sum_{i=1}^{i=N} \int_0^T c_i(t) q_i(t) dt , \tag{11}$$

where U is a constrained set of control action values $c_i(t)$ and

$$q_i(t) = \int_{\Gamma_i} (\partial\phi(\mathbf{x},t)/\partial\nu) d\Gamma_i$$

are functions defining the efficiency and value of control action in the inner loop with index $i = \overline{0,N}$. Then, given the scarcity of shared resources

$$\sum_{i=1}^{i=N} a_i(t) q_i(t) \geq b(t) \tag{12}$$

and differential constraints, that is, equations (1) and inirtial-boundary conditions (2), the optimal control problem lies in seeking the extreme

$$J_* = \operatorname{opt}_{u \in U} J$$

and corresponding values $u^* = u_1^*, \ldots, u_N^*$.

Difficulties brought on by the existence of constraint equation (12) may be obviated by using the following type of functional in place of optimality criterion (10):

$$\hat{J} = J + \lambda \sum_{i=1}^{i=N} d_i(t) q_i(t),$$

in which λ is an indefinite parameter. Next a conjugate problem is considered instead of original optimization, and special features of the function

$$\text{opt}_{u \in U} J = F(\lambda)$$

are defined using the method of continuation by parameter λ. The function $F(\lambda)$ was demonstrated to be a piecewise continuous function. Conditions were defined wherein the original problem has a solution. Algorithms were designed for obtaining the optimal solution $\lambda^* = \lambda_{\max}$ and

$$u^* = \{u_1^*, \ldots, u_N^*\} \in U.$$

Such algorithm presents a multilevel computational process with parallelization of computations at every hierarchical layer through the use of decomposition and partitioning methods combined with spatial multigrid options for approximating operator equations in function spaces. The parallelization itself is oriented to using multiprocessor computer systems.

3. CONCLUSIONS

1. In the sense of theory, proposed results expand the class of effectively decidable problems of large-scale nonlinear multivariable system simulation and optimal control in function spaces.

2. In the sense of applications, proposed results are extremely actual for modeling processes of filtering heterogeneous fluids in porous underground reservoirs, as well as for solving optimization problems in designing and managing gas field development and groundwater level control.

3. The extension of proposed approach to problems of modeling heterogeneous fluid filtration processes in porous underground reservoirs (Akhmetzyanov, 2006, 2007) and to solving optimization problems in design and management of multiphase multicomponent hydrocarbon deposits development can lead to breakthrough achievements in this area. For example, the problem of optimal oil field development control can be treated as minimization of associated water production (optimality criterion) under planned constraints for oil production (shared resources restriction). Such interpretation is especially important for developing oil fields.

4. For a given class of nonlinear multivariable systems, the effective resolving of simulation, identification, optimal design and optimal control problems on multiprocessor computer systems using multilevel (hierarchical) computation concurrency in message passing interface (MPI) environment is not only a preferable, but also the most productive approach.

REFERENCES

Akhmetzyanov, A.V. (2006). Hierarchical parallel computing for modeling of fluid filtration processes in porous media. In: *Proceedings III International conferences "Parallel Computations and Control Problems" (PACO' 2006)*, p. 251-265. V.A. Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia.

Akhmetzyanov, A.V. (2008). Computational aspects in Controlling Filtration of Fluids and Gases in porous media. *Automation and Remote Control*, vol. 69, no. 1, pp. 1-12.