

Optimal Feedback Control of Microalgal Growth Based on the Slow Reduction

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Abstract: A state feedback law is proposed for on-line optimization of microalgal growth in a photobioreactor in the presence of non-measurable disturbances. The objective is to maximize a photosynthetic production rate (specific growth rate of microalgae) by manipulating the irradiance. The model describing the growth of microalgae is based on the mechanistic description in the form of the so-called photosynthetic factory. The reduction to a slow dynamics is used to derive analytically the approximation of the optimal feedback control. The analysis of the obtained explicit formula shows that the optimal feedback control actually performs optimal transfer to a constant optimal irradiance developed earlier but does not achieve principally better performance than the optimal control within the class of constant irradiance. Therefore, by reducing fast dynamics one can not reveal possible more complex optimal solutions. At the same time, this contradicts to a common belief in biotechnological community that the fast phenomena may be neglected. Illustrative simulations are included.

1. INTRODUCTION

The main goal of this study is to derive the optimal feedback control of a bioreactor for cultivation of microalgae (i.e. photobioreactor) which operates under high-irradiance (Masojídek *et al.*, 2003). The model considered here is the lumped parameter model for photosynthesis and photoinhibition, the so-called model of photosynthetic factory - PSF model (Eilers and Peeters, 1988; Eilers and Peeters, 1993; Kmeř *et al.*, 1993; Papáček *et al.*, 2007; Wu and Merchuk, 2001). This phenomenological (mechanistic) five-parameter (q_1, \dots, q_5) model, introduced later on, is the stiff system and the separation between the fast (light and dark reactions) and slow dynamic (photoinhibition) is modelled by a small parameter $\epsilon \approx q_5$. The control signal is the incident irradiance and the on-line measurable quantity is the photosynthetic oxygen production rate (proportional to the specific growth rate μ defined as: $\mu := \dot{c}_x/c_x$, where c_x is the microalgal cell density).

The purpose of this paper is to analyze these two-scale phenomena and to use this analysis to compute explicit optimal feedback control to maximize algal biomass production. Namely, the reduction of the dynamical system to a slow manifold will be developed and then the corresponding less dimensional optimal control problem will be solved analytically. The resulting optimal performance will be compared with the earlier computed (Papáček *et al.*, 2007) optimal performance within the class of constant irradiance inputs. It will be shown that constant irradiance input remain reasonable option, at least it is difficult to achieve significant improvement using the reduction to slow dynamics. This fact would be confirmed by numerous

simulations, changing even the parameters of the system to enhance velocity of the fast dynamics. Basically, this paper shows that neglecting fast transition phenomena equals to resignation on considering dynamical phenomena at all.

This paper is organized as follows. Section 2 presents the dynamical model of the microalgal growth in detail and derives its reduction to a slow manifold. Section 3 repeats the well-known formulation of the end point optimization problem and applies the Pontryagin's maximum principle to derive the feedback relation between a PSF state and the controlled input to maximize the average production rate. Simulation experiments are collected in Section 4 while the final section draws some conclusions and outlooks for further research.

2. DYNAMICAL MODEL OF MICROALGAL GROWTH

The photosynthetic microorganisms growth modelling has long been regarded as a well-defined discipline in algal biotechnology, consisting of the coupling between photosynthesis and irradiance, resulting in the steady-state light response curve (so-called *P-I curve*), which represents the microbial kinetics, see e.g. *Monod* or *Haldane* type kinetics (Schugerl and Bellgardt, 2000). Hence, an adequate model of photosynthesis, observable on basis of photosynthetic oxygen production, is of paramount importance. In Figure 1, the steady-state production curve of *Haldane* type is drawn. For high value of S (substrate) we see the so-called inhibition by the substrate. The governing relation is as follows (using the most usual notation in biotechnological literature):

$$\mu = \frac{\mu^* S}{K_S + S + S^2/K_I}, \quad (1)$$

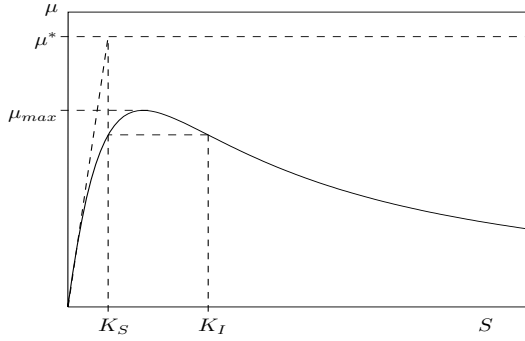


Fig. 1. Steady-state production curve of *Haldane* type or *Substrate inhibition* kinetics.

where μ is specific growth rate, S is a limiting substrate, and μ^* , K_S , K_I are three model constants. The physiological meaning of parameters μ^* , K_S (saturation constant), K_I (inhibition constant) is clearly shown on Figure 1. Note that for $K_I \rightarrow \infty$ the Haldane (substrate inhibition) kinetics converges into Monod (saturation) kinetics. Maximum of the Haldane-type curve occurs when

$$S_{opt} = \sqrt{K_S K_I}, \quad \mu_{max} = \frac{\mu^*}{2\sqrt{K_S/K_I + 1}}. \quad (2)$$

Nevertheless, in order to develop an optimal feedback control, a dynamic model would be more useful. The main difficulty in considering the dynamic behavior of the photosynthetic processes consists in their different time-scales (Rosello *et al.*, 2003). While the characteristic time of microalgal growth, like doubling time, etc., is in order of hours, light and dark reactions occur in milliseconds and photoinhibition in minutes.

Due to the known experiment based knowledge of the relevant processes, one can both formulate the qualitative model behavior and determine the model structure and the number of model parameters. These are the most important qualitative results: (i) the steady state kinetics is of *Haldane* type (Nedbal *et al.*, 1996) and (ii) the microalgal culture in suspension has so-called *light integration* property (Terry, 1986; Nedbal *et al.*, 1996; Janssen *et al.*, 2001), i.e. as the light/dark cycle frequency is going to infinity, the value of resulting production rate (e.g. oxygen evolution rate) goes to a certain limit value, which depends on average irradiance only (Papáček *et al.*, 2007). These features are best comprised by the model described further in detail.

2.1 Model of photosynthetic factory – PSF model

The following model, called as the **model of photosynthetic factory** has been recently studied in the biotechnological literature (Eilers and Peeters, 1988; Eilers and Peeters, 1993; Kmeř *et al.*, 1993; Wu and Merchuk, 2001). Its main features are schematically shown on Figure 2 where three states of the photosynthetic factory are: R resting state, A activated state, B inhibited state. Transition rates measured in s^{-1} are αu , βu , γ , δ while the input variable u is the irradiance. The transition from state A to state B models the photoinhibition process, while the transition from state B to state R models the recovery from the photoinhibition. The photosynthetic growth is proportional to the so-called dark reactions modelled as the transition from state A to state R , see equation (4). Light reactions are modelled as the transition from state R to state A . This scheme can be mathematically modelled as follows

$$\dot{x} = Ax + u(t)Bx + u(t)C,$$

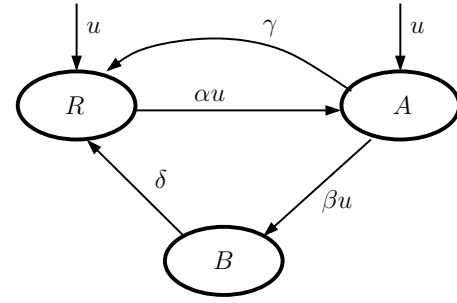


Fig. 2. Scheme of states and transition rates of the photosynthetic factory – Eilers and Peeters PSF model.

where the single scalar input $u(t)$ represents the irradiance in the culture (units: $\mu E m^{-2} s^{-1}$) and A, B, C are matrices and column vector of the appropriate dimensions. The state x of the PSF model is three dimensional, namely, $x = (x_R, x_A, x_B)^T$, where x_R represents the probability that PSF is in the resting state R , x_A the probability that PSF is in the activated state A , and x_B the probability that PSF is in the inhibited state B . The PSF can only be in one of these states, so: $x_R + x_A + x_B = 1$. Taking into account this condition and preferring the states x_A, x_B due to their measurability gives the following two dimensional PSF model (for more detail on the PSF model and its identification, see (Rehák *et al.*, 2008)):

$$\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} = \begin{bmatrix} -\gamma & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u(t) \begin{bmatrix} -(\alpha + \beta) & -\alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u(t) \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad (3)$$

where $\alpha, \beta, \gamma, \delta$ are four rate constants of PSF model and $u(t)$ is the known scalar function (it is assumed that $u(t)$ is at least piecewise continuous). In other words, PSF model is the so-called bilinear controlled system, cf. (Čelikovský, 1988) and references within there.

The PSF model has to be completed by an equation connecting the hypothetical states of PSF model with some quantity related to the cell growth. This quantity is the specific growth rate μ . According to (Eilers and Peeters, 1988; Wu and Merchuk, 2001), the rate of photosynthetic production (specific growth rate) is proportional to the number of transitions from the activated to the resting state, i.e. $\gamma x_A(t)$. Finally, for the average specific growth rate we have the relation:

$$\mu = \kappa \gamma \frac{1}{t_f - t_0} \int_{t_0}^{t_f} x_A(t) dt, \quad (4)$$

where κ is a new dimensionless constant – the fifth PSF model parameter. Equation (4) will be further used as the performance index.

For the constant input signal u , the system (3) becomes linear system of ODE with constant coefficients having two distinct negative eigenvalues λ_F, λ_S . Let $|\lambda_F| \geq |\lambda_S|$, then (for details see Papáček *et al.*, 2007):

$$\lambda_F \cong -[(\alpha + \beta)u + \gamma], \quad \lambda_S \cong -\left[\frac{\alpha\beta u^2}{(\alpha + \beta)u + \gamma} + \delta\right]. \quad (5)$$

As a consequence of $\lambda_F < 0, \lambda_S < 0$, there exists a unique asymptotically stable steady state solution of (3) for the constant input signal $u \geq 0$ given by

$$x_{A_{ss}} = \frac{\delta \cdot \alpha u}{\lambda_F \lambda_S}, \quad x_{B_{ss}} = \frac{\alpha \beta u^2}{\lambda_F \lambda_S}. \quad (6)$$

Moreover, there exists a value of irradiance to maximize growth rate at the steady-state condition (6). Denote in the sequel the input which maximize $x_{A_{ss}}$ with respect to u as $u_{opt_{ss}}$, straightforward computations give $u_{opt_{ss}} = \sqrt{\frac{\gamma \delta}{\alpha \beta}}$. Finally, define the new dimensionless input variable to be used throughout the rest of this paper as $u^* := u/u_{opt_{ss}}$.

Remark 1 Notice that both above mentioned requirements for a process model are accomplished: (i) the PSF model steady state behavior corresponds to *Haldane* type kinetics (let put $x_{A_{ss}}$ from (6) into (4), then the next relation equivalent to (1) is received: $\mu = \frac{\kappa \gamma \delta \alpha u}{\alpha \beta u^2 + \delta (\alpha + \beta) u + \gamma \delta}$), and (ii) the light integration capacity is the inherent property of bilinear system due to the so-called Lipschitz dependence of trajectories on control (Čelikovský, 1988).

Summarizing, the PSF model given by (3,4) will be used as the modelling framework of microalgal growth and will be further investigated in the sequel.

2.2 PSF model re-parameterization

The aim of this short subsection is to rewrite the model (3),(4) introducing a more convenient parametrization. Namely, consider new parameters q_i , $i = 1, \dots, 5$, defined as

$$q_1 := \sqrt{\frac{\gamma \delta}{\alpha \beta}}, \quad q_2 := \sqrt{\frac{\alpha \beta \gamma}{\delta}} \frac{1}{\alpha + \beta}, \quad q_3 := \kappa \gamma \sqrt{\frac{\alpha \delta}{\beta \gamma}}, \quad (7)$$

$$q_4 := \alpha q_1, \quad q_5 := \beta/\alpha. \quad (8)$$

together with earlier introduced dimensionless irradiance $u^* := u/u_{opt_{ss}}$ giving the re-parameterized model

$$\frac{1}{q_4} \begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} = \begin{bmatrix} -q_2(1+q_5) & 0 \\ 0 & -\frac{q_5}{q_2(1+q_5)} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u^* \begin{bmatrix} -(1+q_5) & -1 \\ q_5 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u^* \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (9)$$

$$\mu = q_2 q_3 (1+q_5) \frac{1}{t_f - t_0} \int_{t_0}^{t_f} x_A(t) dt. \quad (10)$$

Notice that q_1 units are those of irradiance ($\mu\text{E m}^{-2} \text{s}^{-1}$), q_2, q_5 are dimensionless, q_3, q_4 are in s^{-1} . The reason to introduce such a re-parameterization is that the role of each new parameter is now much more clearly visible. Namely, parameters q_1, q_2, q_3 correspond to the steady state properties of the PSF as by comparing (13) and (2) one can see that q_2^2 and q_3 plays a similar role as K_S/K_I and μ^* , respectively, while $q_1 := u_{opt_{ss}}$ by definition. Furthermore, q_4 influence the overall dynamics through constant time scaling only, while q_5 is a small parameter quantifying the separation between the fast and slow dynamic; $q_5 \approx 10^{-4}$, based on (Wu and Merchuk, 2001), the following values of PSF model parameters were calculated for the microalga *Porphyridium* sp.: $q_1 := 250.106 \mu\text{E m}^{-2}$, $q_2 := 0.301591$, $q_3 := 0.176498e - 3 \text{s}^{-1}$, $q_4 := 0.483955 \text{s}^{-1}$, $q_5 := 0.298966e - 3$.

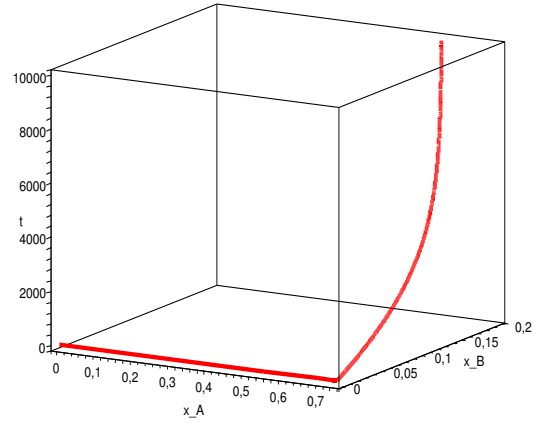


Fig. 3. State trajectory $[x_A(t), x_B(t)]$ for the constant input $u^* = 1$ and the initial condition $[x_A, x_B] = [0, 0]$.

Finally, the expressions for the steady states depending on constant inputs given by (6) has after the above re-parameterization the following more simple form

$$x_{A_{ss}} = \frac{u^*}{q_2(1+q_5)(u^{*2} + u^*/q_2 + 1)}, \quad (11)$$

$$x_{B_{ss}} = \frac{u^{*2}}{u^{*2} + u^*/q_2 + 1}. \quad (12)$$

In particular, by (9-12) the constant input $u^* = 1$ maximizes value of both μ and x_A among all positive constant inputs u^* :

$$\mu_{max} = \frac{q_3}{2+q_2^{-1}}, \quad x_{A_{ss}}^{max} \cong \frac{1}{2q_2 + 1}. \quad (13)$$

2.3 Order reduction of the ODE system (9)

As stated above, the system (9) is a stiff system, i.e., roughly saying, its first equation contains coefficients that are several order higher than those of the second one. To make advantage of that, one can reduce the dynamics to the one dimensional one using the singular perturbation approach with respect to the small parameter $q_5 \approx 10^{-4}$ (Tichonov *et al.*, 1980). This is done in the following way. First, introduce a new faster time scale $\tau = q_5^{-1}t$, so that the system (9) takes the form

$$\frac{q_5}{q_4} \frac{d}{d\tau} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} -q_2(1+q_5) & 0 \\ 0 & -\frac{q_5}{q_2(1+q_5)} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u^* \begin{bmatrix} -(1+q_5) & -1 \\ q_5 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u^* \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (14)$$

Now, after dividing the second equation by q_5 one obtains the singularly perturbed system with respect to the small parameter q_5 . This system thanks to the properties of its right hand side clearly satisfies the sufficient condition for the convergence of the singular perturbation¹ (Khalil, 1987; Tichonov *et al.*, 1980). One can therefore take the limit $q_5 \rightarrow 0$ in (14) to obtain

$$x_A = (1 - x_B) \frac{u^*}{u^* + q_2},$$

¹ Recall, that this condition means geometrically that the slow manifold given by (15) satisfies certain stability properties, namely, $\dot{x}_A < 0$ for $x_A > (1 - x_B) \frac{u^*}{u^* + q_2}$ and $\dot{x}_A > 0$ for $x_A < (1 - x_B) \frac{u^*}{u^* + q_2}$, which is indeed obviously the case here.

$$\frac{1}{q_4} \frac{d}{d\tau} x_B = -\frac{1}{q_2} x_B + u^*(1 - x_B) \frac{u^*}{u^* + q_2}.$$

Now, changing the time scale back to the real time variable t , one has finally the following reduced system

$$x_A = (1 - x_B) \frac{u^*}{u^* + q_2}, \quad (15)$$

$$\frac{d}{dt} x_B = -\frac{q_4 q_5}{q_2} x_B + u^*(1 - x_B) \frac{u^*}{u^* + q_2} q_4 q_5. \quad (16)$$

This means that any solution of the system, no matter what the initial conditions are, quickly satisfies the above relations (15, 16) with great precision. The corresponding convergence is so fast that it is practically unmeasurable (see Figure 3, where the states reach the slow manifold in few seconds). Further convergence to the steady state along the dynamics (16) without breaking the relation (15) is then much slower. The set of all states satisfying (15) is called as the **slow manifold** while the relation (16) is called as the **slow dynamics**. Often, for simplicity, these relations are called as the **slow reduction**. Notice, that the reduction (15,16) guarantees that the natural biological condition $x_A, x_B \in [0, 1]$ is preserved.

Notice, that there are basically two options, how to proceed further. One can consider the slow dynamics (16) and re-compute the functional (10) via (15) to obtain functional depending on x_B and u^* . Advantage of this approach is that no further simplifying requirements on the irradiance input would be needed. Nevertheless, due to the resulting complicated functional depending both on state and input, this approach leads to a still quite complicated optimization problem, hardly solvable in a closed analytical form.

Therefore, the following complementary approach was taken. The functional stays in its simple form (10) but then the slow dynamics (16) should be re-computed into x_A component. That means to differentiate the relation (15) and replace then everywhere x_B expressing it also from the relation (15). Here, the crucial simplifying assumption is that the irradiance is slowly changing, i.e. $|\dot{u}^*(t)|$ is negligible. After performing this plan, the ODE system (9) is reduced into (15), and (17) given bellow. As a matter of fact, it holds $x_A = \lambda_S x_A - \lambda_S x_{A_{ss}}$, and after some evaluation (based on: $(1 + q_5) \cong 1$, and $\lambda_S \cong \frac{-q_4 q_5}{u^* + q_2} [u^{*2} + u^*/q_2 + 1]$), one has

$$\dot{x}_A = \frac{-q_4 q_5}{u^* + q_2} \left([u^{*2} + u^*/q_2 + 1] x_A - u^*/q_2 \right). \quad (17)$$

Summarizing, in the sequel one can concentrate on maximizing the functional (10) subject to constraint given by the one-dimensional controlled system (17).

3. OPTIMAL CONTROL VIA MAXIMUM PRINCIPLE

In this section, the optimization problem will be formulated and its solution in terms of the static state optimizing feedback will be derived in the explicit form.

First, let us specify the type of optimization problem, which is clearly the fixed initial point and free end-point optimization problem with fixed initial and terminal times t_0, t_f . More specifically, one can formulate the corresponding optimization problem as follows (Pontryagin *et al.*, 1962). **Minimize**² the performance index

² To keep notation consistent with a standard minimization formulation, later on, minimization of the algae production with the negative sign would be

$$J = \int_{t_0}^{t_f} f_0(x(t)) dt, \quad u^*(t) \in [0, U], \quad U > 0, \quad (18)$$

subject to the dynamics

$$\dot{x}_1 = f_1(x_1, u^*(t)), \quad t_0 \leq t \leq t_f, \quad x_1(t_0) = x_1^0, \quad (19)$$

where (putting $x_A := x_1$):

$$f_0 = -x_1, \quad f_1 = \frac{q_4 q_5 (u^*/q_2 - [u^{*2} + u^*/q_2 + 1] x_1)}{u^* + q_2}, \quad (20)$$

where t_0 is the initial time, t_f is a given final time, $u^*(t)$ is the scalar controlled input (remember that only a slowly changing u^* is allowed). By Pontryagin's maximum principle (Pontryagin *et al.*, 1962), denoted PMP in the sequel, the necessary condition for some u^* to be optimal may be obtained as follows, see Proposition 1 in the Appendix. Introduce the Hamiltonian:

$$H = -f_0 + \psi_1 f_1, \quad (21)$$

where $\psi = [\psi_1]^\top$ is the so-called adjoint variable, or co-state which is a solution of the following adjoint equation

$$\dot{\psi}_1 = q_4 q_5 \frac{u^{*2} + u^*/q_2 + 1}{u^* + q_2} \psi_1 - 1. \quad (22)$$

Then, if some function $u_{opt}^*(t)$, $t \in [t_0, t_f]$ minimizes the functional then by PMP there should exist a nontrivial solution $\psi(t)$, $t \in [t_0, t_f]$, of (22) with $\psi_1(t_f) = 0$ such that it holds for all $t \in [t_0, t_f]$

$$\max_{u^* \in [0, U]} H(x(t), u^*, \psi(t)) = H(x(t), u_{opt}^*(t), \psi(t)) \equiv 0. \quad (23)$$

Using particular system data, one has

$$\begin{aligned} & \max_{u^* \in [0, U]} \left[-x_1 + \frac{u^*/q_2 - [u^{*2} + u^*/q_2 + 1] x_1}{(q_4 q_5)^{-1} (u^* + q_2)} \psi_1 \right] \\ & = \left[-x_1 + \frac{u_{opt}^*/q_2 - [u_{opt}^{*2} + u_{opt}^*/q_2 + 1] x_1}{(q_4 q_5)^{-1} (u_{opt}^* + q_2)} \psi_1 \right]. \end{aligned} \quad (24)$$

Notice, that (22) with $\psi_1(t_f) = 0$ obviously implies that $\psi_1(t) > 0$, $\forall t \in [t_0, t_f]$. Indeed, since the expression on the right hand side of (22) multiplying ψ_1 is always strictly positive³, condition $\psi_1(t') \leq 0$ for some t' obviously implies $\psi_1(t) < 0 \forall t > t'$ which contradicts $\psi_1(t_f) = 0$. Therefore, the condition (24) is equivalent to

$$\begin{aligned} & \max_{u^* \in [0, U]} \left[\frac{u^*/q_2 - [u^{*2} + u^*/q_2 + 1] x_1}{(q_4 q_5)^{-1} (u^* + q_2)} \right] = \\ & \frac{u_{opt}^*/q_2 - [u_{opt}^{*2} + u_{opt}^*/q_2 + 1] x_1}{(q_4 q_5)^{-1} (u_{opt}^* + q_2)}. \end{aligned}$$

Computing the derivative of the above expression with respect to u^* to be maximized and looking for its zero points gives the following necessary condition

$$\frac{\partial \left[\frac{u^*/q_2 - [u^{*2} + u^*/q_2 + 1] x_1}{(q_4 q_5)^{-1} (u^* + q_2)} \right]}{\partial u^*} (u_{opt}^*) = 0, \quad (25)$$

considered, which is obviously equivalent to the maximization of the algae production.

³ Recall, that all parameters and u^* should be positive due to their biological interpretation.

giving after some tedious but straightforward evaluation that the only positive critical point is

$$u^*_{PMP} = -q_2 + \sqrt{q_2^2 + \frac{1}{x_1}}, \quad (26)$$

moreover, one can check that the above derivative changes sign at that critical point from the positive sign to the negative one. Summarizing, (26) gives **the unique maximum** of the Hamiltonian for $u^* > 0$ and it is the only possibility how to comply with the maximum principle. Due to obvious existence of the optimal solution, one can conclude that (26) defines the unique optimal control.

From the control theory point of view, the relation (26) may be interpreted as a **static state feedback** control maximizing the microalgal growth. This feedback is easily implementable as x_A is directly measured biological quantity.

Remark 2 The feedback (26) has the following property making it coherent with earlier obtained results on constant optimal control (Papáček *et al.*, 2007). Namely, the constant input $u^* = 1$ maximizes the steady state for x_A given by (11); let be $x_1^* := \frac{1}{2q_2+1}$. Therefore, after neglecting the transition process, $u^* = 1$ may be called as the optimal control within the class of constant inputs. Now, notice that substituting steady state (11) into (26) gives exactly $u^* = 1$. Moreover, for $x_1 \geq x_1^*$, (26) gives $u^*_{PMP} \leq 1$, while for $x_1 \leq x_1^*$, (26) gives $u^*_{PMP} \geq 1$, so it actually guarantee the convergence to “optimal” steady state x_1^* (11).

4. SIMULATION EXPERIMENTS

Numerous computer simulations were performed and resulting performance compared with the one of constant control input $u^* \equiv 1$. Recall, that such a constant control is optimal among all possible constant controllers in the following sense. The PSF model is for any constant fixed input a linear system having constant coefficients and constant non-homogeneity. Moreover, it can be easily shown to be asymptotically stable, therefore, after some time the second component involved in the performance index is equal to a certain steady state. In the re-parametrization introduced in this paper, $u^* \equiv 1$ maximizes the value of this steady state, thereby maximizing also its integral.

Unfortunately, in all extensive simulations the control $u^* \equiv 1$ has practically the same performance, or even slightly better one, than the feedback (26). The reason is that the improvement for the slow reduction is not that significant in order to “pay expenses” caused during the transfer to slow dynamics. Moreover, the reduction assumption that the input should vary only slowly somehow pre-determines such an outcome.

One can observe such an outcome on the simulation in Figure 4 where the second component trajectory corresponding to the feedback (26) lies steadily below the one corresponding to $u^* \equiv 1$. Simulations on Figure 4 are for the realistic biologically justified parameters based on the known model identifications, see e.g. (Rehák *et al.*, 2008) or further references within there. Further numerous simulations were performed, for various final times and system parameters (even those biologically unrealistic). All simulations confirm the above observation. In general, for faster fast dynamics and longer time interval, the results were slightly more in favor for (26), but still practically the same as for $u^* \equiv 1$.

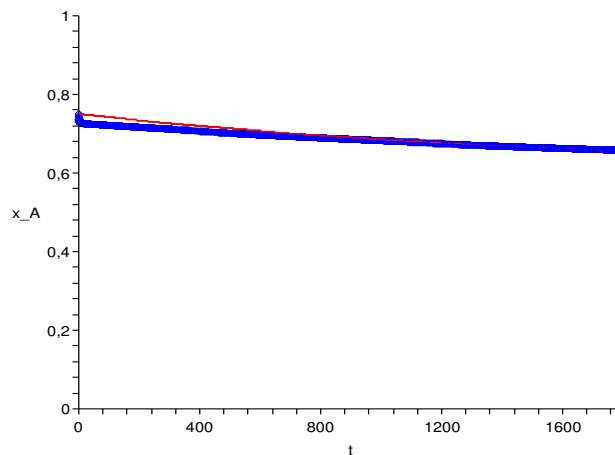


Fig. 4. State feedback law (26) along optimal trajectory (lower bold curve) state trajectory for $u^* \equiv 1$ (upper curve).

5. CONCLUSIONS AND OUTLOOKS

The problem of the optimal control of microalgal growth was considered here and treated by its reduction to a slow dynamics. The resulting formula (26) provides the state feedback law maximizing the growth of the microalgae. Nevertheless, its performance does not improve the one for constant input class which is also simple case of feedback solution. The outlook for future research are therefore to find less simplifying method to solve optimal control problem. Important lesson here is that the fast dynamics phenomena are not negligible and may play important role. This is in total contradiction with a common intuition in the biotechnological literature, cf. (Eilers and Peeters, 1993), and deserves to be promoted. Results of this paper show that reducing the system to a slow dynamics in fact equals to resignation on considering the dynamics phenomena at all, thereby giving basically the same information as the well-known Haldane type curve.

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APPENDIX

To keep the present paper self-contained, this appendix aims to give handy formulation of the Pontryagin Maximum Principle (PMP) for the case of minimizing integral performance index on a **fixed time interval, fixed initial state and free terminal state condition** and recall how it can be derived from more general PMP formulation (Pontryagin *et al.*, 1962). Such a formulation is provided by the following

Proposition 1. Consider the following system and the performance index

$$\dot{x} = f(x, u), \quad x = [x_1, \dots, x_n]^T \in \mathbb{R}^n, \quad u(t) \in U \subset \mathbb{R}^m, \quad (27)$$

$$J = \int_{t_0}^{t_f} f_0(x, u) dt, \quad x(t_0) = x^0, \quad x(t_f) \in \mathbb{R}^n, \quad (28)$$

to be minimized choosing a measurable function $u(t)$ where $x^0 \in \mathbb{R}^n$, $0 \leq t_0 < t_f$ and compact U are given. Suppose u^{opt}

is an optimal control minimizing performance index in problem (27,28) and let $x^{opt}(t), x^{opt}(0) = x^0$, be the corresponding state trajectory. Then there exists nontrivial solution $\psi(t) = [\psi_1(t), \dots, \psi_n(t)]^T$ of the following adjoint equation

$$\dot{\psi} = \left[\frac{\partial f_0}{\partial x}(u^{opt}, x^{opt}) \right]^T - \left[\frac{\partial f}{\partial x}(u^{opt}, x^{opt}) \right]^T \psi, \quad \psi(t_f) = 0,$$

such that

$$\max_{u \in U} \left[-f_0(x^{opt}, u) + \psi^T f(x^{opt}, u) \right] = \left[-f_0(x^{opt}, u^{opt}) + \psi^T f(x^{opt}, u^{opt}) \right].$$

Proof The optimal control u^{opt} existing by the assumption of the proposition obviously solves the following optimal control problem as well:

for given t_0, t_f, x_0, U , find $T_0 < T_f \in \mathbb{R}$ and measurable $u^{opt}(t) \in U, t \in [T_0, T_f]$, to minimize $x_0(T_f)$, where:

$$\begin{aligned} \dot{x}_0 &= f_0(x, u), \quad \dot{x} = f(x, u), \quad \dot{x}_{n+1} = 1, \\ [x_0, x, x_{n+1}](T_0) &= (0, x^0, t_0), \\ [x_0, x, x_{n+1}](T_f) &\in \\ \{\tilde{x} = [\tilde{x}_0, \tilde{x}, \tilde{x}_{n+1}] \in \mathbb{R}^{n+2} \mid \tilde{x}_{n+1} = t_f\}. \end{aligned}$$

By Theorem 3 in (Pontryagin *et al.*, 1962)), if $u^{opt}(t) \in U, t \in [T_0, T_f]$, is a solution of the above reformulated problem then there should exist the nontrivial solution $\bar{\psi}(t)$ of⁴

$$\dot{\psi}_0 = 0, \quad \dot{\psi} = - \left[\frac{\partial f_0}{\partial x} \right]^T \psi_0 - \left[\frac{\partial f}{\partial x} \right]^T \psi, \quad \dot{\psi}_{n+1} = 0 \quad (29)$$

$$\bar{\psi} := [\psi_0, \psi, \psi_{n+1}]^T, \quad \psi(T_f) = 0, \quad (30)$$

such that the following conditions hold

$$H(\bar{\psi}(t), u^{opt}(t), x^{opt}(t)) = \max_{u \in U} H(\bar{\psi}(t), u, x^{opt}(t)) \quad (31)$$

$$\psi_0(T_f) \leq 0, \quad H(\bar{\psi}(T_f), u^{opt}(T_f), x^{opt}(T_f)) = 0, \quad (32)$$

where $H := \psi_0 f_0 + \psi^T f + \psi_{n+1}$ is the Hamiltonian of the extended system. By (30) the conditions (32) equivalent to

$$\psi_{n+1}(T_f) + \psi_0(T_f) f_0(T_f) = 0, \quad \psi_0(T_f) \leq 0,$$

and by (29,30) the previous property is equivalent to

$$\psi_{n+1}(T_f) + \psi_0(T_f) f_0(T_f) = 0, \quad \psi_0(T_f) < 0, \quad (33)$$

otherwise $\psi_0(T_f) = 0$ and the solution of the co-state extended system would be trivial which contradicts the requirement of PMP. Now, notice that for any $\psi_0(T_f) < 0$ the first condition in (33) can be always satisfied choosing proper $\psi_{n+1}(T_f)$, which moreover, does not appear elsewhere, i.e. condition (33) is reduced to $\psi_0(T_f) < 0$. Finally, due to homogeneity of both the adjoint equation (29) and Hamiltonian $H := \psi_0 f_0 + \psi^T f + \psi_{n+1}$ with respect to $\bar{\psi} = [\psi_0, \psi, \psi_{n+1}]^T$, it should exist the co-state solution with $\psi_0(T_f) = -1$ as well. Observing that the remaining conditions constitute those given in the formulation of the proposition to be proved concludes the proof. \square

⁴ The second equality in (30) is due to the so-called transversality condition which requires that the co-state vector at both initial and terminal times should be perpendicular to initial and terminal sets boundaries at the appropriate optimal trajectory points. In our case the terminal set is hyperplane given by the condition that the last component is fixed. Notice also, that in (29) $\frac{\partial f_0}{\partial x}(x, u)$ and $\frac{\partial f}{\partial x}(x, u)$ are evaluated along $u^{opt}(t)$ and the corresponding state trajectory $x^{opt}(t), x^{opt}(0) = x^0$.