Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Regularization of the Limiting Optimal Controller in Robust Stabilization ⋆

Svetlana Iantchenko * **Andrey Ghulchak** **

* *Department of Mathematics, Lund University, Box 118, 221 00 Lund, Sweden (email: sveta@maths.lth.se)*
** *Department of Mathematics, Lund University, Box 118, 221 00 Lund, Sweden (email: ghulchak@maths.lth.se)*

Abstract: In this paper, we consider the problem of robust optimization for a system with uncertainty of rank one. The main result is the regularization procedure of the limiting optimal controller. We derive a method to obtain a low order suboptimal controller that provides a stability margin as close to the optimal one as necessary. The method is illustrated by two examples.

1. INTRODUCTION

The convex duality principle has brought a new insight into robust stabilization. The dual problem for systems with uncertainties of rank one has been introduced in Ghulchak [2004]. Several examples have been studied in Ghulchak and Rantzer [2002] to illustrate the power and simplicity of the principle. A method based on unstable cancelations has been presented to calculate the largest stability margin and to design the optimal controller of low order. Unfortunately, the optimal controller is never robustly stabilizing itself and needs to be approximated somehow by stabilizing ones. An approximation should ideally preserve the low order of the optimal controller. Different *ad hoc* ideas have been used in Ghulchak and Rantzer [2002] to obtain the suboptimal controller, for example, by analyzing Bode/Nyquist plot, however, a general approach to this problem still needs to be developed. In this paper, such an approach is presented. The controller obtained is of low order and provides a stability margin that is arbitrary close to the optimal one.

2. NOTATIONS

By $\mathbb{R}$ (or $\mathbb{C}$) we denote the field of real (or complex) number. The unit circle and the open unit disc in $\mathbb{C}$ are denoted by $\mathbb{T}$ respectively $\mathbb{D}$

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$
$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

For any measurable $Y \subset \mathbb{C}^n$, the notation $L^p(Y)$ stands for the standard Lebesgue space of functions $f : \mathbb{T} \to Y$ equipped with the norm

$$\|f\|_p = \begin{cases} (\int_{\mathbb{T}} |f(z)|^p dm(z))^{\frac{1}{p}}, & 1 \le p < +\infty, \\ \operatorname{ess\,sup}_{z\in\mathbb{T}} |f(z)|, & p = +\infty \end{cases}$$

where by $|\cdot|$, we denote the usual 2-norm in $\mathbb{C}^n$

$$|f| = \sqrt{|f_1|^2 + |f_2|^2 + \ldots + |f_n|^2}.$$

$\mathbf{H}^p(Y)$ denotes Hardy space of functions in $\mathbf{L}^p(Y)$ that can be analytically continued inside the unit disk. $\mathbf{H}^p_0(Y)$ denotes the shifted $\mathbf{H}^p(Y)$

$$\mathbf{H}^p_0(Y) = z\mathbf{H}^p(Y) = \{f \in \mathbf{H}^p(Y) | f(0) = 0\}.$$

The space $\mathcal{C}$ is the space of continuous functions on $\mathbb{T}$.

The disk algebra $\mathbf{A}(Y)$ is the subspace of $\mathbf{H}^\infty$ that consists of analytic functions in $\mathbb{D} \subset Y$ that can be extended continuously to the closed unit disk.

The brief notations $\mathbf{A}$, $\mathbf{H}^\infty$ etc. will be used if $Y = \mathbb{C}^n$ and the dimension of the space is clear from context.

The superscript $^\top$ stands for transposition and † for pseudoinverse. A bar denotes the complex conjugate. The prefix $\mathcal{B}$ denotes the unit ball in the corresponding space.

3. PRELIMINARIES

Given a nominal plant P and an uncertainty set $\Delta \ni 0$, the general robust controller design problem is to find a controller K that robustly stabilizes the whole family of perturbed plants

$$P_\delta = \frac{N + \delta^\top G_1}{M + \delta^\top G_2}, \ \delta \in \nu\Delta$$

for as large ν as possible. Here $F = (N\ M) \in \mathbf{A}_{1\times n}$ is the coprime factorization of the plant P, the weight matrix $G = (G_1\ G_2) \in \mathbf{A}_{m\times n}$ and the set Δ is a convex compact set in $\mathbb{C}^m$. It was shown in Rantzer and Megretski [1994] that the problem of finding a controller is equivalent to the condition in terms of a function $h \in \mathbf{H}_\infty$

$$\text{Re } (F + \delta^\top G(z))h(z) > 0, \ \forall z \in \mathbb{T}, \ \forall \delta \in \nu\Delta. \qquad (1)$$

The problem of finding $h \in \mathbf{H}_\infty$ such that condition (1) holds is considered as a primal problem. We would like to solve it for as large ν as possible, that is, for

$$\nu_{opt} = \sup\{\nu | \exists h \in \mathbf{H}^\infty : \text{Re}\,(F + \delta^\top G)h > 0\ \forall \delta \in \nu\Delta.\}$$

⋆ This work is supported by the Swedish Research Council, Project 2005-3911

In Ghulchak [2004] a dual problem was introduced as follows:

Theorem 1. Let $F \in \mathbf{A}_{1\times n}$, $G \in \mathbf{A}_{m\times n}$ and denote $\Phi_\delta = F + \delta^\top G$. Then the optimal value ν_{opt} takes the following form

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\},$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}_\infty(\nu\Delta) \colon \Phi_\delta w \in \mathbf{H}_0^1\} \quad (2)$$

and the singular part

$$\nu_{opt|s} = \inf\{\nu | \exists z \in \mathbb{T},\ \delta \in \nu\Delta : \Phi_\delta(z) = 0\}. \quad (3)$$

Suppose we have solved the primal/dual problem, i.e. we found the maximal stability margin ν_{max}. We know that we can determine the optimal controller using the alignment principle for convex optimization. According to the alignment principle,see Luenberger [1969], we get

$$(N + \delta_{opt}^\top G_1)h_1 - (M + \delta_{opt}^\top G_2)h_2 \equiv 0,$$

and the optimal controller

$$h_{opt} = \frac{h_1}{h_2} = \frac{M + \delta_{opt}^\top G_2}{N + \delta_{opt}^\top G_1}. \quad (4)$$

It means that the optimal controller is the inverted plant where the worst uncertainty is plugged in.

In Ghulchak [2001] and Iantchenko and Ghulchak [2007] the dual parametrization of plant factors with destabilizing uncertainties was proposed. According to this parametrization *the numerator and denominator of the plant with the worst uncertainty have more common unstable zeros than the total number of unstable poles.* The occurrence of common unstable zeros leads to a zero/pole cancelation.This cancelation results in the optimal controller being of low order. The major drawback of this controller is that the closed loop system is not robustly stabile. We can deal with this problem if we construct a controller that (a) is robustly stabilizing, (b) provides the stability margin close to the optimal one, (c) is still of low order. Formally, we want to find a low order suboptimal controller h_ε such that

$$\text{Re } (F + \delta^\top G(z))h_\varepsilon(z) > 0, \forall \delta \in (\nu_{max} - \varepsilon)\Delta.$$

For this purpose we increase the stability area. Consider $F, G \in \mathbf{A}$ on the unit ball $\mathcal{B}$. Suppose that F and $G \in \mathbf{A}$ on the bigger area $\mathcal{B}_\varepsilon$. By changing the variable z to $w = (1-\varepsilon)z$ we get new F_ε and G_ε. We will solve the primal/dual problem for $F_\varepsilon + \delta^\top G_\varepsilon$ and obtain the maximal stability margin ν_ε and the optimal controller h_ε. Since we use the duality principle and the dual parametrization of plant factors with destabilizing uncertainties the controller h_ε will be of low order. To show that this controller provides a stability margin close to the optimal one it is enough to show that $\nu_\varepsilon \to \nu_{max}$ if $\varepsilon \to 0$. We have the following theorem:

Theorem 2. $\nu_\varepsilon \to \nu_{max}$ if $\varepsilon \to 0$ and

h_ε stabilizes the system $F + \delta^\top G$.

Proof. See Appendix.

Since the controller h_ε has demanded properties (a),(b) and (c), h_ε will be the suboptimal controller we wanted to construct.

According to the method we have described above we can now propose an algorithm to find the suboptimal controller. Usually the system we consider is defined in the right half plan and we will begin with the conformal bilinear transformation of the right half complex plan onto the unit disk.

(1) Transform the right half plan onto the unit disk by changing the variable $z = \frac{1-s}{1+s}$.
(2) Increase the stability area to B_ε by variable $w = (1-\varepsilon)z$
(3) Transform the stability area B_ε onto the right half plan by changing the variable $s_{new} = \frac{1-w}{1+w}$.
(4) Solve the primal/dual problem for $F_\varepsilon + \delta^\top G_\varepsilon$ and obtain the maximal stability margin ν_ε and the optimal controller h.
(5) Change back to the original stability area by variable $s = \frac{((2-\varepsilon)s_{new}+\varepsilon)}{(2-\varepsilon+\varepsilon s_{new})}$ and obtain the suboptimal controller h_ε.

To illustrate how this algorithm works we will solve two examples: the first one is the robust stabilizability of the plant $\frac{s-\delta}{(s-1)^2}$ and second one is the gain margin optimization for the plant $\delta\frac{s-1}{(s+1)(s-2)}$.

4. DESIGN OF SUBOPTIMAL CONTROLLER TO $\frac{S-\delta}{(S-1)^2}$.

The robust stabilizability of $\frac{s-\delta}{(s-1)^2}$ on the uncertainty set $\|\delta\| \le \nu$ was solved in Ghulchak and Rantzer [2002]. The stability margin ν_{max} was calculated and the optimal controller that achieves the optimal level of stability was designed. However the controller was not proper. Now we will show how we can design a robustly stabilizing low order suboptimal controller that provides a stability margin as close to the optimal one as we wish. We will consider the case of complex δ.

According to the proposed algorithm we will start with tree times changing the variable and obtain the following system:

$$G_{\delta,\varepsilon} = \frac{\frac{(2-\varepsilon)s-\varepsilon}{2-\varepsilon-\varepsilon s} - \delta}{\frac{4(s-1)^2}{(2-\varepsilon-\varepsilon s)^2}}$$

$$G_{\delta,\varepsilon} = \frac{((2-\varepsilon)s - \varepsilon)(2-\varepsilon-\varepsilon s) - \delta(2-\varepsilon-\varepsilon s)^2}{4(s-1)^2}$$

We will solve the problem to calculate the maximal stability margin ν_{max} and design the optimal controller for $G_{\delta,\varepsilon}$. We use the duality principle. Recall that according to this principle the numerator and the denominator of the plant with the worst uncertainty have more common unstable zeros than the total number of unstable poles.

The denominator of $G_{\delta,\varepsilon}$ has double zero at $s = 1$. It means that the number of possible common unstable zeros cannot exceed two and the numerator and the denominator can have at most one unstable pole which must be a pole of δ. Apart from this unstable pole δ can have a double pole at $s = \frac{2-\varepsilon}{\varepsilon}$. Summarizing the duality principle we get

$\nu_{max} = \inf \|\delta\|_\infty$ over all δ that have three unstable poles (two of them are known) and provide the plant with two unstable cancelations.

We can choose

$$\delta(s) = \frac{\gamma s^3 + bs^2 + cs + d}{(2-\varepsilon-\varepsilon s)^2(s-a)},\ a > 0. \tag{5}$$

We have to choose δ proper, otherwise $\|\delta\|_\infty$ will be unbounded. By a we denote an unstable pole the plant $G_{\varepsilon,\delta}$ can have. Recall that it has to be the only one unstable pole. Put δ in the plants equation and we get the numerator

$$\frac{s^3(-\varepsilon(2-\varepsilon)-\gamma) + s^2(a\varepsilon(2-\varepsilon)+(2-\varepsilon)^2+\varepsilon^2-b)}{(2-\varepsilon-\varepsilon s)^2(s-a)} +$$

$$\frac{s(-a(2-\varepsilon)^2 - a\varepsilon^2 - \varepsilon(2-\varepsilon)-c) + (a\varepsilon(2-\varepsilon)-d)}{(2-\varepsilon-\varepsilon s)^2(s-a)} =$$

$$\frac{(s-1)^2(\beta s + k)}{(2-\varepsilon-\varepsilon s)^2(s-a)},$$

where the last equality comes from the necessity to have double zero at 1. Then

$$G_{\delta,\varepsilon} = \frac{(\beta s + k)}{4(s-a)}.$$

A suboptimal controller will have the order one.

For our purpose to design a controller we have to find 8 variables. We have 4 equations due to the unstable cancelation condition from above. Another four equations we get using the property of δ. We know that in the complex case δ_{opt} will be an all-pass function with $|\delta(j\omega)| = \nu_{max}$. Now we have a possibility to solve our system and find δ_{opt}.

For the sake of brevity we omit the technical calculations and just present the result for different ε.

For $\varepsilon = 0.1$ we get $a = 2.606$ and $\nu_{max} = 0.361$. The suboptimal controller is

$$h_\varepsilon = \frac{(s-2.959)}{0.005s + 1.124}.$$

Now take $\varepsilon = 0.01$. For this ε we found $a = 2.431$ and $\nu_{max} = 0.409$. The suboptimal controller is

$$h_\varepsilon = \frac{(s-2.728)}{0.048s + 5.033}.$$

The last value we will take is $\varepsilon = 0.001$. Then we get $a = 2.416$ and $\nu_{max} = 0.4137$. The suboptimal controller will be

$$h_\varepsilon = \frac{(s-2.707)}{0.023s + 4.569}.$$

Note that all suboptimal controllers are robustly stabilizing and have the order one.

In Ghulchak [2004], the maximal stability margin was calculated and found the optimal controller that has achieved this level of stability. The unstable pole in the numerator $a = 1 + \sqrt{2}$, the stability margin $\nu_{opt} = \sqrt{2} - 1 \approx 0.414$ and the optimal controller is $K_{opt} = s - a$. If we will compare our results with the optimal one we will see that in case $\varepsilon = 0.1$ the suboptimal controller provides robust stability for $|\delta| \le 0.361 \approx \nu_{opt} - 0.053$, if $\varepsilon = 0.01$ the suboptimal controller achieves robust stability for $|\delta| \le 0.409 \approx \nu_{opt} - 0.005$ and finally if $\varepsilon = 0.001$ the suboptimal controller provides robust stability for $|\delta| \le 0.4137 \approx \nu_{opt} - 0.0003$. We have seen that we can find a first order suboptimal controller that provides a stability margin arbitrary close to the optimal one.

5. GAIN MARGIN OPTIMIZATION.

In this section we will consider the gain margin problem for the plant

$$G_\delta(s) = \delta G(s) = \delta \frac{s-1}{(s+1)(s-2)}. \tag{6}$$

Doyle et al. [1992] have shown that the largest achievable $k_{opt} = 4$ and suggested a sixth order controller for the gain margin $k = 3.5$. Ghulchak [2004] has found the 2-d order optimal controller and regularized it by perturbing the Nyquist plot.

Let us show next that the method suggested in the paper does a regularization of the optimal controller without going too deep into analysis of the Nyquist plot by merely following the five steps of our algorithm. Again we will start with three changes of the variable and obtain the following system:

$$G_{\delta,\varepsilon} = \delta \frac{(s-1)(2-\varepsilon-\varepsilon s)}{(1-\varepsilon)(s+1)((2+\varepsilon)s - 4 + \varepsilon)}. \tag{7}$$

The denominator of (7) has one unstable zero at $s = \frac{4-\varepsilon}{2+\varepsilon}$. It means that the numerator should not have any unstable poles at all and $\delta(s-1)(2-\varepsilon-\varepsilon s)$ should be an analytical function with zero at $s = \frac{4-\varepsilon}{2+\varepsilon}$. It gives us that the function δ must contain the unstable factors $s-1$ and $2-\varepsilon-\varepsilon s$ in the denominator, the factor $(2+\varepsilon)s - (4-\varepsilon)$ in the numerator, be real on the imaginary axis and proper. We have just one possibility for δ :

$$\delta(s) = \gamma \frac{((2+\varepsilon)^2 s^2 - (4-\varepsilon)^2)(b^2 - s^2)}{(s^2-1)((2-\varepsilon)^2 - \varepsilon^2 s^2)}$$

for some constant γ. Then

$$G_{\delta,\varepsilon} = \delta G = \gamma \frac{((2+\varepsilon)s + (4-\varepsilon))(b^2 - s^2)}{(s+1)^2(1-\varepsilon)((2-\varepsilon)+\varepsilon s)}.$$

A suboptimal controller will be of the third order.

On the imaginary axis the function should belong to the interval $[1, k]$.

$$\delta(j\omega) = \gamma \frac{((2+\varepsilon)^2\omega^2 + (4-\varepsilon)^2)(b^2 + \omega^2)}{(\omega^2+1)((2-\varepsilon)^2 - \varepsilon^2\omega^2)} \in [1, k].$$

We solve the problem for different ε. Take first $\varepsilon = 0.1$. The optimal k_ε is 2.7375 ($\approx k_{opt} - 1.26$) which corresponds $\gamma = 0.00621$. The suboptimal controller is

$$h_\varepsilon = \frac{1.2685(0.105s+1)(s+1)^2}{(0.1496s+1)(s+1.74)(1-0.045s)}.$$

For $\varepsilon = 0.01$ we get that the $k_\varepsilon = 3.8272$ ($\approx k_{opt} - 0.17$) which corresponds $\gamma = 0.000095$. The suboptimal controller is

$$h_\varepsilon = \frac{1.0297(0.01005s+1)(s+1)^2}{(0.015s+1)(s+1.970)(1-0.005s)}$$

And finally for $\varepsilon = 0.001$ we will come very close to k_{opt} which is 4. Now we calculate $k_\varepsilon = 3.9821$ ($\approx k_{opt} - 0.018$) which corresponds $\gamma = 0.99452 \cdot 10^{-6}$. The suboptimal controller is

$$h_\varepsilon = \frac{1.003(0.0010s + 1)(s + 1)^2}{(0.0015s + 1)(s + 1.997)(1 - 0.0005s)}$$

Note that we have designed a suboptimal controller of third order. We can obtain the gain margin k_ε arbitrary close to the optimal k without increasing the order of controllers.

6. CONCLUSION

In this paper we have proposed a method to construct a robustly stabilizing controller for a system with parametric uncertainty. The controller obtained is nearly optimal in the sense that the stability margin of the closed loop system approximates the largest possible one for this plant with a given precision. At the same time the order of the controller is low and remains unchanged for all such approximations. The idea of the method is to increase the instability region by small ε, to find the optimal controller there and to shrink it back to the original one. Such a "regularization" is shown to be continuous with respect to the stability margin.

Appendix A. APPENDIX

A.1 Proof of Theorem 2

Proof. Denote $\mathbf{H}_\infty^+ = \mathbf{H}_\infty(\mathcal{B}_\varepsilon)$ and $\mathbb{T}^+$ is the boundary of $\mathcal{B}_\varepsilon$. According to changing the variable as we have proposed above ($w = (1 - \varepsilon)z$) and the assumption that F and G are analytic functions in bigger area $\mathcal{B}_\varepsilon$ that can be extended continuously to the closed disk, the solution to the primal problem with F_ε and G_ε for $h \in \mathbf{H}_\infty$ and $z \in \mathbb{T}$ is the same as the solution to the primal problem with F and G for $h \in \mathbf{H}_\infty^+$ and $z \in \mathbb{T}^+$, i.e.

$$\sup_{h\in\mathbf{H}_\infty^+} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}^+} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) = \sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z).$$

According to the mean value theorem for harmonic functions, see for example W.Rudin [1973],

$$\inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) \geq \inf_{z\in\mathbb{T}^+} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z)$$

and since $\mathbf{H}_\infty^+ \subset \mathbf{H}_\infty$ we get

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) \geq \sup_{h\in\mathbf{H}_\infty^+} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}^+} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z)$$

and

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) \geq \sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z).$$

Call the problem

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G)h(z) > 0$$

as the first problem and the problem

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z) > 0$$

as the second.

To show the first part of the theorem let take ν such that the second problem has solution. We have denoted ν_ε the optimal stability margin in the second problem, i.e. $\mathrm{Re}\,(F_\varepsilon + \delta_{opt}^\top G_\varepsilon)h_{opt} = 0$, where $\|\delta_{opt}\| = \nu_\varepsilon$. It's clear that $\nu_\varepsilon > \nu$. But if the second problem has a solution for ν, according to our inequality from above, the first problem has a solution too and $\nu_{max} > \nu$. We get that

$$\nu_\varepsilon > \nu \Rightarrow \nu_{max} > \nu$$

and it means that $\nu_{max} \geq \nu_\varepsilon$.

We will show that there is ε_ν such that $\nu_\varepsilon \geq \nu_{max} - \varepsilon_\nu$.

$$\mathrm{Re}\,(F_\varepsilon + \delta^\top G_\varepsilon)h = \mathrm{Re}\,(F_\varepsilon - F + F + \delta^\top (G_\varepsilon - G + G))h = \mathrm{Re}\,(F + \delta^\top G)h + \mathrm{Re}\,(F_\varepsilon - F + \delta^\top (G_\varepsilon - G))h.$$

Using the properties of sup and inf we get that

$$\inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z) \geq \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) - \sup_{z\in\mathbb{T}} |\mathrm{Re}\,(F_\varepsilon(z) - F(z) + \delta^\top (G_\varepsilon(z) - G(z)))h(z)| \geq$$

$$\inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) - \sup_{z\in\mathbb{T}} |\mathrm{Re}\,(F_\varepsilon(z) - F(z))h(z)| - \sup_{z\in\mathbb{T}} |\mathrm{Re}\,(\delta^\top (G_\varepsilon(z) - G(z)))h| \geq$$

$$\inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) - \|F_\varepsilon - F\| - |\delta|\|G_\varepsilon - G\|$$

and

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z) \geq \sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) - \|F_\varepsilon - F\| - \sup_{\delta\in\Delta_\nu} |\delta|\|G_\varepsilon - G\|.$$

Since F and G are continuous, $\|F_\varepsilon - F\| \to 0$ and $\|G_\varepsilon - G\| \to 0$ as $\varepsilon \to 0$. It means that

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z) \geq \sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) - \varepsilon_1.$$

Let take ν such that the first problem has a solution. Then there exists ε_2 such that

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) - \varepsilon_2 > 0,$$

i.e. the second problem has a solution too if ε is sufficiently close to 0. It means that $\exists\varepsilon$ such that $\nu_\varepsilon > \nu$.
Take now ν closed to ν_{max}, i.e. $\nu = \nu_{max} - \varepsilon_\nu$ and repeat this argumentation. We get

$$\forall\varepsilon_\nu > 0\ \exists\varepsilon : \nu_\varepsilon > \nu = \nu_{max} - \varepsilon_\nu.$$

We have shown that there exists ε such that $\nu_{max} - \varepsilon_\nu < \nu_\varepsilon \leq \nu_{max}$. It means that if $\varepsilon_\nu \to 0$, $\nu_\varepsilon \to \nu_{max}$. It is clear that ε has to be sufficiently small.

It remains to show that h_ε stabilizes the system $F + \delta^\top G$. From inequality

$$\sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F(z) + \delta^\top G(z))h(z) \geq \sup_{h\in\mathbf{H}_\infty} \inf_{\delta\in\Delta_\nu} \inf_{z\in\mathbb{T}} \mathrm{Re}\,(F_\varepsilon(z) + \delta^\top G_\varepsilon(z))h(z)$$

it is clear that if h stabilizes the second system for fix δ the same h will stabilize the first system for the same δ. It means that if we have found the optimal controller $h_{opt,\varepsilon}$ that stabilizes the second system for all δ such that $|\delta| \leq \nu_\varepsilon$ the same controller will stabilize the first system for the same δ. We have

$$\mathrm{Re}\,(F_\varepsilon(w) + \delta^\top G_\varepsilon(w))h_{opt,\varepsilon}(w) =$$
$$\mathrm{Re}\,(F(\frac{w}{1-\varepsilon}) + \delta^\top G(\frac{w}{1-\varepsilon}))h_{opt,\varepsilon}(w).$$

Now change the variable back $z = \frac{w}{1-\varepsilon}$ and we get

$$\mathrm{Re}\,(F(\frac{w}{1-\varepsilon}) + \delta^\top G(\frac{w}{1-\varepsilon}))h_{opt,\varepsilon}(w) =$$
$$\mathrm{Re}\,(F(z) + \delta^\top G(z))h_\varepsilon(z),$$

where $h_\varepsilon = h_{opt,\varepsilon}((1-\varepsilon)z)$ is the suboptimal controller that stabilizes the system $F + \delta^\top G$ with the stability margin $\nu_\varepsilon > \nu_{max} - \varepsilon_\nu$, i.e. arbitrary close to the optimal one .

REFERENCES

J. Doyle, B.Francis, and A. Tannenbaum. *Feedback Control Theory.* New York: MacMillan, 1992.

A. Ghulchak. Duality in robust control:uncertainty vs. controller. Conference on Decision and Control, Orlando, USA, December 4-7 2001.

A. Ghulchak. Robust stabilizability and low-order controllers: Duality principle in case studies. Conference on Decision and Control, Atlantis, Bahamas, December 14-17 2004.

A. Ghulchak and A. Rantzer. Duality in $\mathbf{H}_\infty$ cone optimization. *SIAM J.Control Optim.*, 41(1):253–277, 2002.

S. Iantchenko and A. Ghulchak. Canonical parametrization of the dual problem in robust optimization:nonrational case. European Control Conference, Kos, Greece, 2007.

David G. Luenberger. *Optimization by Vector Space Methods.* John Wiley and Sons, 1969.

A. Rantzer and A. Megretski. A convex parametrization of robustly stabilizing controllers. *IEEE Transaction on Automatic Control*, 39(9):1802–1808, 1994.

W.Rudin. *Real and Complex Analysis.* McGraw-Hill, New York, 1973.