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Iterative learning control based ramp metering tracking various trajectories[1]

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Abstract: In this work, we apply the iterative learning control approach to address the traffic density control via ramp metering in a macroscopic level freeway environment. The traffic density control problem is formulated into an output tracking problem and the tracking trajectories are variable with time and iteration change. Rigorous analyses and intensive simulations show that the iterative learning control method we proposed can deal with this kind of problem successfully.

Keywords: Iterative Learning control; ramp metering; tracking various trajectories.

1. INTRODUCTION

Freeway traffic congestion is a major problem in today's metropolitan areas. It leads to delays, reduced traffic safety, increased fuel consumption, severe air pollution and the freeway infrastructure under-used. Consequently, freeway traffic control, which aims to solve traffic problems such as making the freeway used effectively, becomes ever increasing important.

Among numerous freeway traffic control methods, ramp metering, which based on monitoring the freeway on-ramps and preventing traffic volume from exceeding freeway capacity, is the major one. It can be implemented by using traffic lights to meter the number of entering vehicles. Ramp metering, when properly applied, is an effective way to ease freeway congestion and improve freeway utilization (M. Papageorgiou, 1983; M. Papageorgiou et al, 1989; M. Papageorgiou et al, 1990a,b; M. Papageorgiou et al, 2002).

Ramp metering strategies includes local ramp metering and coordinated ramp metering. Local ones are much easier to design and implement. Moreover, in many cases, they have been proven to be non-inferior to coordinated approaches (M. Papageorgiou et al, 2002; M. Papageorgiou et al, 1997). The demand-capacity (DC) control, the occupancy (OCC) control (D. P. Masher, et al, 1975) and ALINEA (M. Papageorgiou et al, 1991) are typical local ramp metering strategies.

It is worth noting that the macroscopic traffic flow patterns are in general repeated every day and the congestions typically happen at the same locations. Ruling out the occasional occurrence of accidents, the routine traffic flow on freeway in the macroscopic level will show inherent repeatability everyday. However, all the traffic control methods mentioned above lack the ability to learn and improve the control performance from a repeated traffic process. Without learning, a control system can only produce the same performance and never works better. To solve this problem, we propose to use iterative learning control as we mentioned in (Zhongsheng Hou et al, 2007; Zhongsheng Hou et al, 2004; Zhongsheng Hou et al, submitted to IEEE TVT). However, as we considered in previous research, the tracking trajectory is the same in every time and every iteration. Practically, the tracking target may various based on the actual needs, i.e., the peak hours may be different from other time, the rainy days and the sunny days can not be treated as the same. Here we will discuss this issue.

This paper is organized as follows. Section 2 gives the discrete traffic flow model and formulates the density control into an output tracking problem in the state space. Section 3 the convergence analysis of the proposed ILC controllers is presented. Simulation results are provided in Section 4. Section 5 concludes the paper.

2. TRAFFIC FLOW MODEL AND PROBLEM FORMULATION

2.1 Traffic Model

The space and time discretized traffic flow model for a single freeway lane with one on-ramp and one off-ramp is given by (1)-(4) below.

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{L_i}[q_{i-1}(k) - q_i(k) + r_i(k) - s_i(k)], \quad (1)$$

$$q_i(k) = \rho_i(k) v_i(k), \quad (2)$$

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$$v_i(k+1) = v_i(k) + \frac{T}{\tau}[V(\rho_i(k)) - v_i(k)] + \frac{T}{L_i} v_i(k)[v_{i-1}(k) - v_i(k)] - \frac{\nu T}{\tau L_i} \frac{[\rho_{i+1}(k) - \rho_i(k)]}{[\rho_i(k) + \kappa]}, \quad (3)$$

$$V(\rho_i(k)) = v_{free}(1 - [\frac{\rho_i(k)}{\rho_{jam}}]^l)^m, \quad (4)$$

where T is the sample time interval in hour, $k = \{0,1,\cdots,K\}$ is the k-th time interval; $i = \{1,2,\cdots,N\}$ is the i-th section of a freeway, and N is the total number of sections. Model variables are listed below. $\rho_i(k)$: density in section i at time kT, (veh/lane/km); $v_i(k)$: space mean speed in section i at time kT, (km/h); $q_i(k)$: traffic flow departure section i at time kT, (veh/h); $r_i(k)$: on-ramp flow rate for section i at time kT, (veh/h); $s_i(k)$: off-ramp flow rate for section i at time kT, (veh/h), which is regarded as an unknown disturbance; L_i: Length of section i, (km); v_{free} and ρ_{jam} are the free speed and maximum possible density per lane, respectively; τ, ν, κ, l, m are constant parameters characterizing a given traffic system in terms of the street geometry, vehicle characteristics, drivers' behaviours, etc.

2.2 Boundary

Boundary conditions can be summarized as follows:

$$\rho_0(k) = q_0(k)/v_1(k), \quad (5)$$

$$v_0(k) = v_1(k), \quad (6)$$

$$\rho_{N+1}(k) = \rho_N(k), \quad (7)$$

$$v_{N+1}(k) = v_N(k), \quad \forall k. \quad (8)$$

2.3 Control Objective

Denote the set of sections that have on-ramps by I_p, $I_p = [i_1, i_2, \cdots, i_p]$, where $i_j (j = 1,2,\cdots,p)$ is the number of the section with an on-ramp, p is the total number of sections with on-ramps. $\Re$ is a linear transformation mapping on a norm space X^N, that is, $\Re : X^N \to X^p$, and $\Re(X) = QX$, where $Q = [\varepsilon_{i1}, \varepsilon_{i2}, \cdots, \varepsilon_{ip}]^T$ is a $p \times N$ matrix, $\varepsilon_{ij} = [0, \cdots 0, 1, 0, \cdots, 0]^T$ represents the unit vector with only the i_j th component to be 1. Further define $P = Q^T$. The control objective is to seek an appropriate control profile which specifies the on-ramp traffic flow $r_{I_p}(k)$, that drives traffic density of sections I_p at time k to converge to the desired traffic density $\rho_{I_p,desired}(k)$ for $k \in \{0,1,\cdots,K\}$, despite the modeling uncertainties and disturbances occurred at some off-ramps.

2.4 State Space Representation and Assumptions

The macroscopic traffic flow model described by equations (1) (2) can be written in the following form

$$\rho_i(k+1) = a_i(k)\rho_i(k) + b_i(k)\rho_{i-1}(k) + c_i(k)r_i(k) - c_i(k)s_i(k), \quad (9)$$

where

$$a_i(k) = 1 - \frac{T}{L_i} v_i(k), \; b_i(k) = \frac{T}{L_i} v_{i-1}(k), \; c_i(k) = \frac{T}{L_i}.$$

Denote

$$x(k) = [v_1(k), v_2(k), \cdots, v_N(k)]^T,$$

$$y(k) = [\rho_1(k), \rho_2(k), \cdots, \rho_N(k)]^T,$$

$$u(k) = [r_{i_1}(k), r_{i_2}(k), \cdots, r_{i_p}(k)]^T,$$

$$w(k) = [c_1(k)s_1(k), c_2(k)s_2(k), \cdots, c_N(k)s_N(k)]^T,$$

$$A(x(k)) = \begin{bmatrix} a_1(k) & 0 & & \cdots & 0 \\ b_2(k) & a_2(k) & & \cdots & 0 \\ 0 & b_3(k) & a_3(k) & & \\ \vdots & & & & \\ & & & b_N(k) & a_N(k) \end{bmatrix},$$

$$B = \begin{bmatrix} c_1(k) & & & \\ & c_2(k) & & \\ & & \ddots & \\ & & & c_N(k) \end{bmatrix}, \; \eta(x(k)) = \begin{bmatrix} b_1(k)\rho_0(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

Then the model (1)-(4) can be rewritten in the state space form as

$$x(k+1) = f(x(k), y(k)), \quad (10)$$

$$y(k+1) = A(x(k))y(k) + BPu(k) + \eta(x(k)) - Bs(k), \quad (11)$$

where $f(\cdot,\cdot)$ is a corresponding vector-valued function. $s(k)$ is the unknown leaving traffic flow on off-ramp at time k, which will be considered as the repetitive disturbance.

Throughout the paper, $\|\cdot\|$ denotes the infinite norm, i.e., for an $s \times t$ matrix M, in which $m_{i,j}$ symbolizes its entries, $\|\cdot\| = \max_{1 \le i \le s} \sum_{j=1}^{t} |m_{i,j}|$. And we define the λ norm of a vector $u(k)$ as $\|u(k)\|_\lambda = \sup_{k \in [0,K]} a^{-\lambda k} \|u(k)\|$, where $\lambda > 0$ and $a > 1$.

Assumption 1: Functions $f(\cdot,\cdot)$, $A(x(k))$ and $\eta(x(k))$ are uniformly globally Lipschitz on a compact set $\Omega = X \times Y$ with respect to their arguments for $k \in [0, K]$, i.e.,

$$\begin{aligned} &\|\Re f(x_1(k), y_1(k)) - \Re f(x_2(k), y_2(k))\| \\ &\quad \le k_x \|\Re(x_1(k) - x_2(k))\| + k_y \|\Re(y_1(k) - y_2(k))\|, \\ &\|\Re A(x_1(k)) - \Re A(x_2(k))\| \le k_A \|\Re(x_1(k) - x_2(k))\|, \\ &\|\Re \eta(x_1(k)) - \Re \eta(x_2(k))\| \le k_\eta \|\Re(x_1(k) - x_2(k))\|, \end{aligned} \tag{12}$$

where k_x, k_y, k_A, k_η are Lipschitz constants. X and Y are the ranges of speed and density of the traffic flow on the freeway, respectively.

Assumption 2: The re-initialization condition is satisfied throughout the repeated iterations, i.e,

$$x_n(0) = x_{d,n}(0) = x_{d,n+1}(0),\ y_n(0) = y_{d,n}(0) = y_{d,n+1}(0) \quad \forall n,$$

where n is the iteration number for the ILC, $x_{d,n}(0)$ is the initial value of the desired state of the n-th iteration.

Assumption 3: There exists a control profile $u_{d,n}(k)$ that can exactly drive the system output to track the desired trajectory $\Re y_{d,n}(k)$ for the systems (10) and (11) over the finite time interval [0, K], $\forall n$.

Assumption 4: The variance of the desired input between the adjacent iteration is bounded.

$u_{d,n+1}(k) = u_{d,n}(k) + \alpha_n(k)$, $\|\alpha_n(k)\|_\lambda \le \varepsilon_1$, $\varepsilon_1 > 0$, $k \in [0, K]$, $\forall n$. $\alpha_n(k)$ is the increment.

Assumption 1 requests the traffic model be globally Lipschitz continuous, which is satisfied in our case because the traffic flow model (1-4) is continuously differentiable in all arguments on any compact set Ω. Moreover, the system states (density and mean speed) cannot be infinite in practice. In addition, the time interval is also finite. This leads to the compact set Ω. Assumption 2 demands the initial state values to be consistent with the desired one, and the initial value of the desired state of different iteration is the same. In practice, if this condition is not met, we can always align the target trajectory with the actual one at the initial stage of tracking (Mingxuan Sun, and D. Wang, 2003). Assumption 3 is a reasonable assumption that the task should be solvable and it implies Assumption 4.

3. ILC BASED RAMP METERING TRACKING VARIOUS TRAJECTORIES

The iterative learning control law is:

$$u_{n+1}(k) = u_n(k) + \beta \Re(y_{d,n+1}(k+1) - y_n(k+1)) \tag{13}$$

where n indicates the iteration number, and β is an iterative learning gain matrix. $y_{d,n+1}(k)$ is the desired output signal (density) of the (n+1)-th iteration at the time k.

Theorem 1: Under Assumption 1-4, choosing the learning gain matrix β such that $\|I_{p\times p} - \beta QBP\| < 1$ in the ILC law (13), the output of the traffic system (10) (11) will lead to $\lim_{n\to\infty} \|\Re(y_{d,n}(k) - y_n(k))\|_\lambda \le \varepsilon$, ε for some suitably defined constant $\varepsilon > 0$ that depends on ε_1. In the sequel, we have $\lim_{n\to\infty} \|\Re(y_{d,n}(k) - y_n(k))\|_\lambda = 0$, if $\varepsilon_1 = 0$.

Proof: See the Appendix.

4. SIMULATION STUDIES

In order to verify the effectiveness of the ILC approach, we simulate a freeway traffic flow process in the presence of a large exogenous disturbance (modeled by an exiting flow in an off-ramp during a period). The learning process is iterated for 20 cycles. The desired density is various with time and iteration change, which is illustrated in Fig.1.

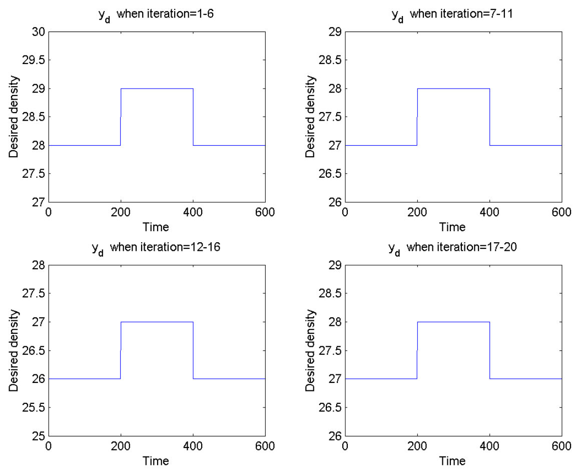


Fig. 1. Desired density

Consider a long segment of freeway that is divided into 12 sections. The length of each section is 500 meters. The initial traffic volume entering section 1 is 1500 veh/h. The initial density and mean speed of each section are set as shown in Table 1. The parameters used in this model are also listed here. $\rho_{jam} = 80(veh/lane/km)$, $v_{free} = 80(km/h)$, $l = 1.8$, $m = 1.7$, $\kappa = 13(veh/km)$, $\tau = 0.1(h)$, $T = 0.00417(h)$, $\nu = 35(km^2/h)$.

In order to show the proposed scheme's robustness, the initial values of density and speed are not in their equilibrium position according to the initial flow volume, that is to say, the initial ILC convergence conditions $\Re\delta x(0) = 0$ is not satisfied strictly.

There are two on-ramps with known traffic demands in sections 2 and 9, and an unknown exiting traffic flow in section 7 as shown in Fig. 2.

Two cases are simulated in this part, one is the no control case, and another is the ILC based control.

Case I. No Control

Without any control, the traffic on the mainstream entering from the traffic demands in on-ramp 2 and 9, and exiting in off-ramp 7, which shown by Fig. 2, are so heavy that there exists a traffic congestion, which is represented by the downstream densities after section 9 getting higher and exceeding the critical density, in the sequel results in slow traffic speed.

Case II. ILC Based Control tracking various trajectories

The ILC gains β_i (i=1, 2) are set to be 15.

Comparing Fig. 4 (a) (b) to Fig. 3 (a) (b), we can see the significant performance improvement in density control. The speed after control is mostly higher than 60 km/h, which means the freeway is unobstructed. From Fig. 4 (c) we can see that the learning error is slight although the desired density is changed with time and iteration change.

5. CONCLUSION

In this paper, ILC based control approach has been successfully applied to solve the traffic density control problem when tracking various trajectories, which has been approved by rigorous analyses. The simulation results show satisfactory responses and confirm the efficacy of the propose approach.

Table 1: Initial values associated with the traffic model

Sections	1	2	3	4	5	6
$\rho_i(0)$	30	30	30	30	30	30
$v_i(0)$	50	50	50	50	50	50
Sections	7	8	9	10	11	12
$\rho_i(0)$	30	30	30	30	30	30
$v_i(0)$	50	50	50	50	50	50

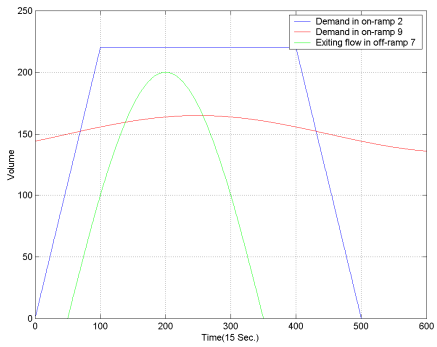
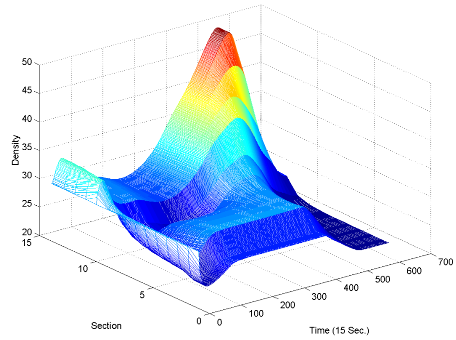
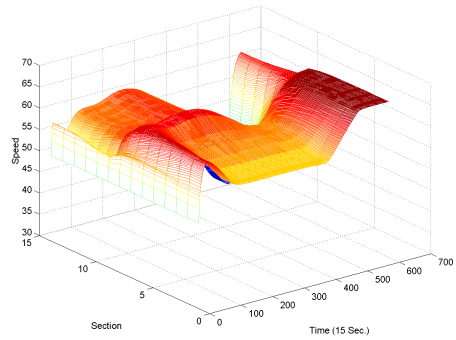


Fig. 2. Known traffic demands in on-ramps and unknown exiting flow in off-ramp

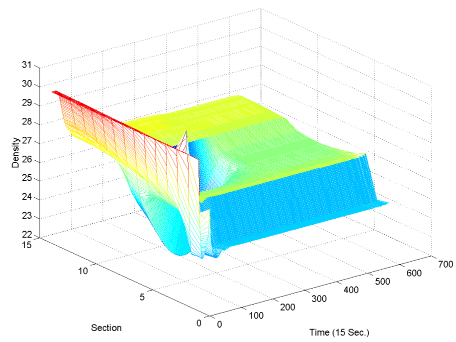


(a)

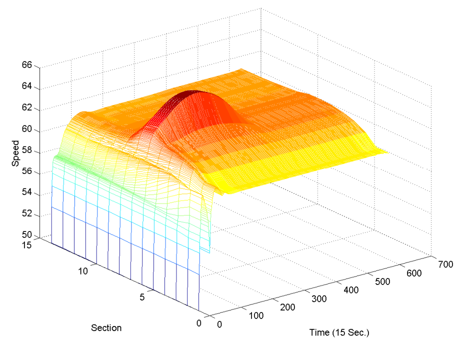


(b)

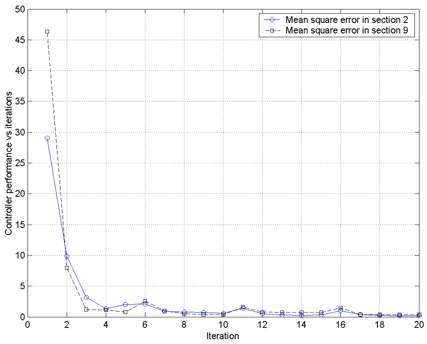
Fig. 3. (a) Density profile with no control. (b) Speed profile with no control



(a)



(b)



(c)

Fig. 4. (a) Traffic volume performances at the 20th iteration. (b) Traffic speed performances at the 20th iteration. (c) Iterative errors in sections 2 and 9.

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APPENDIX

Proof of Theorem 1:

From the iterative learning control law (13) we have

$$\begin{aligned} & u_{d,n+1}(k) - u_{n+1}(k) \\ & = u_{d,n+1}(k) - u_n(k) - \beta\Re(y_{d,n+1}(k+1) - y_n(k+1)). \end{aligned} \tag{14}$$

From (11) we get

$$\begin{aligned} & \Re(y_{d,n+1}(k+1) - y_n(k+1)) \\ & = \Re([A(x_{d,n+1}(k)) - A(x_n(k))]y_{d,n+1}(k)) \\ & +\Re(A(x_n(k))[y_{d,n+1}(k) - y_n(k)]) \\ & +QBP[u_{d,n+1}(k) - u_n(k)] + \Re(\eta(x_{d,n+1}(k)) - \eta(x_n(k))). \end{aligned} \tag{15}$$

Inserting (15) into (14) gives

$$\begin{aligned} & u_{d,n+1}(k) - u_{n+1}(k) = (I - \beta QBP)(u_{d,n+1}(k) - u_n(k)) \\ & -\beta\Re([A(x_{d,n+1}(k)) - A(x_n(k))]y_{d,n+1}(k)) \\ & -\beta\Re(A(x_n(k))[y_{d,n+1}(k) - y_n(k)]) \\ & -\beta\Re(\eta(x_{d,n+1}(k)) - \eta(x_n(k))). \end{aligned} \tag{16}$$

Taking norm operation of (16) gives

$$\begin{aligned} & \|u_{d,n+1}(k) - u_{n+1}(k)\| \le \|I - \beta QBP\| \|u_{d,n+1}(k) - u_n(k)\| \\ & +\sigma_1(\|\Re(x_{d,n+1}(k) - x_n(k))\| + \|\Re(y_{d,n+1}(k) - y_n(k))\|), \end{aligned} \tag{17}$$

where $\sigma_1 = \max\limits_{k\in[0,K]} \{(\beta k_A b_{yd} + \beta k_\eta), \beta b_A\}$.

From (10), we have

$$\begin{aligned} & \|\Re(x_{d,n+1}(k) - x_n(k))\| \le k_x \|\Re(x_{d,n+1}(k-1) - x_n(k-1))\| \\ & + k_y \|\Re(y_{d,n+1}(k-1) - y_n(k-1))\|. \end{aligned} \tag{18}$$

Taking norm operation for (15) yields

$$\begin{aligned} & \|\Re(y_{d,n+1}(k) - y_n(k))\| \\ & \le (k_A b_{yd} + k_\eta)\|\Re(x_{d,n+1}(k-1) - x_n(k-1))\| \\ & +b_A \|\Re(y_{d,n+1}(k-1) - y_n(k-1))\| \\ & +\|QBP\| \|u_{d,n+1}(k-1) - u_n(k-1)\|. \end{aligned} \tag{19}$$

Summing up (18) (19), using Assumption 2, we have

$$\begin{aligned}
&\|\Re(x_{d,n+1}(k)-x_n(k))\|+\|\Re(y_{d,n+1}(k)-y_n(k))\| \\
&\le \sigma_2(\|\Re(x_{d,n+1}(k-1)-x_n(k-1))\| \\
&+\|\Re(y_{d,n+1}(k-1)-y_n(k-1))\|) \\
&+\|QBP\|\|u_{d,n+1}(k-1)-u_n(k-1)\| \\
&\le \sigma_2{}^k(\|\Re(x_{d,n+1}(0)-x_n(0))\|+\|\Re(y_{d,n+1}(0)-y_n(0))\|) \\
&+\sum_{j=0}^{k-1}\sigma_2{}^{k-j-1}\|QBP\|\|u_{d,n+1}(j)-u_n(j)\| \\
&\le \sum_{j=0}^{k-1}\sigma_2{}^{k-j-1}\|QBP\|\|u_{d,n+1}(j)-u_n(j)\|
\end{aligned} \tag{20}$$

where $\alpha_2=\max_{k\in[0,K]}\{(k_x+k_A b_{yd}+k_\eta),(k_\eta+b_A)\}$. Without loss of generality, we assume that $\sigma_2>1$ for simplicity, otherwise similar approach still applies(Chen, Y.Q., Wen, C.Y., 1999).

Substituting (20) into (17) and taking λ norm operation gives

$$\begin{aligned}
&\sup_{k\in[0,K]}\sigma_2{}^{-\lambda k}\|u_{d,n+1}(k)-u_{n+1}(k)\| \\
&\le \|I-\beta QBP\|\sup_{k\in[0,K]}\sigma_2{}^{-\lambda k}\|u_{d,n+1}(k)-u_n(k)\| \\
&+\sigma_1\|QBP\|\sup_{k\in[0,K]}\sigma_2{}^{-\lambda k}\sum_{j=0}^{k-1}\sigma_2{}^{k-j-1}\|u_{d,n+1}(j)-u_n(j)\|,
\end{aligned} \tag{21}$$

Since

$$\begin{aligned}
&\sup_{k\in[0,K]}\sigma_2{}^{-\lambda k}\sum_{j=0}^{k-1}\sigma_2{}^{k-j-1}\|u_{d,n+1}(j)-u_n(j)\| \\
&=\sigma_2{}^{-1}\sup_{k\in[0,K]}(\sum_{j=0}^{k-1}\sigma_2{}^{-\lambda j}\|u_{d,n+1}(j)-u_n(j)\|\sigma_2{}^{(\lambda-1)(j-k)}) \\
&\le \sigma_2{}^{-1}\sup_{k\in[0,K]}(\sum_{j=0}^{k-1}(\sup_{k\in[0,K]}\sigma_2{}^{-\lambda j}\|u_{d,n+1}(j)-u_n(j)\|) \\
&\times\sigma_2{}^{(\lambda-1)(j-k)}) \\
&\le \sigma_2{}^{-1}\|u_{d,n+1}(j)-u_n(j)\|_\lambda\sup_{k\in[0,K]}\sum_{j=0}^{k-1}\sigma_2{}^{(\lambda-1)(j-k)} \\
&=\|u_{d,n+1}(j)-u_n(j)\|_\lambda\frac{1-\sigma_2{}^{-(\lambda-1)K}}{\sigma_2{}^\lambda-\sigma_2},
\end{aligned} \tag{22}$$

(21) becomes

$$\begin{aligned}
\|u_{d,n+1}(k)-u_{n+1}(k)\|_\lambda &\le \|I-\beta QBP\|\|u_{d,n+1}(k)-u_n(k)\|_\lambda \\
&+\sigma_1\|QBP\|\|u_{d,n+1}(j)-u_n(j)\|_\lambda\frac{1-\sigma_2{}^{-(\lambda-1)K}}{\sigma_2{}^\lambda-\sigma_2}.
\end{aligned} \tag{23}$$

Thus there exists a sufficiently large constant λ , such that the following inequality holds when $\|I-\beta QBP\|<1$,

$$\|I-\beta QBP\|+\sigma_1\|QBP\|\frac{1-\sigma_2{}^{-(\lambda-1)K}}{\sigma_2{}^\lambda-\sigma_2}\le\rho<1 \tag{24}$$

We can conclude

$$\|u_{d,n+1}(k)-u_{n+1}(k)\|_\lambda\le\rho\|u_{d,n+1}(k)-u_n(k)\|_\lambda. \tag{25}$$

From Assumption 4 and (25) we get

$$\begin{aligned}
\|u_{d,n+1}(k)-u_n(k)\|_\lambda &\le \|u_{d,n}(k)-u_n(k)\|_\lambda+\varepsilon_1 \\
&\le \rho\|u_{d,n}(k)-u_{n-1}(k)\|_\lambda+\varepsilon_1 \\
&\le \rho^n\|u_{d,1}(k)-u_0(k)\|_\lambda+\varepsilon_1\sum_{j=0}^{n-1}\rho^j.
\end{aligned} \tag{26}$$

Inserting (26) into (25), we have

$$\begin{aligned}
&\|u_{d,n+1}(k)-u_{n+1}(k)\|_\lambda \\
&\le \rho^{n+1}\|u_{d,1}(k)-u_0(k)\|_\lambda+\varepsilon_1\frac{\rho(1-\rho^n)}{1-\rho}
\end{aligned} \tag{27}$$

From (27) we get

$$\lim_{n\to\infty}\|u_{d,n+1}(k)-u_{n+1}(k)\|_\lambda\le\varepsilon_1\frac{\rho}{1-\rho}. \tag{28}$$

Equation (19) gives

$$\begin{aligned}
&\|\Re(y_{d,n+1}(k)-y_{n+1}(k))\| \\
&\le \frac{\sigma_1}{\beta}(\|\Re(x_{d,n+1}(k-1)-x_{n+1}(k-1))\| \\
&+\|\Re(y_{d,n+1}(k-1)-y_{n+1}(k-1))\|) \\
&+\|QBP\|\|u_{d,n+1}(k-1)-u_{n+1}(k-1)\|.
\end{aligned} \tag{29}$$

From (20) and Assumption 2, we get

$$\begin{aligned}
&\|\Re(x_{d,n+1}(k-1)-x_{n+1}(k-1))\| \\
&+\|\Re(y_{d,n+1}(k-1)-y_{n+1}(k-1))\| \\
&\le \sum_{j=0}^{k-2}\sigma_2{}^{k-j-1}\|QBP\|\|u_{d,n+1}(j)-u_{n+1}(j)\|.
\end{aligned} \tag{30}$$

Inserting (30) into (29) and taking λ norm operation yields

$$\begin{aligned}
&\|\Re(y_{d,n+1}(k)-y_{n+1}(k))\|_\lambda \\
&\le \sigma_3\|u_{d,n+1}(k)-u_{n+1}(k)\|_\lambda,
\end{aligned} \tag{31}$$

where $\sigma_3=\|QBP\|(\frac{\sigma_1(1-\sigma_2{}^{(K-1)(1-\lambda)})}{\beta\sigma_2{}^\lambda(\sigma_2{}^{\lambda-1}-1)}+\sigma_2{}^{-\lambda})$, when λ is sufficiently large, ε_3 can be slight.

We can drive from (28) and (31)

$\lim_{n\to\infty}\|\Re(y_{d,n+1}(k)-y_{n+1}(k))\|_\lambda\le\varepsilon$, $\varepsilon=\sigma_3\varepsilon_1\frac{\rho}{1-\rho}$, when $\varepsilon_1=0$, $\varepsilon=0$.

Then $\lim_{n\to\infty}\|\Re(y_{d,n+1}(k)-y_{n+1}(k))\|_\lambda=0$.