Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Corrected Mathematical Model of Quadruple Tank Process

Tomi Roinila, Matti Vilkko, Antti Jaatinen

Institute of Automation and Control, Tampere University of Technology, Tampere, Finland (e-mail: tomi.roinila@tut.fi, matti.vilkko@tut.fi, antti.jaatinen@tut.fi)

Abstract: A quadruple tank apparatus has been developed in many universities for use in undergraduate chemical engineering laboratories. The control experiment illustrates the performance limitations for multivariable systems posed by ill-conditioning, right half plane transmission zeros, and model uncertainties. The experiment is suitable for teaching how to select among multiloop, decoupling, and fully multivariable control structures. A number of these reports are, however, based on erroneous mathematical modeling and thus resulting incorrect results. Obviously all these reports refer originally to the one and same paper which includes this incorrect part of modeling. The error is significant if the pumps used in the experiment are not identical. If they are identical the error is, however, negligible. Mathematical derivation and simulation results are provided to give a corrected model and illustrate the effect of the widespread incorrect modeling.

1. INTRODUCTION

A quadruple tank process (Johansson, 2000) was designed and constructed to give undergraduate chemical engineers laboratory experience with key multivariable control concepts. By changing two flow ratios in the apparatus, a range of multivariable interactions can be investigated using only one experimental apparatus. Quadruple tank process has been used to teach students the following skills (Rusli, et al., 2002):

1. linearizing the nonlinear dynamics and constructing transfer functions for multivariable systems,
2. designing decentralized controllers,
3. implementing decouplers to reduce the effect of interactions,
4. implementing a fully multivariable control systems and
5. selecting the best control structure, based on the characteristics of the multivariable process.

A schematic diagram of the quadruple tank process is illustrated in Figure 1. The target is to control the level in the lower two tanks with two pumps. The process inputs are v_1 and v_2 (input voltages to the pump 1 and 2, respectively) and the outputs y_1 and y_2 (voltages from level measurement devices of tank 1 and 2, respectively).

The experiment has two inputs (pump speeds) which can be manipulated to control the two outputs (tank levels). The system exhibits multivariable dynamics because each of the pumps affects both of the outputs. The system has an adjustable multivariable zero that can be set to a right-half or left-half plane value by changing the valve settings of the experiment. Unmeasured disturbances can be applied by pumping water out of the top tanks and into the lower reservoir.

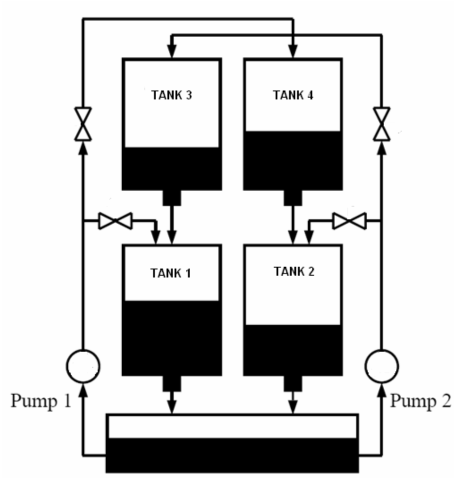

Figure 1. Schematic diagram of the quadruple tank process.

The quadruple tank process with the schematic above has been designed and studied in many universities. It is a widespread and illustrative experiment which can be applied in many control courses. The investigation of this system has also yielded plenty of conference and journal papers. A number of these reports, for instance (Alavi and Hayes, 2006; Alavi et al., 2005; Gatzke et al., 2002; Johansson, 2000;

Johansson et al., 1999; Johansson and Nunes, 1998; Numsomran et al., 2004; Rusli et al., 2002) are, however, based on erroneous mathematical modelling and thus resulting incorrect results. Obviously all these reports refer to the one and same paper (Johansson, 2000) which includes this incorrect part of modelling. Let us next derive the mathematical expression of this system and hence show the erroneousness of the original model.

The mathematical modelling of the quadruple tank process can be obtained by using Bernoulli's law. The constants are denoted in Table 1.

Table 1. Constants of quadruple tank process

Symbol	Constant
h_i	level of water in tank i
a_i	area of the pipe flowing out from tank i
A_i	area of tank i
γ_1	ratio of water diverting to tank 1 and 4
γ_2	ratio of water diverting to tank 2 and 3
k_1	gain of pump 1
k_2	gain of pump 2
v_1	manipulated input 1 (pump 1)
v_2	manipulated input 2 (pump 1)
g	gravitational constant

Now the nonlinear equations are obtained by

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}v_1 \quad (1)$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}v_2 \quad (2)$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}v_2 \quad (3)$$

$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}v_1 \quad (4)$$

Hence the linear model can be expressed by

$$\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} u \quad (5)$$

and

$$y = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} x \quad (6)$$

where

$$u = \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix},\ x = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{bmatrix},\ y = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} \text{ and } T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$$

2. TRANSFER FUNCTION MATRIX

The inherited erroneousness of the modelling of the quadruple tank process occurs in the derivation of a transfer function matrix. Let us derive the transfer function matrix of the model and hence show the incorrectness of the original model given by Johansson.

The transfer function matrix between input control and output is given by (Dorf and Bishop, 2001)

$$G(s) = C(sI - A)^{-1}B + D \quad (7)$$

By using expression (7)

$$G(s) = PRS \quad (8)$$

where

$$P = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix}, \quad (9)$$

$$R = \begin{bmatrix} s+\frac{1}{T_1} & 0 & -\frac{A_3}{A_1T_3} & 0 \\ 0 & s+\frac{1}{T_2} & 0 & -\frac{A_4}{A_2T_4} \\ 0 & 0 & s+\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & s+\frac{1}{T_4} \end{bmatrix}^{-1}, \quad (10)$$

and

$$S = \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \quad (11)$$

Simplifying these yields

$$G(s) = \begin{bmatrix} \frac{k_c\gamma_1 k_1 T_1}{A_1(sT_1+1)} & \frac{k_c T_1(1-\gamma_2)k_2}{A_1(sT_3+1)(sT_1+1)} \\ \frac{k_c T_2(1-\gamma_1)k_1}{A_2(sT_4+1)(sT_2+1)} & \frac{k_c T_2\gamma_2 k_2}{A_2(sT_2+1)} \end{bmatrix} \quad (12)$$

$\Leftrightarrow$

$$G(s) = \begin{bmatrix} \frac{\gamma_1 c_1}{sT_1+1} & \frac{k_c T_1(1-\gamma_2)k_2}{A_1(sT_3+1)(sT_1+1)} \\ \frac{k_c T_2(1-\gamma_1)k_1}{A_2(sT_4+1)(sT_2+1)} & \frac{\gamma_2 c_2}{sT_2+1} \end{bmatrix} \quad (13)$$

where $c_1 = \frac{T_1 k_1 k_c}{A_1}$ and $c_2 = \frac{T_2 k_2 k_c}{A_2}$

2.1 Significance of the error

The transfer function matrix (4) in (Johansson, 2000) can be written in the form of

$$G(s) = \begin{bmatrix} \frac{\gamma_1 T_1 k_1 k_c}{A_1(sT_1+1)} & \frac{(1-\gamma_2)T_1 k_1 k_c}{A_1(sT_3+1)(sT_1+1)} \\ \frac{(1-\gamma_1)T_2 k_2 k_c}{A_2(sT_4+1)(sT_2+1)} & \frac{\gamma_2 T_2 k_2 k_c}{sT_2+1} \end{bmatrix} \quad (15)$$

whereas the correct transfer function matrix has a form of

$$G(s) = \begin{bmatrix} \frac{\gamma_1 T_1 k_1 k_c}{A_1(sT_1+1)} & \frac{(1-\gamma_2)T_1 k_2 k_c}{A_1(sT_3+1)(sT_1+1)} \\ \frac{(1-\gamma_1)T_2 k_1 k_c}{A_2(sT_4+1)(sT_2+1)} & \frac{\gamma_2 T_2 k_2 k_c}{sT_2+1} \end{bmatrix} \quad (16)$$

The variables k_1 and k_2 have been changed in the elements two and three. A practical significance of this error is quite negligible if the parameters of pumps are relatively close to each other. In that case k_1 and k_2 are quite the same and the error is difficult to observe. This is obviously the reason why this erroneousness has not been noticed so far. The error will be marked and the difference is considerable if different pumps are used. This is illustrated in Section 3.

3. SIMULATION

The quadruple tank process can be simulated either as minimum or non-minimum phase system. This state can be chosen by adjusting the position of the valves (Johansson, 2000). Let us simulate the process by using the original incorrect mathematical model as well as the corrected one. By varying the values of k_1 and k_2 the significance of the error can be illustrated.

Let us first use the constant values given by Johansson and simulate the minimum phase as well as the non-minimum phase case. Simulation results for minimum and non-minimum phase systems can be seen in Figures 2-3 and 4-5, respectively. Step values of inputs vary around 0.5 volts on both sides of equilibrium values (i.e. in this case from 2.5 to 3.5).

As Figures 2 – 5 show the results of correct and incorrect models are relatively close to each other. Now the values of k_1 and k_2 are almost equal and as it was mentioned in this case the erroneousness of the original model is difficult to see.

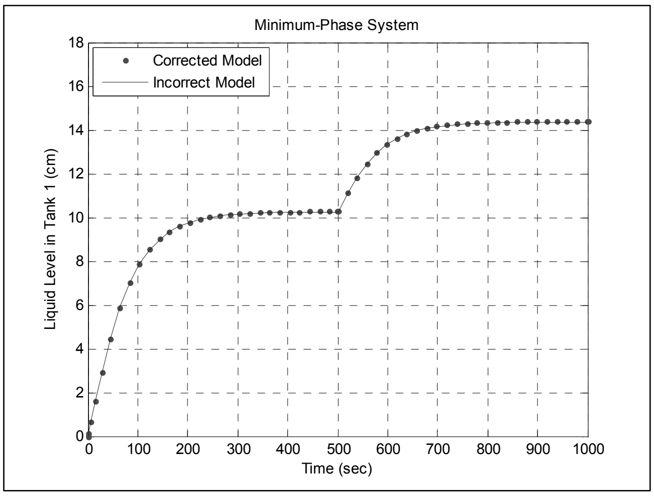

Figure 2. Liquid level of Tank 1

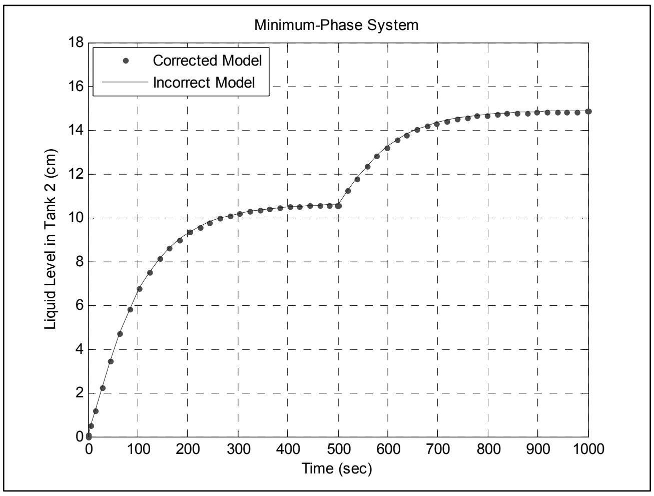

Figure 3. Liquid level of Tank 2

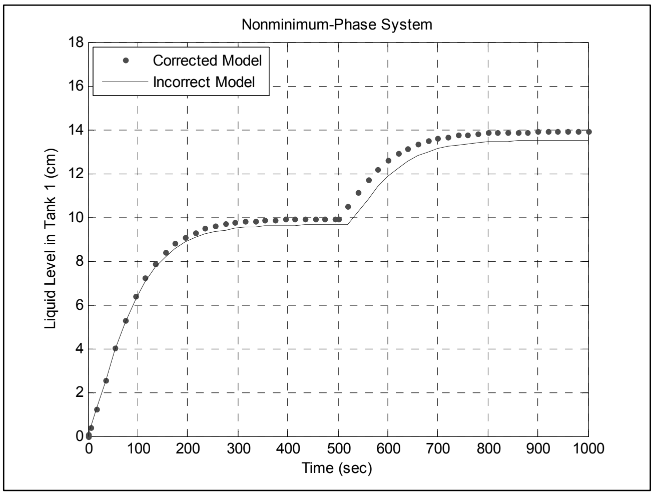

Figure 4. Liquid level of Tank 1

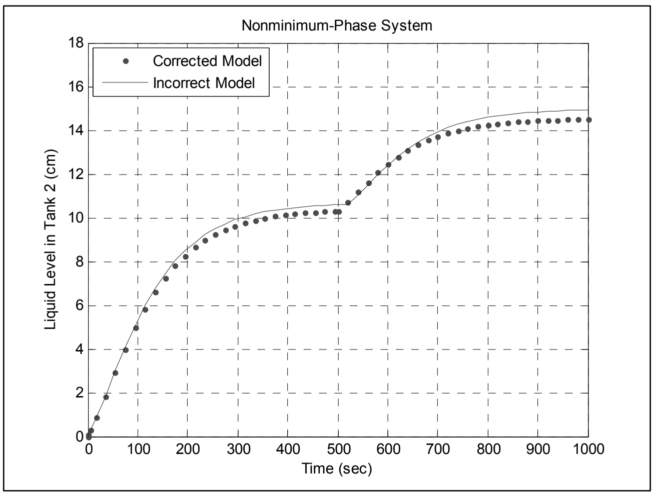

Figure 5. Liquid level of Tank 2

Let us then illustrate a case where significantly different pumps are used. Let us put $k_1 = 2$ and $k_2 = 6$ which are quite realistic values. In practice these values could correspond to maximum flow of 6 cm^3/s and 18 cm^3/s, respectively. Now by using equations (1)-(4) new equilibrium values can be calculated and apply for the simulator. Now the step values of inputs vary 0.5 volts on both sides of their new equilibrium values (i.e. from 4.5 to 5.5 (u_1) and from 1.3 to 2.1 (u_1)).

As it can be clearly seen in Figures 6-9 now the results of the correct and incorrect models are significantly different. The values of liquid levels are far from their equilibrium values when incorrect model is used.

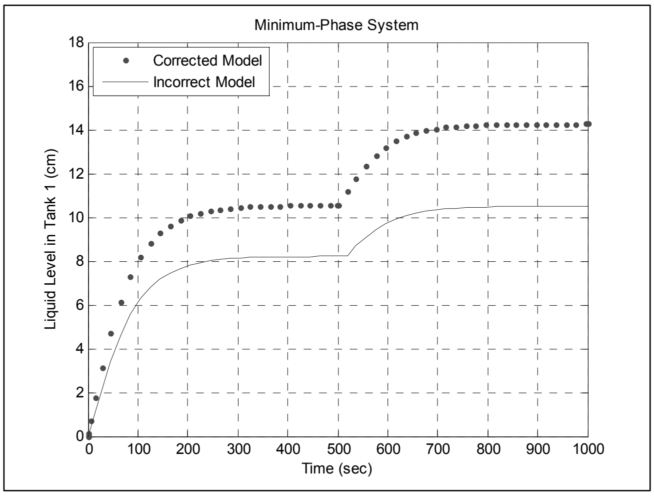

Figure 6. Liquid level of Tank 1.

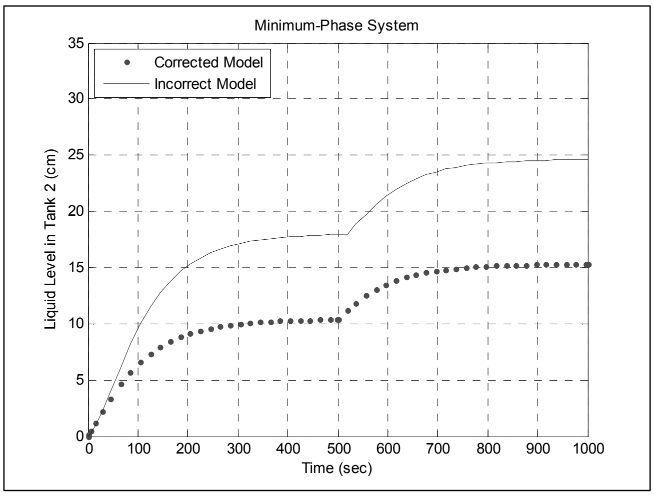

Figure 7. Liquid level of Tank 2.

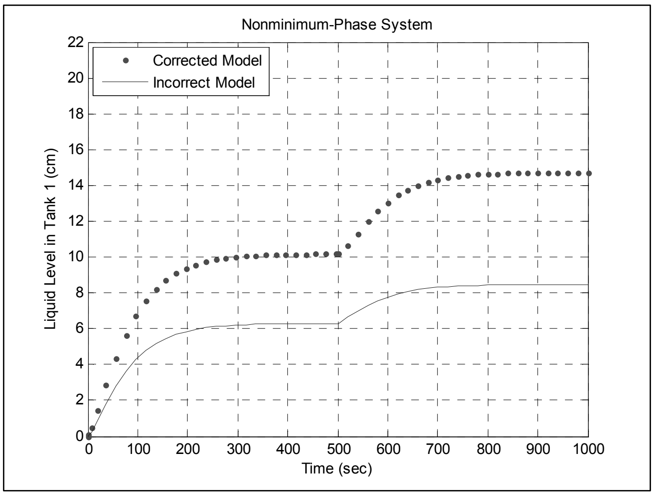

Figure 8. Liquid level of Tank 1.

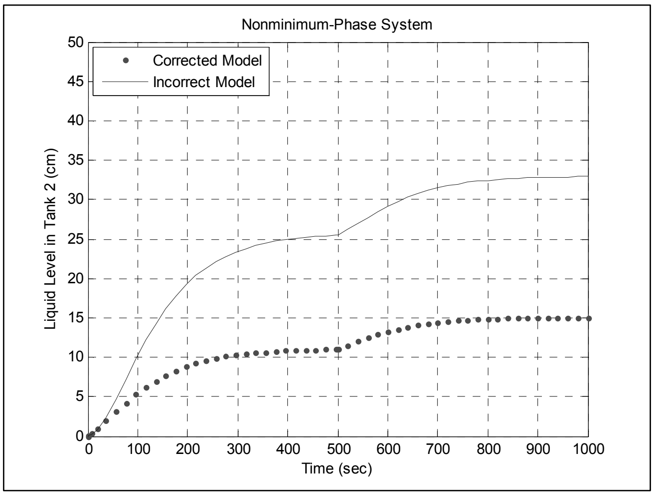

Figure 9. Liquid level of Tank 2.

4. VISUAL VERIFICATION OF MIMO-MODEL

The erroneousness of the original transfer function matrix is easy to notice without any further calculation. Let us show this and also illustrate some practical rules of thumb when dealing with transfer function matrix. Let us consider a general MIMO-model with two inputs and two outputs. This can be denoted by

$$Y = GU \tag{17}$$

$$\Leftrightarrow$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{18}$$

where Y is the output vector, G the transfer function matrix and U the input vector. Now, using general denotation, the first lower index of the components indicates whether the transfer function affects the output corresponding to this index. The second lower index indicates whether the transfer function is affected by the input corresponding to this index. For instance the transfer function of a_{12} affects the output of y_1 with the input of u_2. Now since the original model

$$G(s) = \begin{bmatrix} \dfrac{\gamma_1 T_1 k_1 k_c}{A_1 (sT_1 + 1)} & \dfrac{(1-\gamma_2) T_1 k_1 k_c}{A_1 (sT_3 + 1)(sT_1 + 1)} \\ \dfrac{(1-\gamma_1) T_2 k_2 k_c}{A_2 (sT_4 + 1)(sT_2 + 1)} & \dfrac{\gamma_2 T_2 k_2 k_c}{sT_2 + 1} \end{bmatrix}$$

is contradictory with these deductions (for instance the second transfer function in the first row contains k_1 which is a component of pump 1 and hence related to u_1) it can be considered incorrect without further calculations.

5. CONCLUSIONS

A widespread and inherited erroneousness in relation of mathematical modelling of quadruple tank process has been noticed. The incorrect part of modelling occurs in transfer function matrix and is related to the two pumps which are used in the quadruple tank apparatus. The error has occurred when the transfer function matrix has been derived from the linearized model. When the properties of the pumps are relatively close to each other the erroneousness of the modelling is difficult to notice. However when different pumps are used the incorrect model may produce significantly inaccurate results. In many reports the properties of the pumps are quite similar and thus the incorrect modelling has not been observed. In this paper a mathematical derivation and simulation results were provided to give a corrected model and illustrate the effect of the widespread incorrect modelling.

REFERENCES

Alavi S.M. Mahdi, Hayes Martin J., *Quantitative Feedback Design for a Benchmark Quadruple-Tank Process*, Irish Signals and Systems Conference, 2006. IET, Jun. 2006, Page(s): 401 – 406.

Alavi S.M. Mahdi, Sedigh A.K., Labibi B., *Pre-Filter Design for Tracking Error Specifications in MIMO-QFT, Decision and Control*, 2005 and 2005 European Control Conference, CDC-ECC '05, 44th IEEE Conference on 12 – 15 Dec. 2005 Page(s):6466 – 6471.

Dorf R.C., Bishop R.H., *Modern Control Systems*, Prentice-Hall, Upper Saddle River, New Jersey, 2001, 831 s.

Gatzke Edward P., Meadows Edward S., Wang Chung, Doyle Francis J., *Model Based Control of a Four-Tank System*, Computers and Chemical Engineering, 2000, 24(2), 1503-1509

Johansson K.H., The Quadruple-Tank Process: *A Multivariable Laboratory Process with an Adjustable Zero*, IEEE Transactions on Control Systems Technology, **Vol. 8, No. 3**, May 2000.

Johansson K.H., Horch A., Wijk O., Hansson A., *Teaching Multivariable Control Using the Quadruple-Tank Process*, , Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, Arizona USA, Dec 1999, **Vol. 1**, 7 – 10 Dec., Page(s):807 – 812.

Johansson K.H., Nunes J.L.R., *A multivariable process with an adjustable zero,* American Control Conference, 1998. Proceedings of the 1998, **Vol. 4**, 24 – 26 June 1998 Page(s): 2045 – 2049.

Numsomran A., Suksri T., Kongratana V., Trisuwannawat T., Tirasesth T., *I-P controller design for quadruple-tank system*, Tencon 2004, 2004 IEEE Region 10 Conference, **Vol. D**, 21 – 24 Nov. 2004 Page(s):471 – 474 Vol. 4.

Numsomran A., Witheephanich K., Trisuwannawat T., Tirasesth T., *I-P controller design for quadruple-tank system*, Control Conference, 2004 5th Asian, **Vol. 2**, 20 – 23 July 2004 Page(s):1296 – 1301.

Rusli Effendi, Ang Siong, Braatz Richard D., *A Quadruple Tank Process Control Experiment*, Department of Chemical and Biomolecular Engineering University of Illinois at Urbana-Champaign, Illinois.