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A Unified Solution to Unbiased Minimum-Variance Estimation for Systems with Unknown Inputs ⋆

Chien-Shu Hsieh *

* *Department of Electrical Engineering, Ta Hwa Institute of Technology, 30740 Hsinchu, Taiwan, R.O.C. (e-mail: cshsieh@ee.thit.edu.tw).*

Abstract: A parameterized three-stage Kalman filter (PTSKF) is proposed, serving as a unified solution to unbiased minimum-variance estimation for systems with unknown inputs that affect both the system and the outputs. The PTSKF is characterized by two design parameters and includes three parts: one is for the main system state estimate, the second is for the optimal unknown inputs estimate, and the last is added to further enhance the robust filtering performance. It is shown that the extended robust two-stage Kalman filter (ERTSKF), which is an extension of the previously proposed RTSKF, and the optimal two-stage Kalman filter (OTSKF) are special cases of this new filter. Simulation results show that not only the filtering performance of the PTSKF is compatible to that of the previous proposed parameterized minimum-variance filter (PMVF) but also the computational complexity of the former is less intensive than that of the latter.

Keywords: Estimation and filtering; Robust estimation; Kalman filtering; Unknown input estimation; Minimum-variance estimation.

1. INTRODUCTION

Unknown inputs filtering has played a significant role in many applications, e.g. bias compensation (Hsieh & Chen, 1999; Ignagni, 2000), geophysical and environmental applications (Kitanidis, 1987), and fault detection and isolation problems (Chen, 1996). As is known (Hsieh & Chen, 1999), when the statistics of the unknown inputs is perfectly known, then the optimal state estimator can be represented by the optimal two-stage Kalman filter (OTSKF), which can be seen as a special implementation of the KF via the TSKF structure. However, this TSKF is limited to systems with unknown inputs that have a prescribed statistical model. To be more general, it may not be any knowledge concerning the model of the unknown inputs.

A general approach to solve the aforementioned state estimation for unknown inputs that have arbitrary statistics is to apply unknown-input decoupled state estimation. Among these, three major approaches have been used in the literature. The first is unbiased minimum-variance estimation (UMVE) (Kitanidis, 1987; Chen & Patton, 1996; Darouach & Zasadzinski, 1997; Hsieh, 2000; Darouach, Zasadzinski, & Boutayeb, 2003). In this approach, the filter parameters are first determined to satisfy some algebraic constraints according to the unbiasedness requirements of the filter. In general, the solutions of the algebraic constraints are parametrized by some parameter matrices. Next, the parameter matrices are determined such that the estimation error variance is minimum. The second is the equivalent system description (ESD) method (Hou & Müller, 1994; Hou & Patton, 1998). In this approach, an equivalent system description for designing an unknown-input decoupled filter for the considered system is first given. Then, the linear minimum-variance estimation is derived by making use of the innovations filtering technique. The system considered in Hou and Patton (1998) has the general form where the unknown inputs affect both the system model and the measurements and the system and measurement noises are correlated. In the last, the filter is designed based on state estimation techniques for descriptor systems (Nikoukhah, Campbell, & Delebecque, 1999).

This paper considers the optimal filtering for systems with unknown inputs that affect both the system model and the measurements via the UMVE method. In an early paper (Hsieh, 2006), we proposed the optimal unbiased minimum-variance filter (OUMVF) through which the direct relationship between the UMVE and the ESD approaches is clearly illustrated. Specifically, the relationships with the existing literature results, i.e., Chen and Patton (1996), Hou and Müller (1994), and Darouach, Zasadzinski, and Boutayeb (2003) are addressed. However, the optimal linear minimum-variance estimator (OLMVE) in Hsieh (2006), which was derived by using the ESD and the innovations filtering method, can not be re-derived via the OUMVF. In Hsieh (2007a), we proposed an extended version of the OUMVF to exactly implement the OLMVE. Moreover, a robust version of the OUMVF and a parameterized minimum-variance filter (PMVF) were also proposed to solve the robust filtering problem for uncertain systems with unknown inputs (Hsieh, 2007b).

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Recently, a specific extension of the robust two-stage Kalman filter (RTSKF) (Hsieh, 2000) has been proposed serving as a useful mean to derive the unified structure of unbiased minimum-variance reduced-order filters (Hsieh, 2007d). In Hsieh (2007c), an extension of the RTSKF, named as the ERTSKF, has also been proposed to serve as an alternative to the OUMVF. This gives a direct connection between the two existing filtering methods for systems with unknown inputs: the UMVE method and the RTSKF technique. Moreover, it was shown that the ERTSKF is computationally more attractive than the OUMVF. Besides, the ERTSKF has the potential advantage over the OUMVF that the filter yields the optimal estimate of the unknown inputs (see Gillijns and Moor (2007) for details).

In this paper, we continue the research in order to deriving a unified filter structure for systems with unknown inputs that may or may not have a prescribed statistical model, and further purpose a parameterized three-stage Kalman filter (PTSKF), which serves as a unified solution to UMVE for systems with unknown inputs that affect both the system and the outputs. The PTSKF is characterized by two design parameters and includes three parts: one is for the main system state estimate, the second is for the optimal unknown inputs estimate, and the last is added to further enhance the robust filtering performance. It is shown that the aforementioned ERTSKF and OTSKF are special cases of this new filter.

This paper is organized as follows. In Section 2, the statement of the problem is addressed. Section 3 previews two ERTSKFs that are related to the PTSKF. An alternative derivation of the OTSKF by using UMVE is presented in Section 4. Next, the proposed PTSKF is derived in Section 5, and its unbiasedness property is addressed in Section 6. Finally, illustrative examples showing the usefulness of the proposed results are provided in Section 7.

2. STATEMENT OF THE PROBLEM

Consider the linear discrete-time stochastic time-varying system with unknown inputs in the form

$$x_{k+1} = A_k x_k + B_k u_k + F_k d_k + w_k \tag{1}$$

$$y_k = H_k x_k + G_k d_k + v_k \tag{2}$$

where $x_k \in R^n$ is the state vector, $u_k \in R^m$ is the known input vector, $y_k \in R^p$ is the measurement vector, and $d_k \in R^q$ is an unknown input vector. Matrices A_k, B_k, F_k, H_k, and G_k have appropriate dimensions. Without loss of generality, it is assumed that $rank(H_k) = m$. The process noise w_k and the measurement noise v_k are zero-mean white noise sequences with the following covariances: $E\{w_k w_l'\} = Q_k \delta_{kl}$, $E\{v_k v_l'\} = R_k \delta_{kl}$, and $E\{w_k v_l'\} = 0$, where $'$ denotes transpose and δ_{kl} denotes the Kronecker delta function. The initial state x_0 is of mean $\hat{x}_0$ and covariance P_0 and is independent of w_k and v_k.

The filtering problem considered in this paper is to estimate x_k from the measurements $\{y_t\}$, where $0 \leq t \leq k$, such that $E[e_k] = 0$ and $tr(E[e_k e_k'])$ is minimized, where $e_k = x_k - \hat{x}_k$. No prior knowledge about the unknown inputs d_k is assumed to be available, and hence the unknown inputs can be any type of signal. In Hsieh (2007b), the PMVF, which is listed in Appendix A, was derived to solve the above-mentioned unknown inputs filtering problem. The PMVF can achieve an optimal compromise between the optimal estimator filter (OEF) (Darouach, Zasadzinski, & Boutayeb; 2003) and the conventional Kalman filter.

The aim of this paper is to extend the previous research in order to deriving a unified filter structure, which is in the form of TSKF, for systems with unknown inputs that may or may not have a prescribed statistical model. Specifically, we intend to explore the possibility of deriving the following TSKF:

$$\hat{x}_k = \bar{x}_{k|k} + V_k d_{k|k} \tag{3}$$

$$P_k = P^{\bar{x}}_{k|k} + V_k P^d_{k|k} V_k' \tag{4}$$

where $\bar{x}_{k|k}$, $d_{k|k}$, $P^{\bar{x}}_{k|k}$, $P^d_{k|k}$, and V_k are to be determined, such that the filter structure of the TSKF is more compact than that of the PMVF, while the filtering performance of the former is still compatible to that of the latter.

3. EXTENSIONS OF THE ROBUST TWO-STAGE KALMAN FILTER

To facilitate the development, we briefly preview two extended RTSKFs (ERTSKFs) (Hsieh, 2007e), which give the optimal system state that is unaffected by the values of the unknown inputs, via the direct application of the previously proposed RTSKF (Hsieh, 2000).

The first is obtained by augmenting the unknown inputs as follows:

$$d_k \to d_k^a = \left[\, d_k' \;\; d_k' \,\right]'. \tag{5}$$

Then, the system model (1) and (2) can be represented as follows:

$$x_{k+1} = A_k x_k + B_k u_k + \bar{F}_k d_k^a + w_k \tag{6}$$

$$y_k = H_k x_k + \bar{G}_k d_k^a + v_k \tag{7}$$

where

$$\bar{F}_k = [\,0 \;\; F_k\,], \quad \bar{G}_k = [\,G_k \;\; 0\,]. \tag{8}$$

The obtained ERTSKF is named as the ARTSKF, and is listed as follows:

$$\hat{x}_k = \bar{x}_{k|k} + V_k d_{k|k} \tag{9}$$

$$P_k = P^{\bar{x}}_{k|k} + V_k P^d_{k|k} V_k' \tag{10}$$

where $\bar{x}_{k|k}$ is given by

$$\bar{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} \tag{11}$$

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K^{\bar{x}}_k (y_k - H_k \bar{x}_{k|k-1}) \tag{12}$$

$$P^{\bar{x}}_{k|k-1} = A_{k-1} P_{k-1} A_{k-1}' + Q_{k-1} \tag{13}$$

$$K^{\bar{x}}_k = P^{\bar{x}}_{k|k-1} H_k' C_k^{-1} \tag{14}$$

$$P^{\bar{x}}_{k|k} = (I - K^{\bar{x}}_k H_k) P^{\bar{x}}_{k|k-1} \tag{15}$$

$d_{k|k}$ is given by

$$d_{k|k} = K^d_k (y_k - H_k \bar{x}_{k|k-1}) \tag{16}$$

$$K^d_k = P^d_{k|k} \bar{S}_k' C_k^{-1} \tag{17}$$

$$P^d_{k|k} = \{\bar{S}_k' C_k^{-1} \bar{S}_k\}^+ \tag{18}$$

in which M^+ denotes any one-condition generalized inverse of M, i.e., $MM^+M = M$, and

$$V_k = \bar{F}_{k-1} - K_k^{\bar{x}}\bar{S}_k \tag{19}$$
$$\bar{S}_k = H_k\bar{F}_{k-1} + \bar{G}_k = [\,G_k \;\; S_k\,], \quad S_k = H_kF_{k-1} \tag{20}$$
$$C_k = H_kP_{k|k-1}^{\bar{x}}H_k' + R_k. \tag{21}$$

The unbiasedness constraint of the above ARTSKF is given as follows:

$$\bar{F}_{k-1} - \bar{F}_{k-1}K_k^d\bar{S}_k = 0. \tag{22}$$

The other, on the other hand, is obtained by decoupling the unknown inputs from the measurement equation. This is achieved by pre-multiplying (2) by

$$T_k = I - G_kG_k^+ \tag{23}$$

which leads to

$$T_ky_k = T_kH_kx_k + T_kv_k. \tag{24}$$

Note that (24) exists only if $rank[G_k] < p$. The obtained ERTSKF is named as the DRTSKF, which is the alternative of the one presented in (Hsieh, 2007c), and is listed as follows:

$$\hat{x}_k = \bar{x}_{k|k} + \bar{V}_kd_{k|k} \tag{25}$$
$$P_k = P_{k|k}^{\bar{x}} + \bar{V}_kP_{k|k}^d\bar{V}_k' \tag{26}$$

where $\bar{x}_{k|k}$ is given by

$$\bar{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} \tag{27}$$
$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K_k^{\bar{x}}(y_k - H_k\bar{x}_{k|k-1}) \tag{28}$$
$$P_{k|k-1}^{\bar{x}} = A_{k-1}P_{k-1}A_{k-1}' + Q_{k-1} \tag{29}$$
$$K_k^{\bar{x}} = P_{k|k-1}^{\bar{x}}H_k'\bar{C}_k^+, \quad \bar{C}_k = T_kC_kT_k' \tag{30}$$
$$P_{k|k}^{\bar{x}} = (I - K_k^{\bar{x}}H_k)P_{k|k-1}^{\bar{x}} \tag{31}$$

$d_{k|k}$ is given by

$$d_{k|k} = K_k^d(y_k - H_k\bar{x}_{k|k-1}) \tag{32}$$
$$K_k^d = P_{k|k}^dS_k'\bar{C}_k^+ \tag{33}$$
$$P_{k|k}^d = \{S_k'\bar{C}_k^+S_k\}^+ \tag{34}$$

and $\bar{V}_k$ is given by

$$\bar{V}_k = F_{k-1} - K_k^{\bar{x}}S_k. \tag{35}$$

The unbiasedness constraint of the above DRTSKF is given as follows:

$$F_{k-1} - F_{k-1}K_k^dS_k = 0. \tag{36}$$

The optimality issues of the above ARTSKF and DRTSKF are fully explored in Hsieh (2007e). To simplify the derivations, in the following discussions we only consider the ARTSKF.

4. OTSKF DESIGN FOR UNKNOWN INPUTS DESCRIBED BY RANDOM-WALK PROCESS

In this section, we present the optimal unknown inputs filtering via the TSKF filter structure for systems with unknown inputs by using the similar approach as given in Hsieh (2007b). Assume that the unknown inputs can be described by the following random-walk process:

$$d_{k+1} = d_k + w_k^d \tag{37}$$

where w_k^d is a zero-mean white noise sequence with the following covariances: $E\{w_k^d(w_l^d)'\} = Q_k^d\delta_{kl}$, $E\{w_k^d(w_l)'\} = 0$, and $E\{w_k^dv_l'\} = 0$. The initial state d_0 is of mean $\bar{d}_0$ and covariance P_0^d, and is independent of w_k, w_k^d, and v_k.

Defining the estimation error of the unknown-input filter as $e_k^d = d_k - d_{k|k}$, the estimator $d_{k|k}$ must satisfy a) unbiasedness: $E[e_k^d] = 0$ and b) minimum-variance: min $tr(P_{k|k}^d)$. In this paper, we consider the following unknown-input filter form:

$$\begin{aligned} d_{k|k} &= (I - K_k^d(\breve{S}_k + H_k\Psi_{k-1}))d_{k-1|k-1} \\ &\quad + K_k^d(y_k - H_k(A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1})) \end{aligned} \tag{38}$$

where K_k^d, $\breve{S}_k$, and Ψ_{k-1} are to be determined. From (1), (2), (37), and (38), the estimation error e_k^d can be written as

$$\begin{aligned} e_k^d &= \left(I - K_k^d(H_kF_{k-1} + G_k)\right)e_{k-1}^d - K_k^dv_k \\ &\quad + K_k^d\left(\breve{S}_k - H_k(F_{k-1} - \Psi_{k-1}) - G_k\right)d_{k-1|k-1} \\ &\quad - K_k^dH_k(A_{k-1}e_{k-1} + w_{k-1}) + (I - K_k^dG_k)w_{k-1}^d. \end{aligned} \tag{39}$$

Using unbiasedness requirement in (39) yields

$$\breve{S}_k = \hat{S}_k - H_k\Psi_{k-1}, \quad \hat{S}_k = H_kF_{k-1} + G_k. \tag{40}$$

From (54) of Hsieh and Chen (1999), one has

$$E\{e_k(e_k^d)'\} = V_kP_{k|k}^d. \tag{41}$$

Using (40) and (41), we can obtain the following error covariance matrix:

$$P_{k|k}^d = P_{k|k-1}^d + K_k^d\{\bullet\}(K_k^d)' - \{\bullet\}_1(K_k^d)' - K_k^d\{\bullet\}_1' \tag{42}$$

where

$$P_{k|k-1}^d = P_{k-1|k-1}^d + Q_{k-1}^d \tag{43}$$
$$\begin{aligned} \{\bullet\} &= (H_k\breve{F}_{k-1} + G_k)P_{k-1|k-1}^d(H_k\breve{F}_{k-1} + G_k)' \\ &\quad + G_kQ_{k-1}^dG_k' + C_k \\ &\quad - H_kA_{k-1}V_{k-1}P_{k-1|k-1}^dV_{k-1}'A_{k-1}'H_k' \end{aligned} \tag{44}$$
$$\{\bullet\}_1 = P_{k-1|k-1}^d\breve{F}_{k-1}'H_k' + P_{k|k-1}^dG_k' \tag{45}$$
$$\breve{F}_{k-1} = A_{k-1}V_{k-1} + F_{k-1}. \tag{46}$$

Finding K_k^d which minimizes the trace of (42), one obtains

$$K_k^d = \{\bullet\}_1\{\bullet\}^{-1} \tag{47}$$
$$P_{k|k}^d = P_{k|k-1}^d - \{\bullet\}_1(K_k^d)'. \tag{48}$$

Using (40), (43), and (46), and choosing Ψ_{k-1} as follows:

$$\Psi_{k-1} = F_{k-1} - \breve{F}_{k-1} P^d_{k-1|k-1} (P^d_{k|k-1})^{-1} \tag{49}$$

(45) can be simplified as

$$\{\bullet\}_1 = P^d_{k|k-1} \breve{S}'_k. \tag{50}$$

Using (40) and (49), (44) can be simplified as

$$\{\bullet\} = \breve{S}_k P^d_{k|k-1} \breve{S}'_k + \breve{C}_k \tag{51}$$

where

$$\breve{C}_k = H_k \breve{Q}_{k-1} H'_k + C_k \tag{52}$$

$$\breve{Q}_k = (F_k - \Psi_k) Q^d_k \breve{F}'_k - A_k V_k P^d_{k|k} V'_k A'_k. \tag{53}$$

Thus, using (47), (48), (50), and (51), the gain matrix K^d_k and the error covariance matrix $P^d_{k|k}$ are given, respectively, as follows:

$$K^d_k = P^d_{k|k-1} \breve{S}'_k \{\breve{S}_k P^d_{k|k-1} \breve{S}'_k + \breve{C}_k\}^{-1} \tag{54}$$

$$P^d_{k|k} = (I - K^d_k \breve{S}_k) P^d_{k|k-1}. \tag{55}$$

Next, we show how to determine the system state estimate $\hat{x}_k$. Assume that $\bar{x}_{k|k}$ and V_k in (3) to be taken by the following forms:

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K^{\bar{x}}_k (y_k - H_k \bar{x}_{k|k-1}) \tag{56}$$

$$V_k = U_k - K^{\bar{x}}_k \breve{S}_k \tag{57}$$

where

$$\bar{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1} + \Psi_{k-1} d_{k-1|k-1}. \tag{58}$$

Using (1)-(3), (37)-(38), (40), and (56)-(58), we obtain the error dynamics e_k as follows:

$$\begin{aligned} e_k = {} & (I - L_k H_k)(A_{k-1} e_{k-1} + w_{k-1}) \\ & - L_k (G_k w^d_{k-1} + v_k) + (F_{k-1} - L_k \hat{S}_k) e^d_{k-1} \\ & + (F_{k-1} - \Psi_{k-1} - U_k) d_{k-1|k-1} \end{aligned} \tag{59}$$

where

$$L_k = U_k K^d_k + K^{\bar{x}}_k (I - \breve{S}_k K^d_k). \tag{60}$$

Using the unbiasedness constraint in (59) yields

$$U_k = F_{k-1} - \Psi_{k-1}. \tag{61}$$

Using (40), (43), (44), (46), and (61), the error covariance matrix of (59) is described by

$$\begin{aligned} P_k = {} & L_k \{\bullet\} L'_k - (P^{\bar{x}}_{k|k-1} H'_k + \breve{F}_{k-1} P^d_{k-1|k-1} \breve{S}'_k) L'_k \\ & - L_k (P^{\bar{x}}_{k|k-1} H'_k + \breve{F}_{k-1} P^d_{k-1|k-1} \breve{S}'_k)' \\ & + P^{\bar{x}}_{k|k-1} + \breve{F}_{k-1} P^d_{k-1|k-1} U'_k \end{aligned} \tag{62}$$

where $P^{\bar{x}}_{k|k-1}$ is given by (70). Finding L_k which minimizes the trace of (62) and using (47), (50), (51), and (57) in (60), one obtains

$$P^{\bar{x}}_{k|k-1} H'_k + \breve{F}_{k-1} P^d_{k-1|k-1} \breve{S}'_k = K^{\bar{x}}_k \breve{C}_k + U_k P^d_{k|k-1} \breve{S}'_k. \tag{63}$$

Solving (63) for $K^{\bar{x}}_k$, one obtains

$$K^{\bar{x}}_k = P^{\bar{x}}_{k|k-1} H'_k \breve{C}_k^{-1}. \tag{64}$$

The error covariance matrix (62) corresponding to (64) is given as follows:

$$\begin{aligned} P_k &= P^{\bar{x}}_{k|k-1} - K^{\bar{x}}_k \breve{C}_k (K^{\bar{x}}_k)' + \breve{F}_{k-1} P^d_{k-1|k-1} U'_k \\ &\quad - U_k P^d_{k|k-1} U'_k + V_k (P^d_{k|k-1} - K^d_k \{\bullet\} (K^d_k)') V'_k \\ &= (I - K^{\bar{x}}_k H_k) P^{\bar{x}}_{k|k-1} + V_k P^d_{k|k} V'_k \end{aligned} \tag{65}$$

where (42), (47), and (57) are used.

Finally, the obtained OTSKF is summarized as follows:

$$\hat{x}_k = \bar{x}_{k|k} + V_k d_{k|k} \tag{66}$$

$$P_k = P^{\bar{x}}_{k|k} + V_k P^d_{k|k} V'_k \tag{67}$$

where $\bar{x}_{k|k}$ is given by

$$\bar{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1} + \Psi_{k-1} d_{k-1|k-1} \tag{68}$$

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K^{\bar{x}}_k (y_k - H_k \bar{x}_{k|k-1}) \tag{69}$$

$$P^{\bar{x}}_{k|k-1} = A_{k-1} P_{k-1} A'_{k-1} + Q_{k-1} + \breve{Q}_{k-1} \tag{70}$$

$$K^{\bar{x}}_k = P^{\bar{x}}_{k|k-1} H'_k \breve{C}_k^{-1} \tag{71}$$

$$P^{\bar{x}}_{k|k} = (I - K^{\bar{x}}_k H_k) P^{\bar{x}}_{k|k-1} \tag{72}$$

$d_{k|k}$ is given by

$$d_{k|k} = (I - K^d_k \breve{S}_k) d_{k-1|k-1} + K^d_k (y_k - H_k \bar{x}_{k|k-1}) \tag{73}$$

$$K^d_k = P^d_{k|k-1} \breve{S}'_k \{\breve{S}_k P^d_{k|k-1} \breve{S}'_k + \breve{C}_k\}^{-1} \tag{74}$$

$$P^d_{k|k} = (I - K^d_k \breve{S}_k) P^d_{k|k-1} \tag{75}$$

and

$$V_k = U_k - K^{\bar{x}}_k \breve{S}_k, \quad U_k = \breve{F}_{k-1} P^d_{k-1|k-1} (P^d_{k|k-1})^{-1} \tag{76}$$

$$\breve{S}_k = H_k U_k + G_k \tag{77}$$

$$\breve{C}_k = H_k P^{\bar{x}}_{k|k-1} H'_k + R_k. \tag{78}$$

Remark 1: The above OTSKF is exactly the one given in Hsieh and Chen (1999) and Ignagni (2000), where the OTSKF are derived by using a two-stage $U-V$ transformation approach and from basic estimating principles, respectively. Hence, the proposed unbiased minimum-variance filtering given in this section can be seen as an alternative to those given in Hsieh and Chen (1999) and Ignagni (2000) to derive the OTSKF.

5. THE DERIVATION OF THE PTSKF

It is known that the ARTSKF is an optimal filter that is not affected by the values of the unknown inputs, which is an important consideration when the unknown-input model is highly non-Gaussian or has unknown statistics. However, due to the limitation of the existence condition, the ARTSKF may need to include some information of the unknown inputs, e.g., random-walk process, in order to enhance the filtering performance. On the other hand, the OTSKF can produce the global optimal system state estimates. Nevertheless, this is achieved only when the unknown-input model is exactly known. Thus, the OTSKF is not a robust filter and its optimality can be compromised by a poor estimation of the unknown inputs. In this section, we shall derive the PTSKF, which may achieve an optimal compromise between the ARTSKF and the OTSKF.

First, we rewrite the original system (1) and (2) as follows:

$$x_{k+1} = A_k x_k + B_k u_k + \tilde{F}_k d_k + \mathcal{F}_k d_k + w_k \tag{79}$$

$$y_k = H_k x_k + \tilde{G}_k d_k + \mathcal{G}_k d_k + v_k \tag{80}$$

where

$$\tilde{F}_k = F_k - \mathcal{F}_k, \quad \tilde{G}_k = G_k - \mathcal{G}_k \tag{81}$$

in which $\mathcal{F}_k$ and $\mathcal{G}_k$ are design parameters that signify the terms $\mathcal{F}_k d_k$ and $\mathcal{G}_k d_k$ should be decoupled from the estimation. Assuming the unknown inputs described by (37), the system (79) and (80) can be rewritten as follows:

$$X_{k+1} = \bar{A}_k X_k + \bar{B}_k u_k + \bar{\mathcal{F}}_k d_k + W_k \tag{82}$$

$$y_k = \bar{H}_k X_k + \mathcal{G}_k d_k + v_k \tag{83}$$

where $X_k = \left[\, x_k' \;\; d_k' \,\right]'$ is an augmented state and

$$\bar{A}_k = \begin{bmatrix} A_k & \tilde{F}_k \\ 0 & I \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} B_k \\ 0 \end{bmatrix}, \quad \bar{\mathcal{F}}_k = \begin{bmatrix} \mathcal{F}_k \\ 0 \end{bmatrix}$$

$$W_k = \begin{bmatrix} w_k \\ w_k^d \end{bmatrix}, \quad \bar{H}_k = \left[\, H_k \;\; \tilde{G}_k \,\right]. \tag{84}$$

Second, applying the ARTSKF to (82) and (83), we obtain

$$\hat{X}_k = \bar{X}_{k|k} + V_k^a \tilde{d}_{k|k} \tag{85}$$

$$P_k^X = P_{k|k}^{\bar{X}} + V_k^a P_{k|k}^{\tilde{d}} (V_k^a)' \tag{86}$$

where $\bar{X}_{k|k}$ is given by

$$\bar{X}_{k|k-1} = \bar{A}_{k-1} \hat{X}_{k-1} + \bar{B}_{k-1} u_{k-1} \tag{87}$$

$$\bar{X}_{k|k} = \bar{X}_{k|k-1} + K_k^{\bar{X}} (y_k - \bar{H}_k \bar{X}_{k|k-1}) \tag{88}$$

$$P_{k|k-1}^{\bar{X}} = \bar{A}_{k-1} P_{k-1|k-1}^X \bar{A}_{k-1}' + Q_{k-1}^X \tag{89}$$

$$K_k^{\bar{X}} = P_{k|k-1}^{\bar{X}} \bar{H}_k' (C_k^X)^{-1} \tag{90}$$

$$P_{k|k}^{\bar{X}} = (I - K_k^{\bar{X}} \bar{H}_k) P_{k|k-1}^{\bar{X}} \tag{91}$$

$\tilde{d}_{k|k}$ is given by

$$\tilde{d}_{k|k} = K_k^{\tilde{d}} (y_k - \bar{H}_k \bar{X}_{k|k-1}) \tag{92}$$

$$K_k^{\tilde{d}} = P_{k|k}^{\tilde{d}} \tilde{S}_k' (C_k^X)^{-1} \tag{93}$$

$$P_{k|k}^{\tilde{d}} = \{\tilde{S}_k' (C_k^X)^{-1} \tilde{S}_k\}^+ \tag{94}$$

and

$$V_k^a = \bar{\mathcal{F}}_{k-1}^a - K_k^{\bar{X}} \tilde{S}_k, \quad \bar{\mathcal{F}}_{k-1}^a = \left[\, 0 \;\; \bar{\mathcal{F}}_{k-1} \,\right] \tag{95}$$

$$\tilde{S}_k = \left[\, \mathcal{G}_k \;\; H_k \mathcal{F}_{k-1} \,\right] \tag{96}$$

$$C_k^X = \bar{H}_k P_{k|k-1}^{\bar{X}} \bar{H}_k' + R_k \tag{97}$$

$$Q_k^X = diag\{Q_k, Q_k^d\}. \tag{98}$$

The system state estimate $\hat{x}_k$ and its error covariance matrix P_k are then obtained as follows:

$$\hat{x}_k = \left[\, I \;\; 0 \,\right] \hat{X}_k, \quad P_k = \left[\, I \;\; 0 \,\right] P_k^X \left[\, I \;\; 0 \,\right]'. \tag{99}$$

Third, noting that $\bar{X}_{k|k}$ given by (87)-(91) is an augmented state Kalman filter (ASKF), and hence using the following notations:

$$\tilde{x}_{k|k} = \left[\, I \;\; 0 \,\right] \bar{X}_{k|k}, \quad \tilde{P}_{k|k} = \left[\, I \;\; 0 \,\right] P_{k|k}^{\bar{X}} \left[\, I \;\; 0 \,\right]' \tag{100}$$

and applying the OTSKF design, which is given in the previous section, to the ASKF, $\bar{X}_{k|k}$, we may obtain

$$\tilde{x}_{k|k} = \bar{x}_{k|k} + \bar{V}_k \bar{d}_{k|k} \tag{101}$$

$$\tilde{P}_{k|k} = P_{k|k}^{\bar{x}} + \bar{V}_k P_{k|k}^{\bar{d}} \bar{V}_k' \tag{102}$$

where $\bar{x}_{k|k}$ is given by

$$\bar{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1} + \bar{\Psi}_{k-1} \bar{d}_{k-1|k-1} \tag{103}$$

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K_k^{\bar{x}} (y_k - H_k \bar{x}_{k|k-1}) \tag{104}$$

$$P_{k|k-1}^{\bar{x}} = A_{k-1} P_{k-1} A_{k-1}' + Q_{k-1} + \bar{U}_k Q_{k-1}^d \breve{F}_{k-1}' - A_{k-1} \bar{V}_{k-1} P_{k-1|k-1}^{\bar{d}} \bar{V}_{k-1}' A_{k-1}' \tag{105}$$

$$K_k^{\bar{x}} = P_{k|k-1}^{\bar{x}} H_k' \bar{C}_k^{-1}, \quad \bar{C}_k = H_k P_{k|k-1}^{\bar{x}} H_k' + R_k \tag{106}$$

$$P_{k|k}^{\bar{x}} = (I - K_k^{\bar{x}} H_k) P_{k|k-1}^{\bar{x}} \tag{107}$$

and the optimal unknown-input filter is given as follows:

$$\bar{d}_{k|k} = (I - K_k^{\bar{d}} \breve{S}_k) \bar{d}_{k-1|k-1} + K_k^{\bar{d}} (y_k - H_k \bar{x}_{k|k-1}) \tag{108}$$

$$K_k^{\bar{d}} = P_{k|k-1}^{\bar{d}} \breve{S}_k' \{\breve{S}_k P_{k|k-1}^{\bar{d}} \breve{S}_k' + \bar{C}_k\}^{-1} \tag{109}$$

$$P_{k|k}^{\bar{d}} = (I - K_k^{\bar{d}} \breve{S}_k) P_{k|k-1}^{\bar{d}} \tag{110}$$

where

$$\bar{V}_k = \bar{U}_k - K_k^{\bar{x}} \bar{S}_k \tag{111}$$

$$\bar{U}_k = \tilde{F}_{k-1} - \bar{\Psi}_{k-1} \tag{112}$$

$$\bar{S}_k = H_k \bar{U}_k + \tilde{G}_k \tag{113}$$

$$\bar{\Psi}_{k-1} = \tilde{F}_{k-1} - \breve{F}_{k-1} P_{k-1|k-1}^{\bar{d}} (P_{k|k-1}^{\bar{d}})^{-1} \tag{114}$$

$$\breve{F}_{k-1} = A_{k-1} \bar{V}_{k-1} + \tilde{F}_{k-1} \tag{115}$$

$$P_{k|k-1}^{\bar{d}} = P_{k-1|k-1}^{\bar{d}} + Q_{k-1}^d \tag{116}$$

$$\breve{S}_k = \hat{S}_k - H_k \bar{\Psi}_{k-1}. \tag{117}$$

Then, using the following relationships (see Hsieh and Chen (1999) for details):

$$\bar{X}_{k|k-1} = \begin{bmatrix} \bar{x}_{k|k-1} + \bar{U}_k \bar{d}_{k-1|k-1} \\ \bar{d}_{k-1|k-1} \end{bmatrix} \tag{118}$$

$$K_k^{\bar{X}} = \begin{bmatrix} K_k^{\bar{x}} + \bar{V}_k K_k^{\bar{d}} \\ K_k^{\bar{d}} \end{bmatrix} \tag{119}$$

$$P_{k|k-1}^{\bar{X}} = \begin{bmatrix} P_{k|k-1}^{\bar{x}} + \bar{U}_k P_{k|k-1}^{\bar{d}} \bar{U}_k' & \bar{U}_k P_{k|k-1}^{\bar{d}} \\ P_{k|k-1}^{\bar{d}} \bar{U}_k' & P_{k|k-1}^{\bar{d}} \end{bmatrix} \tag{120}$$

Eqs. (92), (95), and (97) can be rewritten as follows:

$$\tilde{d}_{k|k} = K_k^{\tilde{d}} (y_k - H_k \bar{x}_{k|k-1} - \bar{S}_k \bar{d}_{k-1|k-1}) \tag{121}$$

$$V_k^a = \begin{bmatrix} 0 & \mathcal{F}_{k-1} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} K_k^{\bar{x}} + \bar{V}_k K_k^{\bar{d}} \\ K_k^{\bar{d}} \end{bmatrix} \tilde{S}_k \tag{122}$$

$$C_k^X = \bar{S}_k P_{k|k-1}^{\bar{d}} \bar{S}_k' + \bar{C}_k. \tag{123}$$

Finally, summarizing the above results and using (99) and (121)-(122), we obtain the PTSKF as follows:

$$\hat{x}_k = \bar{x}_{k|k} + \bar{V}_k \bar{d}_{k|k} + \tilde{V}_k \tilde{d}_{k|k} \tag{124}$$

$$P_k = P^{\bar{x}}_{k|k} + \bar{V}_k P^{\bar{d}}_{k|k} \bar{V}'_k + \tilde{V}_k P^{\tilde{d}}_{k|k} \tilde{V}'_k \tag{125}$$

where $\bar{x}_{k|k}$, $\bar{d}_{k|k}$, $\tilde{d}_{k|k}$, $P^{\bar{x}}_{k|k}$, $P^{\bar{d}}_{k|k}$, $P^{\tilde{d}}_{k|k}$, and $\bar{V}_k$ are given by (104), (108), (121), (107), (110), (94), and (111), respectively, and

$$\tilde{V}_k = [\, I \;\; 0 \,]\, V^a_k = [\, 0 \;\; \mathcal{F}_{k-1} \,] - (K^{\bar{x}}_k + \bar{V}_k K^{\bar{d}}_k)\tilde{S}_k. \tag{126}$$

Note that the above PTSKF can be easily rewritten as the prescribed TSKF form (3)-(4), which is listed as follows:

$$\hat{x}_k = \bar{x}_{k|k} + V_k d_{k|k} \tag{127}$$

$$P_k = P^{\bar{x}}_{k|k} + V_k P^{d}_{k|k} V'_k \tag{128}$$

where $V_k = \begin{bmatrix} \bar{V}_k & \tilde{V}_k \end{bmatrix}$, $d_{k|k} = \begin{bmatrix} \bar{d}'_{k|k} & \tilde{d}'_{k|k} \end{bmatrix}'$, and $P^d_{k|k} = diag\{P^{\bar{d}}_{k|k}, P^{\tilde{d}}_{k|k}\}$.

Remark 2: The proposed PTSKF may be seen as an alternative to the previous proposed HTSKF (see Hsieh (2000) for details) and is characterized by two design parameters: $\mathcal{F}_k$ and $\mathcal{G}_k$. If one chooses $\mathcal{F}_k = F_k$ and $\mathcal{G}_k = G_k$, then the obtained PTSKF will be equivalent to the ARTSKF. However, if one chooses $\mathcal{F}_k = 0$ and $\mathcal{G}_k = 0$, then the obtained PTSKF, on the other hand, will be equivalent to the OTSKF.

6. ON THE UNBIASEDNESS OF THE PTSKF

To facilitate the following discussions, the PTSKF is rewritten, by using (104), (111), (108), (121), and (124), as follows:

$$\begin{aligned}\hat{x}_k &= (I - L_k H_k)\bar{x}_{k|k-1} + L_k y_k \\ &\quad + \left(\bar{U}_k - L_k \bar{S}_k - \bar{V}_k K^{\bar{d}}_k (\breve{S}_k - \bar{S}_k)\right) \bar{d}_{k-1|k-1}\end{aligned} \tag{129}$$

$$L_k = K^{\bar{x}}_k + \bar{V}_k K^{\bar{d}}_k + \tilde{V}_k K^{\tilde{d}}_k. \tag{130}$$

First, we show that the gain matrix L_k, given by (130), satisfies the following constraint:

$$L_k \tilde{S}_k = [\, 0 \;\; \mathcal{F}_{k-1} \,] \tag{131}$$

which is the unbiasedness constraint of the PMVF. Using (93), (94), (126), and (130), we obtain

$$\begin{aligned}L_k \tilde{S}_k &= (K^{\bar{x}}_k + \bar{V}_k K^{\bar{d}}_k)(C^X_k)^{1/2}(M_k - M_k M^+_k M_k) \\ &\quad + [\, 0 \;\; \mathcal{F}_{k-1} \,]\, M^+_k M_k\end{aligned} \tag{132}$$

where $M_k = (C^X_k)^{-1/2}\tilde{S}_k$. One can easily check that under the following condition:

$$rank \begin{bmatrix} M_k \\ [\, 0 \;\; \mathcal{F}_{k-1} \,] \end{bmatrix} = rank\, [\, M_k \,] \tag{133}$$

the constraint (131) is satisfied.

Next, we show the unbiasedness requirement of the filter (129). Using (1)-(2), (81), (103), (113), and (129)-(131), we obtain the error dynamics e_k as follows:

$$\begin{aligned}e_k &= (I - L_k H_k)(A_{k-1} e_{k-1} + w_{k-1}) - L_k(\tilde{G}_k w^d_{k-1} + v_k) \\ &\quad + \left((I - L_k H_k)\tilde{F}_{k-1} - L_k \tilde{G}_k\right) e^d_{k-1} \\ &\quad + \bar{V}_k K^{\bar{d}}_k (\breve{S}_k - \bar{S}_k)\bar{d}_{k-1|k-1}\end{aligned}$$

which is unbiased if $\bar{V}_k K^{\bar{d}}_k (\breve{S}_k - \bar{S}_k) = \bar{V}_k K^{\bar{d}}_k \tilde{S}_k \,[\, I \;\; I \,]' = 0$. This is always satisfied if one has

$$\bar{V}_k K^{\bar{d}}_k \tilde{S}_k = 0. \tag{134}$$

Note that the ARTSKF and the OTSKF both satisfy the unbiasedness requirement (134), which are given by $\bar{V}_k = 0$ and $\tilde{S}_k = 0$, respectively.

7. ILLUSTRATIVE EXAMPLES

To show the proposed results, the numerical example given by Darouach, Zasadzinski, and Boutayeb (2003) is considered, where the parameters of system (1) and (2) are given as follows:

Case 1:

$$A_k = \begin{bmatrix} -0.0005 & -0.0084 \\ 0.0517 & 0.8069 \end{bmatrix}, \quad H_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_k = \begin{bmatrix} 0.0129 & 0 \\ -1.2504 & 0 \end{bmatrix}, \quad G_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q_k = \begin{bmatrix} 0.0036 & 0.0342 \\ 0.0342 & 0.3249 \end{bmatrix}, \quad R_k = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.16 \end{bmatrix}.$$

Without loss of generality, the known input u_k is not considered in this simulation case. The initial state and its estimate are both assumed to be zero, and the initial covariance is given by $P_0 = diag(10, 200)$. The unknown inputs are given by

$$d_k = \begin{bmatrix} 5u_s[k] - 5u_s[k-20] + 5u_s[k-70] \\ 4u_s[k] - 4u_s[k-30] + 4u_s[k-65] \end{bmatrix}$$

where $u_s[k]$ is the unit-step function. In this simulation example, we assume that $Q^d_k = diag\{0.025, 0.016\}$, and the simulation time is 100 time steps.

As considered in Hsieh (2007b), four forms of $\mathcal{F}_{k-1}$ and two of $\mathcal{G}_k$ are chosen as follows:

$$\mathcal{F}^1_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \mathcal{F}^2_{k-1} = \begin{bmatrix} 0 & 0 \\ -1.2504 & 0 \end{bmatrix}, \; \mathcal{G}^1_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{F}^3_{k-1} = \begin{bmatrix} 0.0129 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{F}^4_{k-1} = \begin{bmatrix} 0.0129 & 0 \\ -1.2504 & 0 \end{bmatrix}, \mathcal{G}^2_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

from which 8 different models of the PTSKF are obtained. In the following discussions, we will name the PTSKF which is characterized by $\mathcal{F}^i_{k-1}$ and $\mathcal{G}^j_k$ as model $k (= 2(i-1) + j)$, denoted by the PTSKF^k. Note that the PTSKF^1 and the PTSKF^8 correspond to the OTSKF and the ARTSKF, respectively. We list the root-mean-square-errors (rmse) and the trace of the state error covariance matrix, i.e. tr$[P_k]$, for these models in Table 1. Moreover, we list the performances of the corresponding PMVFs in Table 2.

From Table 1, we obtain the following observation: the PTSKF^3 filter has the best filtering performance, which is slightly better than the KF (PTSKF^1) and is much better than the ARTSKF/OEF (PTSKF^8). The filtering degradation of the PTSKF^8 is due to the fact that the matrix $\tilde{S}_k$ is of full row rank, which renders the obtained filter to a degenerated filter (see Darouach, Zasadzinski, and Boutayeb (2003) for more details), and hence results in an unacceptable state error covariance. This simulation suggests a possible way to apply a filter other than existing ones to give potentially the global minimum-variance state estimates. It is also clear from Table 2 that the best filtering performance of the PMVF is given by the PMVF^3. This illustrates a compatible result between the PTSKF and the PMVF. Specifically, the filtering performances of the models 1 and 8 of the PTSKF are as the same as those of the corresponding PMVFs.

Next, we consider the complexity issue. In this regard, we use floating point operations, or "flops," in Matlab as a measure of the computational complexity. The flops of the aforementioned filters are also listed in Tables 1 and 2, from which we obtain that all the flops of the PTSKFs are fewer than the corresponding results of the PMVFs. This shows that the PTSKF is potentially more compact, in the sense of less computational complexity, than the PMVF. This is due to the fact that the sub-filters imbedded in the former are all in the form of the KF, which in general has a more compact filter structure than the UMVF.

Finally, to further show the filtering performance of the PTSKF is indeed compatible to that of the PMVF, the numerical example given by Chen and Patton (1996) is considered, where the parameters of system (1) and (2) are given as follows:

Case 2:

$$A_k = \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix}, \quad H_k = I_{3\times 3}$$

$$B_k = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, \quad F_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The covariance matrices are given as $R_k = 0.1^2 I_{3\times 3}$ and $Q_k = diag\{0.1^2, 0.1^2, 0.01^2\}$. In the simulation, we set $u_k = 10$, $x_0 = 0$, $P_0 = 0.1^2 I_{3\times 3}$, and

$$d_k = \begin{bmatrix} \Delta a_{11} & \Delta a_{12} & \Delta a_{13} \\ \Delta a_{21} & \Delta a_{22} & \Delta a_{23} \end{bmatrix} x_k + \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} u_k$$

where $\Delta a_{ij} = -0.5 a_{ij}$ and $\Delta b_j = 0.5 b_j$. The simulation time is 100 time steps.

The following $\mathcal{F}_{k-1}$ and $\mathcal{G}_k$ are considered.

$$\mathcal{F}^1_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathcal{F}^2_{k-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathcal{F}^3_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{F}^4_{k-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{G}^1_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{G}^2_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We list the rmse in the state estimates, the trace of the state error covariance matrix, and the flops of the PTSKFs and the PMVFs in Tables 3 and 4, respectively. From the tables, we conclude that the filtering performance of the PTSKF is compatible to that of the PMVF and the computational superiority of the former over the latter is definitely true.

Table 1. The rmse performance, the trace of the state error covariance matrix, and the flops of the PTSKFs for case 1

filter	rmse(e^1)	rmse(e^2)	tr[P_k]	flops
PTSKF^1	0.0852	2.4416	1.5725	1259
PTSKF^2	0.0725	5.4302	4.8001	1385
PTSKF^3	0.0647	2.4285	1.4332	1365
PTSKF^4	0.0670	5.3024	0.1101	1462
PTSKF^5	0.0994	2.4751	1.5756	1365
PTSKF^6	0.0993	5.4534	4.8040	1600
PTSKF^7	0.0656	2.4459	1.4217	1365
PTSKF^8	0.0993	11.4414	134.7506	1600

Table 2. The rmse performance, the trace of the state error covariance matrix, and the flops of the PMVFs for case 1

filter	rmse(e^1)	rmse(e^2)	tr[P_k]	flops
PMVF^1	0.0852	2.4416	1.5725	1559
PMVF^2	0.0685	4.7539	2.9303	1505
PMVF^3	0.0637	2.4338	1.6191	1605
PMVF^4	0.0657	5.2918	0.1101	1611
PMVF^5	0.0993	2.4503	1.5835	1698
PMVF^6	0.0993	4.6288	2.9480	1856
PMVF^7	0.0656	2.4315	1.6154	2021
PMVF^8	0.0993	11.4414	134.7506	2050

Table 3. The rmse performance, the trace of the state error covariance matrix, and the flops of the PTSKFs for case 2

filter	rmse(e^1)	rmse(e^2)	rmse(e^3)	tr[P_k]	flops
PTSKF^1	0.1053	0.6243	1.6177	0.0218	2273
PTSKF^2	0.1053	0.6074	1.6922	0.0218	2355
PTSKF^3	0.1060	0.6248	1.6169	0.0219	2335
PTSKF^4	0.1060	0.6074	1.6924	0.0219	2574
PTSKF^5	0.1053	0.1489	1.5356	0.0289	2320
PTSKF^6	0.1053	0.0949	1.8080	0.0267	2590
PTSKF^7	0.1075	0.1484	1.5374	0.0289	2509
PTSKF^8	0.1060	0.0949	1.8080	0.0268	3184

Table 4. The rmse performance, the trace of the state error covariance matrix, and the flops of the PMVFs for case 2

filter	rmse(e^1)	rmse(e^2)	rmse(e^3)	tr[P_k]	flops
PMVF^1	0.1053	0.6243	1.6177	0.0218	3848
PMVF^2	0.1053	0.6203	1.6235	0.0219	3732
PMVF^3	0.1060	0.6243	1.6179	0.0219	3779
PMVF^4	0.1060	0.6203	1.6237	0.0220	3760
PMVF^5	0.1053	0.0949	1.8098	0.0267	3826
PMVF^6	0.1053	0.0949	1.8078	0.0267	3734
PMVF^7	0.1060	0.0949	1.8100	0.0268	3871
PMVF^8	0.1060	0.0949	1.8080	0.0268	4058

8. CONCLUSION

In this paper, an attempt to define a unified solution to unbiased minimum-variance filtering for systems with unknown inputs is presented. It is shown that, with a novel parameterized design method, the PTSKF serving as a dedicated unified solution to unknown inputs filtering problem is derived. The proposed PTSKF is characterized by two design parameters: $\mathcal{F}_{k-1}$ and $\mathcal{G}_k$ and is an extension of the conventional TSKF. Simulation results show that not only the filtering performance of the PTSKF is compatible to that of the PMVF but also the computational complexity of the former is less intensive than that of the latter. This research suggests a possible way other than existing methods to derive a minimum-variance filter in order to enhance filtering performance for systems with unknown inputs.

The extended work of finding the adaptive model switching rule that can achieve the optimal filtering performance in general conditions via optimally determining the design parameters is under investigation.

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Appendix A. A PARAMETERIZED MINIMUM-VARIANCE FILTER

For easy reference, the parameterized minimum-variance filter (PMVF) derived in Hsieh (2007b) for the special case $\Delta A_k = 0$ and $\Delta B_k = 0$ is listed as follows.

1) System State Estimator:

$$\begin{aligned}\hat{x}_k &= (I - L_k H_k)(A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1}) \\ &\quad + V_k \hat{d}_{k-1} + L_k y_k \qquad \text{(A.1)}\end{aligned}$$

$$P_k = \bar{P}^a_{k-1} - \breve{\mathcal{G}}_k \bar{P}^b_k - \left((\bar{P}^b_k)' - \breve{\mathcal{G}}_k\{\bullet\}\right) L'_k \qquad \text{(A.2)}$$

where

$$L_k = \breve{\mathcal{G}}_k + \breve{\mathcal{K}}_k \breve{\mathcal{T}}_k \qquad \text{(A.3)}$$

$$\breve{\mathcal{G}}_k = [\,0 \;\; \mathcal{F}_{k-1}\,]\,[\,\mathcal{G}_k \;\; H_k\mathcal{F}_{k-1}\,]^+ \qquad \text{(A.4)}$$

$$\breve{\mathcal{K}}_k = \left((\bar{P}^b_k)' - \breve{\mathcal{G}}_k\{\bullet\}\right)\breve{\mathcal{T}}'_k\left(\breve{\mathcal{T}}_k\{\bullet\}\breve{\mathcal{T}}'_k\right)^+ \qquad \text{(A.5)}$$

$$\breve{\mathcal{T}}_k = I - [\,\mathcal{G}_k \;\; H_k\mathcal{F}_{k-1}\,]\,[\,\mathcal{G}_k \;\; H_k\mathcal{F}_{k-1}\,]^+ \qquad \text{(A.6)}$$

$$V_k = \tilde{F}_{k-1} - L_k(H_k\tilde{F}_{k-1} + \tilde{G}_k) \qquad \text{(A.7)}$$

$$\begin{aligned}\{\bullet\} &= H_k\bar{P}^a_{k-1}H'_k + R_k + \tilde{G}_k(P^d_{k-1} + Q^d_{k-1})\tilde{G}'_k \\ &\quad + \tilde{G}_k\{\bullet\}'_1 H'_k + H_k\{\bullet\}_1\tilde{G}'_k \qquad \text{(A.8)}\end{aligned}$$

$$\{\bullet\}_1 = A_{k-1}\tilde{P}_{k-1} + \tilde{F}_{k-1}P^d_{k-1} \qquad \text{(A.9)}$$

$$\{\bullet\}_2 = \bar{P}_{k-1} + \tilde{F}_{k-1}\tilde{P}'_{k-1}A'_{k-1} \qquad \text{(A.10)}$$

$$\bar{P}^a_{k-1} = \{\bullet\}_1\tilde{F}'_{k-1} + \{\bullet\}_2 \qquad \text{(A.11)}$$

$$\bar{P}^b_k = H_k\bar{P}^a_{k-1} + \tilde{G}_k\{\bullet\}'_1 \qquad \text{(A.12)}$$

$$\bar{P}_k = A_k P_k A'_k + Q_k. \qquad \text{(A.13)}$$

2) Unknown Inputs Estimator:

$$\begin{aligned}\hat{d}_k &= (I - K^d_k S_k)\hat{d}_{k-1} \\ &\quad + K^d_k(y_k - H_k(A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1})) \qquad \text{(A.14)}\end{aligned}$$

$$P^d_k = P^d_{k-1} + Q^d_{k-1} - \Xi_k(K^d_k)' \qquad \text{(A.15)}$$

$$\text{(A.16)}$$

where

$$K^d_k = \Xi_k(C^d_k)^{-1} \qquad \text{(A.17)}$$

$$\Xi_k = P^d_{k-1}\mathcal{S}'_k + Q^d_{k-1}G'_k + \tilde{P}'_{k-1}A'_{k-1}H'_k \qquad \text{(A.18)}$$

$$\begin{aligned}C^d_k &= H_k\bar{P}_{k-1}H'_k + R_k + \mathcal{S}_k P^d_{k-1}\mathcal{S}'_k + G_k Q^d_{k-1}G'_k \\ &\quad + \mathcal{S}_k\tilde{P}'_{k-1}A_{k-1}H'_k + H_k A_{k-1}\tilde{P}_{k-1}\mathcal{S}'_k \qquad \text{(A.19)}\end{aligned}$$

$$\begin{aligned}\tilde{P}_k &= \Phi_k\{\bullet\}_1 - \Phi_k(\{\bullet\}_1\mathcal{S}'_k + \{\bullet\}_2 H'_k)(K^d_k)' \\ &\quad + L_k R_k (K^d_k)' - L_k\tilde{G}_k P^d_k \qquad \text{(A.20)}\end{aligned}$$

$$\mathcal{S}_k = H_k F_{k-1} + G_k \qquad \text{(A.21)}$$

$$\Phi_k = I - L_k H_k. \qquad \text{(A.22)}$$