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Gain Scheduling using Time-varying Kalman Filter for a class of LPV Systems ⋆

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Abstract: Current gain-scheduling approaches only assure stability when the underlying parameter varies sufficiently slowly, and hence stable closed-loop is not *guaranteed* for more general (i.e. faster) parameter variations. Shamma and Athans (1992) provides a solution to overcome this by computing Riccati Differential Equation (RDE) online for the current parameter value with the offline Algebraic Riccati Equation (ARE) solutions for every parameter value, which is computationally demanding. This paper achieves a very significant simplification, by showing how only a finite number of AREs need be used and the online RDE solutions can be computed by table look-up and matrix inversion. In simulations the method yields results indistinguishable from those achieved in Shamma and Athans (1992).

1. INTRODUCTION

Over the last two decades, gain scheduling has been one of the most promising control designs for nonlinear systems. The idea is well suited to systems where their operating points or associated physical parameters - such as friction and mass, etc - are frequently changing over time. Some successful early examples in flight control (Stein, 1980) and engine control (Gumbleton and Bowler, 1982) demonstrate its ability to handle parameter changes adequately. Among many different gain scheduling approaches, the most common setting characterises a given plant using a set of linear systems for different operating points. For each operating point the existing, possibly cutting-edge, linear control tools will be chosen to design and analyse local controllers with all desired properties (Shamma and Athans, 1990). A mechanism is then introduced which changes the controller in accord with the perceived change of operating point, either by switching or continuous change.

There are, however, some ad hoc aspects of nearly all gain scheduling approaches. If a finite number of operating points is to be used, there is a lack of a systematic methodology to determine an adequate number and their location. Again, performance characterisation given operating point changes is generally absent, and indeed there may be no formal assurance of stability. Due to general stability results often only being applicable to very slowly varying systems, control designers may face a practical limitation that "scheduled variables must vary slowly" (Shamma and Athans, 1992). Not many results without this slow variation assumption have been published on the stability and performance (Rugh and Shamma, 2000), but there is one paper (Shamma and Athans, 1992) that suggests a particular way of synthesising the controller with potentially fast variations of parameters. The operating point is assumed to vary continuously, and the plant is assumed to be minimum-phase at every operating point. A Linear Quadratic Gaussian (LQG) design is performed at each operating point using Loop Transfer Recovery (LTR) ideas (Doyle and Stein, 1981). At any one operating point, the state feedback gain is arranged to achieve very fast eigenvalues (considerably faster than the average rate of change of operating point with a known upper bound) by solving the Algebraic Riccati Equation (ARE) with suitable weighting matrices. Given uniform observability and uniform controllability of key matrix pairs, the observer gain is used to stabilise the plant, even in the presence of fast (though not arbitrarily fast) variations in operating point by using the solution of a time-varying Riccati Differential Equation (RDE). Examples of the efficacy of this procedure, and the inadequacies of competing procedures, are shown in Shamma and Athans (1992). While the end result is attractive, this method requires a RDE to be solve online, which is computationally extremely demanding. The contribution of this paper is to introduce a significant modification to the scheme of Shamma and Athans (1992), and it allows much more calculation to be done offline, and much less calculation online. The modest and controllable penalty flows from the fact that only a finite number of operating points are used to determine controller designs, and the controller at any instant of time will not necessarily be precisely tuned to the operating point, but rather

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to a nearby one. Hence, instead of solving the Kalman filter (KF) RDE online, we can use an explicit transient formula that just looks up the pre-determined solutions of the associated AREs at a finite set of given system operating points, and connects them in a standard way. In addition, only a finite number of AREs are solved to determine a finite set of controller gains.

The outline of this paper is as follows: in section 2, we provide a mathematical overview of state estimation and RDE. The main results are provided in Section 3, followed by a stability analysis in Section 4. We present simulated results with a numerical example in Section 5, and give closing remarks in Section 6.

2. BACKGROUND

2.1 Overview of State Estimation and RDE

Consider a time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t); \quad y(t) = C(t)x(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^k$ are state, system input and output, respectively, and the initial state value $x(t_0)$ is given. The matrices $A : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ and $C : \mathbb{R}^+ \to \mathbb{R}^{k\times n}$ are assumed to have bounded entries. Then, the standard state observer

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + \hat{K}(t)[y(t) - C(t)\hat{x}(t)] \quad (2)$$

takes $y(t)$ and $u(t)$ as its inputs, and produces the state estimate $\hat{x}(t)$ that ensures the error between the true state and its estimate $e(t) := x(t) - \hat{x}(t)$ converges asymptotically, namely exponentially, to zero. One technique to find $\hat{K}$ is to use KF theory, which are not fully summarised.

Definition 1. For the homogeneous part of (1) with an initial time t_0, the ***state-transition matrix***, $\Phi(t, t_0)$ is defined to be a $n \times n$ matrix that produces the state vector $x(t) \in \mathbb{R}^n$, $\forall\ t \geq t_0$ using $x(t) = \Phi(t, t_0)x(t_0)$ given the initial state $x(t_0)$ with standard properties that $\Phi(t, t) = I, \forall\ t$ and $\Phi(t, s) = \Phi(t, w)\Phi(w, s),\ \forall\ t, s, w$

Definition 2. The pair $(A(t), C(t))$ is defined to be ***uniformly observable*** *iff* $\exists \delta_o, \alpha_o, \beta_o > 0$,

$$\alpha_o I \ \leq\ W(s, s + \delta_o)\ \leq\ \beta_o I,\ \forall s \in \mathbb{R}, \quad (3)$$

where the observability gramian $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is

$$W(s, t) = \int_s^t \Phi'(\tau, s)C'(\tau)C(\tau)\Phi(\tau, s)d\tau \quad (4)$$

Similarly, ***uniform controllability*** is defined as

$$\alpha_c I \ \leq\ M(s - \delta_c, s)\ \leq\ \beta_c I \quad (5)$$

for some $\delta_c, \alpha_c, \beta_c > 0$ and all s in a dual manner to the above using the controllability gramian

$$M(s, t) = \int_s^t \Phi(t, \tau)B(\tau)B'(\tau)\Phi'(t, \tau)d\tau. \quad (6)$$

Let $Q(t)$ be a symmetric and positive semidefinite matrix for all t, and suppose $Q(t)$ is bounded. Then there exists a matrix $L(t)$ such that $L(t)L'(t) = Q(t), \forall t$. The standard filtering solution is given as the following lemma:

Lemma 3. Consider the linear system given by (1). Suppose $(A(t), C(t))$ is uniformly observable and $(A(t), L(t))$ is uniformly controllable on the interval $[t_0, \infty)$, where $L(t)L'(t) = Q(t),\ \forall t$. Then, there exists a standard state observer (2) which ensures the exponential convergence of the estimation error $e(t)$ to zero, where the estimation system matrix $\hat{A}(t) = A(t) - \hat{K}(t)C(t)$, filter gain $\hat{K}(t) = \Sigma(t)C'(t)R^{-1}(t)$ and $\Sigma(t)$ is the solution, which is bounded, of the matrix RDE

$$\begin{aligned}\dot{\Sigma}(t) =& \Sigma(t)A'(t) + A(t)\Sigma(t) \\ &- \Sigma(t)C'(t)(R(t))^{-1}C(t)\Sigma(t) + Q(t)\end{aligned} \quad (7)$$

with any boundary condition $\Sigma(t_0) = \Sigma'(t_0) = J \geq 0$. Here, the matrices $R(t) = R'(t) > 0$ and $Q(t) = Q'(t) \geq 0$ are design parameters to be chosen with continuous bounded, and bounded entries also for $(R(t))^{-1}$.

Proof. See Anderson and Moore (1981) for proof of the discrete time case.

Let $\Sigma(t; t_0)$ denote the solution of the RDE (7) with initial condition $\Sigma(t_0) = 0$ and assume that coefficients in (7) are time-invariant; then $\Sigma(t; t_0) = \Sigma(t - t_0)$. Also, $\Sigma(t - t_0)$ is monotone increasing in t and bounded (Anderson and Moore, 1989). Then the limiting value $\bar{\Sigma}$ exists and is given as

$$\bar{\Sigma} := \lim_{t\to\infty} \Sigma(t; t_0) = \lim_{t\to\infty} \Sigma(t - t_0) = \lim_{t_0\to-\infty} \Sigma(t - t_0) \quad (8)$$

For the case where the given system (1) is time-invariant, the following lemma gives the time-invariant solution:

Lemma 4. Consider the time-invariant case of the linear system (1) with constant matrices $\bar{A}, \bar{C}, \bar{Q}$ and $\bar{R}$ and suppose that $(\bar{A}, \bar{C})$ is detectable and $(\bar{A}, \bar{L})$ is stabilisable, where $\bar{L}\bar{L}' = \bar{Q}$. Then the limiting solution $\bar{\Sigma}$ (8) satisfies the associated ARE

$$\bar{\Sigma}\bar{A}' + \bar{A}\bar{\Sigma} - \bar{\Sigma}\bar{C}'\bar{R}^{-1}\bar{C}\bar{\Sigma} + \bar{Q} = 0 \quad (9)$$

and $\bar{\Sigma}$ is also the solution of (7). The gain of the standard observer $\hat{K} = \bar{\Sigma}\bar{C}'\bar{R}^{-1}$ is constant and if B is also constant, the standard observer (2) given as a time-invariant system ensures the exponential convergence of $e(t) = x(t) - \hat{x}(t)$ to zero.

Proof. See Anderson and Moore (1989).

Finding solutions for the RDE (7) is computationally demanding, but the associated ARE (9) can be easily solved. Callier et al. (1992) proposes an explicit formula for time-invariant systems that provides the transient solution of the RDE (when $A(t), C(t), R(t)$ and $Q(t)$ are constant) computed *using knowledge of the steady state solution*, i.e. the solution of the associated ARE. A dual filtering result is given as the following:

Lemma 5. Consider the time-invariant case of the linear system (1) with constant matrices $\bar{A}, \bar{B}, \bar{C}, \bar{R}$ and $\bar{Q}$, where $(\bar{A}, \bar{C})$ is observable. Let $\bar{\Sigma} = \bar{\Sigma}' \geq 0$ be a stabilising solution of the associated ARE (9), $\hat{A} := (\bar{A} - \bar{\Sigma}\bar{C}'\bar{R}^{-1}\bar{C})$ and let $\bar{W}$ satisfy

$$\bar{W}\hat{A}' + \hat{A}\bar{W} - \bar{C}\bar{R}^{-1}\bar{C}' = 0. \quad (10)$$

Then the transient solution of the RDE (7) from any initial value $\Sigma(t_0) = J = J' \geq 0$ to the stabilising solution $\bar{\Sigma}$ with $\tau := t - t_0, \tau \geq 0$ can be obtained by

$$\begin{aligned}\Sigma(\tau) = \bar{\Sigma} + \mathrm{e}^{\hat{A}\tau}[(J - \bar{\Sigma}) \\ \{I + (\bar{W} - \mathrm{e}^{\hat{A}'\tau}\bar{W}\mathrm{e}^{\hat{A}\tau})(J - \bar{\Sigma})\}^{-1}]\mathrm{e}^{\hat{A}'\tau}.\end{aligned} \quad (11)$$

Proof. The dual theorem is proved in Callier et al. (1992).

2.2 System of Interest

Suppose the system of interest has $p \geq 0$ physical parameters. If we assume that the parameters remain bounded, then we may restrict the range of each parameter to be symmetric about zero within $[-1, 1]$ by variable substitutions and scaling. We then consider the time-varying system parameter $\Lambda : \mathbb{R}^+ \to \Lambda_{\text{box}}$ where $\Lambda_{\text{box}} := [-1, 1]^p$, which can be measured online. In the remainder of this paper, our aim is to provide a gain scheduling design method for a Linear Parameter Varying (LPV) system, described by

$$\begin{aligned} \dot{x}(t) &= A(\Lambda(t))x(t) + B(\Lambda(t))u(t); \\ y(t) &= C(\Lambda(t))x(t), \end{aligned} \tag{12}$$

where $x(t)$, $u(t)$ and $y(t)$ are the state variable, (control) input and system output, respectively. The matrix functions $A : \Lambda_{\text{box}} \to \mathbb{R}^{n\times n}$, $B : \Lambda_{\text{box}} \to \mathbb{R}^{n\times m}$ and $C : \Lambda_{\text{box}} \to \mathbb{R}^{k\times n}$ are assumed to have bounded entries at all times. We shall introduce a temporary assumption that the system (12) is uniformly controllable and uniformly observable in Section 3.2.

3. CONTROL SYNTHESIS

3.1 Existing Shamma and Athans Design Approach

Given the assumption that for every parameter value the plant is minimum phase, the following approach ensures a stable closed loop even with fast parameter variation:

Let Γ, $Q(\kappa)$ and $R(\kappa)$, $\forall \kappa \in \Lambda_{\text{box}}$ be matrix design parameters to be chosen. For the given LPV system (12), the gain scheduling controller is given as the standard observer/state feedback system

$$\begin{aligned} \dot{\hat{x}} = [&A(\Lambda(t)) - B(\Lambda(t))G(\Lambda(t)) \\ &- H(t)C(\Lambda(t))]\hat{x} - H(t)(r - y), \end{aligned} \tag{13}$$

with the control law $u := -G(\Lambda(t))\hat{x}$, where r, $\hat{x}$ and y are the reference signal, state estimate and the system output, respectively. The given observer gain $H : \mathbb{R}^+ \to \mathbb{R}^{n\times k}$ is

$$H(t) = (R(\Lambda(t)))^{-1}\Sigma(t)C'(\Lambda(t)) \tag{14}$$

and the given control feedback gain $G : \Lambda_{\text{box}} \to \mathbb{R}^{m\times n}$ is

$$G(\kappa) = \Gamma^{-1}B'(\kappa)Z(\kappa), \tag{15}$$

where $\Sigma(t)$ is the solution of the time-varying KF RDE

$$\begin{aligned} \dot{\Sigma}(t) =&\Sigma(t)A'(\Lambda(t)) + A(\Lambda(t))\Sigma(t) + Q(\Lambda(t)) \\ &- \Sigma(t)C'(\Lambda(t))(R(\Lambda(t)))^{-1}C(\Lambda(t))\Sigma(t), \end{aligned} \tag{16}$$

$\Sigma(0) = \Sigma'(0) > 0$, and $Z(\kappa)$ is the stabilising solution of the ARE

$$\begin{aligned} &Z(\kappa)A(\kappa) + A'(\kappa)Z(\kappa) + C'(\kappa)C(\kappa) \\ &\qquad - Z(\kappa)B(\kappa)\Gamma^{-1}B'(\kappa)Z(\kappa) = 0. \end{aligned} \tag{17}$$

Here, given that a bound on the rate of variation of the physical parameter is available the fact that the closed-loop feedback dynamics associated with the matrix

$$A_{\text{cont}}(\kappa) := A(\kappa) - B(\kappa)\Gamma^{-1}B'(\kappa)Z(\kappa) \tag{18}$$

are stable for each fixed κ, and have modes which are considerably faster than the rate of parameter variations means that also the closed-loop dynamics associated with $A_{\text{cont}}(\Lambda(t))$ are stable. This stability result follows from the fact that a system with time-varying parameters is stable if it is stable for all fixed values of the parameters and the parameters vary much more slowly than the system dynamics (Desoer and Vidyasagar, 1975). Note also that it is standard in LTR design to ensure that either the dynamics associated with the state feedback law or those associated with the estimator are very fast; here, it is the dynamics associated with the state feedback law, and this ensures that the overall closed-loop dynamics obtained by using the controller based on feedback of a state estimate are mainly dominated by the observer dynamics, which become a form of target loop for the closed-loop dynamics. Of course, to use the LTR design approach, it is required that the original system has a minimum phase or stable invertibility property.

Next, in relation to arguing the stability of the overall loop, we note that the time-varying dynamics associated with the KF are stable under uniform controllability and uniform observability assumptions (which will be examined in Section 4.2), with stability not being related to the speed of parameter variations (Bucy and Joseph, 1968). When both the state feedback dynamics and observer dynamics are stable, the entire closed-loop becomes stable. However, even if this gain scheduling design using loop transfer recovery works with fast parameter variation, there is still a computational problem where the cost of solving the RDE online is concerned. A separate computational problem arises from the fact that an infinite number of control AREs will, or at least may, have to be solved.

3.2 New Design Procedure

The new gain scheduling controller for the LPV system of interest in (12) with the time-varying system parameter $\Lambda(t)$ requires the following assumptions and offline calculations:

1. Suppose a set of N operating points $\Lambda_{\text{oper}} := \{\Lambda_1, \Lambda_2, \ldots, \Lambda_N\} \subset \Lambda_{\text{box}}$ are identified a priori and that for each $\Lambda_i \in \Lambda_{\text{oper}}$ there is an associated open subset $\Xi_i \subset \Lambda_{\text{box}}$ such that $\Lambda_i \in \Xi_i$ and $\bigcup_{i=1}^{N} \Xi_i = \Lambda_{\text{box}}$. Let the set of all subsets be $\Xi_{\text{oper}} := \{\Xi_1, \Xi_2, \ldots, \Xi_N\}$.
2. Choose matrix design parameters $\Gamma \in \mathbb{R}^{m\times m}$ and $Q_i \in \mathbb{R}^{n\times n}$, $R_i \in \mathbb{R}^{k\times k}$ for each $i \in \{1, \ldots N\}$.
3. At each $\Lambda_i \in \Lambda_{\text{oper}}$, assume that the corresponding "frozen" form of the given system is a minimal realisation with $A_i := A(\Lambda_i), B_i := B(\Lambda_i)$, $C_i := C(\Lambda_i)$ Hence, we have (A_i, B_i) is controllable and (A_i, C_i) is observable for all $i \in \{1, \ldots N\}$.
4. For each operating point $\Lambda_i \in \Lambda_{\text{oper}}$, determine a stabilising solution $\bar{\Sigma}_i$ of the observer ARE (9) for the corresponding values A_i, C_i, Q_i, R_i. Let the collection of all stabilising solutions be $\bar{\Sigma}_{\text{oper}} := \{\bar{\Sigma}_1, \bar{\Sigma}_2, \ldots, \bar{\Sigma}_N\}$.

We can then use the following procedure to determine the observer gain $H : \mathbb{R}^+ \to \mathbb{R}^{k\times k}$ for the observer/state feedback controller (13), as well as a sequence of switching times $\{t_j\}_{j\in\mathbb{Z}_+}$ and an index function $c : \mathbb{R}^+ \to \{1, \ldots N\}$. Note that the control feedback gain is completely determined by the index function as $G(\Lambda_{c(t)})$ using (15).

1. Initialise by letting $j = 0$, $t_0 = 0$, and choosing $c(t_0)$ such that $\Lambda(t_0) \in \Xi_{c(t_0)}$.
2. Suppose that at time t, we will not sample $\Lambda(t)$ again until time t_+.

- set $c(\tau) = c(t)$ for all $\tau \in (t, t_+)$
- for all $\tau \in [t, t_+)$, compute $\Sigma(\tau)$ using the transient formula (10) with the initial value $J = \Sigma(t)$ and the final value $\bar{\Sigma} = \bar{\Sigma}_{c(t)}$. Note that it would *not* be expected that the final value would be attained at time t_+ or t_{j+1}.
- set the observer gain for all $\tau \in [t, t_+)$ as $H(\tau) = (R_{c(t)})^{-1}\Sigma(\tau)C'_{c(t)}$

(3) Then suppose that we sample $\Lambda(t)$ at time t, and that the last time for which we sampled $\Lambda(t)$ was t_-.
  (a) If $\Lambda(t) \in \Xi_{c(t_-)}$, then
    - set $c(t) = c(t_-)$
  (b) If $\Lambda(t) \notin \Xi_{c(t_-)}$, then
    - choose $c(t)$ such that $\Lambda(t) \in \Xi_{c(t)}$
    - set $t_{j+1} = t$
    - increment $j = j + 1$

Note that it would not be required that we knew the next sampling time t_+ at time t, the point being that c, and thus, $\bar{\Sigma}$ and G, may remain fixed until further notice; that is, until the next switching time t_{j+1}. It follows from the above procedure that

$$c(t) = c(t_j) \quad \forall\, t \in [t_j, t_{j+1}),\ \forall\, j \in \mathbb{Z}_+.$$

In this paper, we assume that we can sample Λ continuously, and the above procedure then further yields

$$\Lambda(t) \in \Xi_{c(t)} \quad \forall\, t \in \mathbb{R}^+ \tag{19}$$

and

$$t_{j+1} = \inf\{t > t_j \mid \Lambda(t) \notin \Xi_{c(t_j)}\} \quad \forall\, j \in \mathbb{Z}^+.$$

Since the regions $\Xi_i \subset \Lambda_{\text{box}}$ are open and $A(t), B(t), C(t)$ are bounded for all $t \in \mathbb{R}^+$, it also follows that

$$\exists\, \delta_j > 0 \ \text{ such that } \ t_{j+1} - t_j \geq \delta_j \quad \forall\, j \in \mathbb{Z}^+.$$

We then define

$$\delta_c := \inf_{j \in \mathbb{Z}^+}(t_{j+1} - t_j) \tag{20}$$

It further follows from the uniform boundedness of A, B, C, the openness of Ξ_i for all i, and the fact that N is finite that $\delta_c > 0$. This value will be crucial in proving uniform controllability in Lemma 7.

Lemma 6. The proposed observer/state feedback controller (13) with the control feedback gain $G(\Lambda_{c(t)})$ using (15) and observer gain $H(t)$ defined as above guarantees its stability for arbitrarily fast variation of the system parameter $\Lambda(t)$.

Proof. See Section 4 for the analysis.

The resulting RDE solution $\Sigma(t)$ will be a continuous and piece-wise differentiable function that is determined by interpolating the pre-calculated stabilising constant values $\bar{\Sigma}_{\text{oper}}$ with the transient formula (Lemma 5) that makes each transient converge towards the next stabilising solution exponentially.

By using the new design principle, there is no need to compute the solution of the actual RDE online, but only two simple online computation are required: determination of the current subset $\Xi_i \in \Xi_{\text{oper}}$ for the measured system parameter $\Lambda(t)$; scheduling $\Sigma(t)$ using the pre-determined values $\bar{\Sigma}_i \in \bar{\Sigma}_{\text{oper}}$ and the transient formula given in Lemma 5. In addition, the set of state feedback gains is pre-computed, and the appropriate gain to use is again determined by the current subset Ξ_i.

4. STABILITY ANALYSIS

Many systems of interest (12) operate near a particular operating point, say Λ_i, with small variations most of the time, and they occasionally make an abrupt switching to an other operating point, namely $\Lambda_j, i \neq j$ (Anderson, 2005; Rugh and Shamma, 2000). For this reason, and for the pedagogical benefit, the stability of the proposed design will be verified in the following three regimes. Firstly, the fixed-point case where plant and controller both assume a common constant value of Λ_i for $\Lambda(t)$ will be examined. Secondly, we consider the transition case when the parameter is constrained only to assumed values in the set Λ_{oper} irrespective of the switch rate, but so that the controller is always 'tuned' to the plant parameter (apart from the time-variation arising in the observer gain part of the controller). The switching may be frequent. Lastly, we consider the parameter-varying case when the plant and controller do not assume in general the same value for $\Lambda(t)$ within the same compact subset Ξ_i at other than isolated times. The plant parameter is $\Lambda(t)$, and the controller parameter is Λ_i for some i, albeit with the time-varying observer gain depending on the past history of the set of $\Lambda_j \in \Lambda_{\text{oper}}$ encountered.

4.1 Fixed-point Analysis

Consider the system of interest (12) and the proposed controller (13) with a constant system parameter $\Lambda(t)$, say Λ_i, for all t. Then the plant and controller will be time-invariant: the controller can be designed via normal LQG methods (albeit using LTR) and the closed-loop system will be stable. Instead of the observer gain $H(t)$ (14) within the controller being determined using the solution of a RDE (7), it will be a constant as $H_i := (R_i)^{-1}\bar{\Sigma}_i C'_i,\ \bar{\Sigma}_i \in \bar{\Sigma}_{\text{oper}}$.

4.2 Finite Parameter Set Analysis

Suppose that the underlying parameter value $\Lambda(t)$ switches among values drawn from the set Λ_{oper}. The state feedback gain incorporated in the controller will switch instantaneously between corresponding values. The associated dynamics will be considerably faster than the average rate of parameter variation and hence, we only need to focus on the stability of the observer dynamics.

Lemma 7. Let c be defined as in Section 3.2, let $\mathcal{A}(t) = A_{c(t)}$ for all t, and let $\mathcal{B}(t) = B_{c(t)}$ for all t. Then, $(\mathcal{A}(t), \mathcal{B}(t))$ is uniformly controllable.

Proof. Let $\delta_c > 0$ be defined as in (20), with the consequence that $t_{j+1} - t_j \geq \delta_c$ for all $j \in \mathbb{Z}^+$. Now consider an arbitrary time interval of length δ_c, $[s - \delta_c, s)$. We need to show that $\exists\ \alpha_c,\ \beta_c > 0$ such that (5) holds $\forall\ s \in \mathbb{R}$. It follows from $0 < \delta_c < \infty$, and $\mathcal{A}$, $\mathcal{B}$ uniformly bounded, that $\exists\ \beta_c > 0$ such that $M(s - \delta_c, s) \leq \beta_c I,\ \forall\ s \in \mathbb{R}$. Since (A_i, B_i) is controllable, if we consider a time interval that does not contain a switching time (that is, for s, t such that $\exists\, j \in \mathbb{Z}^+$ for which

$t_j \leq s < t < t_{j+1}$, with $c(t_j) = i$) and if $\frac{\delta_c}{2} \leq t - s \leq \delta_c$, then $\exists\ \alpha_i > 0$ such that $M(s,t) \geq \alpha_i I$. Let $\tilde{\alpha}_c = \min_i \alpha_i$.

An interval of the form $[s - \delta_c, s)$ will intersect either one or two intervals of the form $[t_j, t_{j+1})$. If one, then we have already established $\tilde{\alpha}_c I \ \leq \ M(s - \delta_c, s)$. If two, then we can express the gramian as

$$\begin{aligned} M(s-\delta_c, s) & \\ &= \int_{s-\delta_c}^{s} \Phi(s,t)\mathcal{B}(t)\mathcal{B}'(t)\Phi'(s,t)dt \\ &= \int_{s-\delta_c}^{t_j} \Phi(s,t)B_{c(t_{j-1})}B'_{c(t_{j-1})}\Phi'(s,t)dt \\ &\qquad + \int_{t_j}^{s} \Phi(s,t)B_{c(t_j)}B'_{c(t_j)}\Phi'(s,t)dt \\ &= \mathrm{e}^{[A_{c(t_j)}(s-t_j)]}M(s-\delta_c, t_j)\mathrm{e}^{[A'_{c(t_j)}(s-t_j)]} + M(t_j, s). \end{aligned}$$

Noting that either $s - t_j \geq \frac{\delta_c}{2}$ or $t_j - (s - \delta_c) \geq \frac{\delta_c}{2}$, we then have $\forall\ s \in \mathbb{R},\ \alpha_c I \ \leq \ M(s - \delta_c, s)$, where

$$\alpha_c = \tilde{\alpha}_c \cdot \min\{1, \min_{i, 0 \leq w \leq \frac{\delta_c}{2}} \lambda_{\min}[\mathrm{e}^{A_i w}\mathrm{e}^{A'_i w}]\} \tag{21}$$

and $\lambda_{\min}[\chi]$ for a symmetric χ denotes the minimum eigenvalue. It follows that $(\mathcal{A}(t), \mathcal{B}(t))$ is uniformly controllable. $\square$

Similarly, the dual result for the uniform observability is also true. Because the given system is now proved to be uniformly controllable and uniformly observable, the dynamics of the observer are stable. Hence, by the separation principle, the overall system is stable.

4.3 Continuously Varying Parameter Analysis

Suppose that the parameter $\Lambda(t)$ is continuously varying. The controller however is built using just the knowledge of the compact set, say $\Xi_i \in \Xi_{\text{oper}}$ at each time, in which $\Lambda(t)$ lies. At any instant of time, this set determines a nominal parameter value in Λ_{oper}, say Λ_i, which is used to set the state-feedback gain part of the controller. The observer gain is however determined from the transient Riccati equation, the solution of which depends on past values as well as the current value of Λ_i.

We shall now write down equations for the closed-loop system comprising plant plus controller, which are presented to highlight the fact that the controller is connected to a plant which is close to, but not identical to that which it would (in accordance with section 4.2) clearly stabilise. The plant with varying $\Lambda(t)$ is close to a plant with piecewise constant $\Lambda_{c(t)}$, of the sort treated in section 4.2. We have already established the stability of such an interconnection. We can then appeal to classical results in Lyapunov theory on the robust stability of systems undergoing a perturbation. By rearranging (12) with $\Lambda(t)$ and (13) with $G(\Lambda_{c(t)})$ (15) and $H(t)$ (14), the full observer-plant arrangement with $r(t) = 0$ can be given as follows:

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} A_{\text{cont}}(\Lambda_{c(t)}) & B_{c(t)}G(\Lambda_{c(t)}) \\ 0 & A_{c(t)} - H(t)C_{c(t)} \end{bmatrix} X(t) + R(X,t), \\ y(t) &= [C_{c(t)} \ \ 0]\ X(t), \end{aligned} \tag{22}$$

where $e(t) = x(t) - \hat{x}(t)$, $X(t) := [x(t) \ \ e(t)]'$, $c(t)$ denotes the index function defined in Section 3.2 and $A_{\text{cont}}(\kappa)$ denotes the control system matrix (18) and

$$R(X,t) := \begin{bmatrix} \delta A(t) - \delta B(t)G(\Lambda_{c(t)}) & 0 \\ \delta A(t) - \delta B(t)G(\Lambda_{c(t)}) - H(t)\delta C(t) & \delta B(t)G(\Lambda_{c(t)}) \end{bmatrix} X(t)$$

reflects the difference between the constant model with $\Lambda_{c(t)}$ and the true plant parameter $\Lambda(t)$ with $\delta A(t) := A(\Lambda(t)) - A_{c(t)}$, $\delta B(t) := B(\Lambda(t)) - B_{c(t)}$, $\delta C(t) := C(\Lambda(t)) - C_{c(t)}$.

In fact, (22) can be classified as a perturbed system and there exist several standard stability analysis techniques for this class of system (Krasovskii, 1963; Hahn, 1963). In particular, we use a theorem from Krasovskii (1963) where the perturbed system is analysed with an application of Lyapunov's methods, and which we now restate:

Theorem 8. (Krasovskii, 1963) Consider an auxiliary system

$$\dot{X}(t) \ = \ f(X,t) \tag{23}$$

and a perturbed system

$$\dot{X}(t) \ = \ f(X,t) + R(X,t). \tag{24}$$

If there exist $\mathfrak{a}, B > 0$ such that solutions of the auxiliary system (23) satisfy

$$\|X(t)\|_2 \ \leq \ B\|X_0\|_2 \exp[-\mathfrak{a}(t-t_0)] \quad \forall\ t \geq t_0 \tag{25}$$

for all initial conditions t_0, X_0, or equivalently, if there exist $c_1, c_2, c_3, c_4 > 0$ and a function $v(X,t)$ such that solutions of the auxiliary system (23) satisfy

$$c_1\|X\|_2^2 \ \leq \ v(X,t) \ \leq \ c_2\|X\|_2^2, \tag{26}$$

$$\dot{v} \ \leq \ -c_3\|X\|_2^2, \tag{27}$$

$$\left\|\frac{\partial v}{\partial X}\right\|_\infty \ \leq \ c_4\|X\|_2, \tag{28}$$

and if there further exists $q \in (0,1)$ such that

$$\|R(X,t)\|_\infty \ < \ \frac{(1-q)c_3\|X\|_2}{c_4} \tag{29}$$

then another set of positive constants exist such that solutions of the perturbed system (24) satisfy (25), or equivalently, (26)-(28).

We now use this result to prove exponential asymptotic stability of our full system (22). We already know that the closed-loop dynamics of the fixed-point system with a $\Lambda_i \in \Lambda_{\text{oper}}$ are exponentially asymptotically stable, and thus, that we can find appropriate constants to satisfy (25), and equivalently, that there exist appropriate constants and a Lyapunov function $v(X,t)$ that satisfy (26)-(28). Now note that it follows from (19) that

$$\sup_t \|\Lambda(t) - \Lambda_{c(t)}\|_\infty \leq \max_i \ \text{diam}(\Xi_i), \tag{30}$$

where $\text{diam}(S) = \sup_{A,B \in S}\|A - B\|_\infty$. It is then clear that given a fixed bound, we can choose a number of regions N large enough so that we can then place the associated regions Ξ_i such that the above sup is within that given bound. Given the uniformly continuous dependence of the entries of $R(X,t)$ on the parameter $\Lambda(t)$, it then further follows that given a $q \in (0,1)$, we can similarly choose large enough N and associated regions Ξ_i such that (29) is satisfied. By choosing such a large enough number of regions N, Theorem 8 then provides the exponential asymptotic stability of the closed-loop observer dynamics.

5. NUMERICAL EXAMPLE

Let us consider the same numerical example illustrated in Shamma and Athans (1992). The system is given as

$$\dot{x}(t) = A(\Lambda(t))x(t) + Bu(t); \quad y(t) = Cx(t), \tag{31}$$

with $x(t_0) = [0\ 0\ 0\ 0\ 0]'$ and

$$A(\Theta) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1\cos\Theta - 1 & 1 & 0 \\ 0 & 100 & 0 & 0 & 0 \\ 0 & -100 & 0 & 0 & 1 \\ 0 & 0 & 10\cos\Theta & 0 & 0 \end{bmatrix} \tag{32}$$

and $B = [0\ 0\ 0\ 0\ 1]'$, $C = [1\ 0\ 0\ 0\ 0]$, where $-1 \leq \Lambda(t) \leq 1 \in \mathbb{R}$. To observe clear switchings with less computations, N has been chosen to be 4. Also, to satisfy (29), the four *equally-spaced* operating points are chosen as $\{-1, -1/3, 1/3, 1\}$ to form Λ_{oper}. In fact, as long as (29) is satisfied, N and Ξ_{oper} can be arbitrarily chosen. The used design parameters are $\Gamma = 10^{-14}$, $R = 10^{-8}$ and $L = [0.011426\ \ 0.044311\ \ 0.388490\ \ -0.062159\ \ 0.918510]'$ as given in Shamma and Athans (1992). Given the plant time constants $\tau = .01$, the time-varying parameter has been chosen with relatively fast variations as $\Lambda(t) = \sin(3t)$ to compare the proposed design against the Shamma and Athans, which is, in fact, faster than the one used in Shamma and Athans (1992). The step responses (a step at $t = 0.3$s) for the solving RDE directly (Shamma and Athans' case) with $\Lambda(t)$ and our new design with $\Lambda_{c(t)}$ are shown in Figure 1 below:

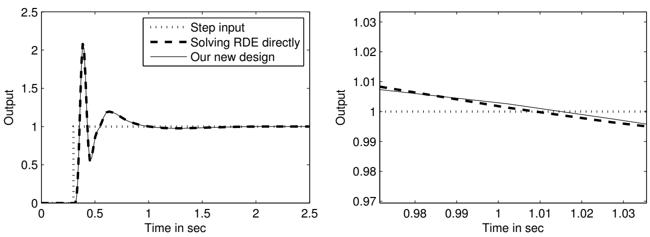

Fig. 1. The step responses for existing and new designs

For the first 2.5-seconds (Fig. 1:left), both outputs behave hardly distinguishably with an overshoot just after the step and the exponential convergence as $t \to \infty$. From the zoomed figure (Fig. 1:right), there is actually a slight deviation between two design outputs, but the difference is negligible and our new design still satisfies (29); hence the stability of the closed loop dynamics is not affected. As N is increased with any equally-spaced operating points, more computation was required, but the same performance and stability property is observed for the given $\Lambda(t)$. Also, from several other simulations with various mixtures of slow and fast time-varying physical parameters, results from the new design have been observed to have the acceptable stability and no significant performance degradation, compared to the Shamma and Athans' controller, which solves the RDE directly.

6. DISCUSSION

In this paper, we have indicated a substantial modification of the gain scheduling design of Shamma and Athans (1992). Their design offers the rather rare advantage within the set of approaches to gain scheduled design of permitting fast (though not infinitely fast) parameter variation. However, this existing approach demands an extreme computational power as it requires an ARE to be solved at each operating point and a RDE to be solved online. Our modification addresses this computational burden. Only a finite number of AREs is needed to be solved offline, and the RDE solution can be formed using table look-up together with simple matrix operations. The procedures depend on the controlled plant being minimum-phase, so that a LQG/LTR controller can be used for each fixed parameter value.As shown in simulations, the same performance can be achieved with what might in advance be conjectured as quite a rough approximation.

Various open questions remain. For example, could one envisage a design guaranteeing ability to handle fast parameter variations in the event that the plant is not minimum phase? Or is the minimum phase property intrinsic to the ability to do fast gain scheduling, no matter what algorithm is used? What are the possible extensions to nonlinear plants? Could H_∞ designs replace the LQG based designs for local controllers?

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