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Stability Analysis and Design for Polynomial Nonlinear Systems Using SOS with Application to Aircraft Flight Control

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Abstract: In this paper, aircraft with control surface impairment faults is modelled as a set of affine parameter-dependent nonlinear systems with polynomial vector fields. Based on the Lyapunov stability theorem, sufficient conditions to test the stability of the closed-loop system are presented. The synthesis problem of stabilizing feedback controllers to enlarge the region of attraction is converted to an optimization problem based on sum-of-squares polynomial. Design and simulation results for the longitudinal model of an F-8 aircraft are presented to illustrate the effectiveness of proposed approach.

1. INTRODUCTION

The stability analysis and feedback controller design for nonlinear systems have received considerable attention over the decades and many important advances in nonlinear design have been made [1, 8]. However, a general methodology for the construction of stabilizing nonlinear control laws remains a challenging task. The main difficulty lies in that the construction of Lyapunov functions and the test for non-negativeness of Lyapunov stability conditions is in fact a NP-hard problem [2]. However, a new computationally tractable nonlinear analysis methodology based on the sum of squares decomposition and semidefinite programming was proposed recently [2]. The advantage of this methodology is that it provides a computational relaxation: instead of checking for the nonnegativeness of a polynomial, the polynomial is only required to satisfy the condition that it is a sum of squares of polynomials.

The sum of squares based methodology has been successfully applied to several control theory problems. These problems include stability analysis for nonlinear systems, time-delay systems, switched and hybrid systems [13, 10, 11], and robust stability analysis [2, 13, 7]. In [3] regions of attraction and attractive invariant sets for nonlinear systems with polynomial vector fields was estimated. In [6] state feedback design for linear parameter-varying (LPV) systems was proposed. Moreover the nonlinear synthesis was also discussed in [5, 12, 14].

The application of sum of squares based method to the aircraft flight control was also proposed in [9], where sum of squares programming approach was used to analyze the stability and robustness properties of the controlled pitch axis of a nonlinear model of an A/C.

In this paper, parameter-dependent nonlinear systems with polynomial vector fields are used to model an F-8 aircraft with control surface impairment faults. The stability analysis and state feedback controller design problem is considered based on the Lyapunov stability results and sufficient conditions for the stability of the closed-loop system are presented. A fixed state feedback polynomial controller is synthesized to realize the system's stability and enlarge the region of attraction for the nonlinear aircraft model at the same time. Robust stability can be guaranteed for the nonlinear aircraft model in normal operation and in the event of control surface faults. In order to reduce the conservativeness involved in the controller design, a parameter-dependent Lyapunov function (instead of a fixed one) is used. Finally, the proposed stability analysis and feedback controller synthesis problem is converted to a sum-of-squares based optimization problem, which can be solved via semidefinite programming with the new software SOSTOOLS [13].

The remainder of the paper is organized as follows: Section 2 provides preliminary material on multivariate polynomial and some related mathematical concepts. Section 3 gives the F-8 aircraft longitudinal flight dynamics with control surface fault. In Section 4, the state feedback design approach based on the sum of squares decomposition and semidefinite programming are presented. Flight control example is provided in Section 5. Finally, the concluding remarks are given in Section 6.

2. PRELIMINARIES

Let $\mathbb{R}$ denote the set of the real numbers, $\mathbb{R}_+ := [0, \infty) \subset \mathbb{R}$. $\mathbb{R}^n$ is the n-dimensional real space. $\mathbb{Z}_+$ denotes the set of nonnegative integers.

2.1 Polynomial Definitions

We define $\mathcal{R}_n$ to be the set of all polynomials in n variables with real coefficients. A polynomial vector field $f : \mathbb{R}^n \to \mathbb{R}^n, f(x) = [f_1(x), \ldots, f_n(x)]^T$ is a vector field with $f_i \in \mathcal{R}_n$, i.e., the entries of the vector field are polynomial functions in $x \in \mathbb{R}^n$, and is denoted as $f(x) \in \mathcal{R}_n^n$.

Definition 1 (Sum of Squares Polynomial):A polynomial $p(x)$ in n variables is a sum of squares (SOS) polynomial if there exist polynomials $f_i(x)$, $i = 1, \ldots, m$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x) \tag{1}$$

We define $\mathbf{\Sigma}_n$ to be the set of sum of squares polynomials in n variables, which is a very important subset of the set of all polynomials $\mathcal{R}_n$. Obviously if $p \in \mathbf{\Sigma}_n$, then $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$.

Another subset of $\mathcal{R}_n$ is the set of positive definite polynomials.

$$\mathcal{P}_n := \{p \in \mathcal{R}_n | p(x) > 0, \forall x \in \mathbb{R}^n \backslash \{0\}\} \tag{2}$$

It should be noted that there are positive definite polynomials that are not sum of squares polynomials, i.e., $\mathcal{P}_n \nsubseteq \mathbf{\Sigma}_n$ [2].

2.2 The Positivstellensatz

A central theorem from real algebraic geometry, the Positivstellensatz, is a powerful theorem which generalizes many known results, for example, the $\mathcal{S}$-procedure [2].

Definition 2: Given $\{g_1, \ldots, g_t\} \in \mathcal{R}_n$, the **Multiplicative Monoid** generated by g_j's is the set of all finite products of g_j's, including the empty product, which is defined to be 1. It is denoted as $\mathcal{M}(g_1, \ldots, g_t)$.

Definition 3: Given $\{f_1, \ldots, f_s\} \in \mathcal{R}_n$, the **Cone** generated by f_i's is

$$\mathcal{P}(f_1, \ldots, f_s) := \\ \{s_0 + \sum s_i b_i | s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \ldots, f_s)\} \tag{3}$$

Definition 4: Given $\{h_1, \ldots, h_u\} \in \mathcal{R}_n$. the **Ideal** generated by h_k's is

$$\mathcal{I}(h_1, \ldots, h_u) := \{ \ \sum h_k p_k \mid p_k \in \mathcal{R}_n \ \} \tag{4}$$

With these elements, the Positivstellensatz for the reals can be formulated.

Theorem 1 (Positivstellensatz) [4]: Let $(f_i)_{i=1,\ldots,s}$, $(g_j)_{j=1,\ldots,t}$, and $(h_k)_{k=1,\ldots,u}$ be finite families of polynomials in $\mathcal{R}_n$. Denote by $\mathcal{P}(f_1, \ldots, f_s)$ the cone generated by $(f_i)_{i=1,\ldots,s}$, $\mathcal{M}(g_1, \ldots, g_t)$ the multiplicative monoid generated by $(g_j)_{j=1,\ldots,t}$, and $\mathcal{I}(h_1, \ldots, h_u)$ the ideal generated by $(h_k)_{k=1,\ldots,u}$. Then the following are equivalent:

1. The set

$$\left\{ x \in \mathbb{R}^n \ \middle| \ \begin{array}{l} f_i(x) \geq 0, i = 1, \ldots, s, \\ g_j(x) \neq 0, j = 1, \ldots, t, \\ h_k(x) = 0, k = 1, \ldots, u \end{array} \right\} \tag{5}$$

is empty,

2. There exist polynomials $f \in \mathcal{P}(f_1, \ldots, f_s)$, $g \in \mathcal{M}(g_1, \ldots, g_t)$, $h \in \mathcal{I}(h_1, \ldots, h_u)$ such that

$$f + g^2 + h = 0 \tag{6}$$

3. THE F-8 AIRCRAFT DYNAMICAL MODEL

The candidate nonlinear aircraft model for this study is the F-8 aircraft longitudinal flight dynamics which consists of both phugoid and short period modes. Ignoring drag, the basic nonlinear equations describing the longitudinal flight dynamics are used [15]

$$\dot{u} = -uq \tan\alpha - g \sin\theta + \frac{L_w}{m} \sin\alpha + \frac{L_t}{m} \sin\alpha_t \tag{7}$$

$$\dot{\alpha} = q + \frac{g}{u} \cos\alpha \cos(\alpha - \theta) - \frac{L_w}{um} \cos\alpha \\ - \frac{L_t}{um} \cos\alpha \cos(\alpha - \alpha_t) \tag{8}$$

$$\dot{\theta} = q \tag{9}$$

$$\dot{q} = (M_w + l L_w \cos\alpha - l_t L_t \cos\alpha_t - cq)/I_y \tag{10}$$

where

$$\alpha_t = (1 - a_\epsilon)\alpha + \delta_e$$
$$L_w = C_L(\alpha)\bar{q}S$$
$$L_t = C_{L_t}(\alpha_t, \delta_e)\bar{q}S_t$$
$$\bar{q} = \frac{\rho u^2}{2\cos^2\alpha}$$

Cubic approximation is used for the lift coefficient cruves

$$C_L(\alpha) = (C_L^1 \alpha - C_L^2 \alpha^3)$$
$$C_{L_t}(\alpha_t, \delta_e) = (C_L^1 \alpha_t - C_L^2 \alpha_t^3 + a_e \delta_e)$$

Consider an altitude of $30,000ft$ (i.e., with $\rho = 0.00089$ $slug/ft^3$ and a speed of sound of 994.85 ft/s), and a level unaccelerated flight at $Mach = 0.85$. The system coefficients for the aircraft model in normal operation are taken as follows:

$$C_L^1 = 4.0, \quad C_L^2 = 12, \quad a_e = 0.1, \quad a_\epsilon = 0.75,$$
$$S = 375ft^2, \quad S_t = 93.4ft^2, \quad m = 667.7slugs,$$
$$I_y = 96800slug\ ft^2, \quad l = 0.189ft, \quad l_t = 16.7ft,$$
$$M_w = 0\ lb\ ft, \quad c = 38332.8\ lb\ ft\ s, \quad g = 32.2ft/s^2$$

With C_L^1 and C_L^2 given as above, the stall angle of attack at the wing and at the tail can be calculated easily to be about 19.1 deg (1/3 rad). An elevator defection limit of 25 deg and elevator rate limit of 100 deg/s are applied.

Control surface faults are commonly seen in fighter aircraft. The usual control surface fault is the control surface impairment which will change the aerodynamic characteristics of the aircraft. Control surface impairment can be characterized by the percentage loss of the total control surface area.

With the system coefficients, the trim conditions for the aircraft models in normal operation and in the operation with 25% loss of control surface can be calculated as follows respectively:

Table 1. Operating points

	Nominal	25% loss
$u_{trim}(ft/s)$	845	845
$q_{trim}(rad/s)$	0	0
$\alpha_{trim}(rad)$	0.044786190	0.044786176
$\theta_{trim}(rad)$	0.044311	0.044318
$\delta_{e_{trim}}(rad)$	-0.0089519	-0.0082947

Substituting the new variables $x_1 = \alpha - \alpha_{trim}$, $x_2 = \theta - \theta_{trim}$, $x_3 = q$ as the states, $u_c = \delta_e - \delta_{e_{trim}}$ as the control input and airspeed $u =$ const into (7)-(10), and using the system coefficients, we obtain the following shot period aircraft models in normal operation and in operation with 25% loss of control surface:

Aircraft model in normal operation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.878x_1 + x_3 - x_1^2 x_3 - 0.0896x_1x_3 \\ x_3 \\ -4.209x_1 - 0.396x_3 - 0.408x_1^2 \end{bmatrix.$$

$$\begin{matrix} -0.019x_2^2 + 0.473x_1^2 + 3.813x_1^3 \\ +0 \\ -2.137x_1^3 \end{matrix}\Bigg] + \begin{bmatrix} -0.216 \\ 0 \\ -20.991 \end{bmatrix} u_c$$

$$\triangleq f_1(x) + g_1(x)u_c \tag{11}$$

Aircraft model with 25% loss of control surface:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.865x_1 + x_3 - x_1^2 x_3 - 0.0896x_1x_3 \\ x_3 \\ -2.929x_1 - 0.396x_3 - 0.409x_1^2 \end{bmatrix.$$

$$\begin{matrix} -0.019x_2^2 + 0.473x_1^2 + 3.81x_1^3 \\ +0 \\ -2.417x_1^3 \end{matrix}\Bigg] + \begin{bmatrix} -0.162 \\ 0 \\ -15.742 \end{bmatrix} u_c$$

$$\triangleq f_2(x) + g_2(x)u_c \tag{12}$$

For loss of control surface between 0% and 25%, interpolation of the above $f_1(x)$ and $f_2(x)$, and $g_1(x)$ and $g_2(x)$ can be used

$$f(x;\vartheta) = f_1(x)\vartheta_1 + f_2(x)\vartheta_2 \tag{13}$$

$$g(x;\vartheta) = g_1(x)\vartheta_1 + g_2(x)\vartheta_2 \tag{14}$$

where $f_1(x), f_2(x) \in \mathcal{R}_3^3$ and $g_1(x), g_2(x) \in \mathcal{R}_3^{3\times 1}$ are as in (11) and (12), with $f_1(0) = 0$ and $f_2(0) = 0$, which represent the vertices of possible control surface impairment for aircraft model. The parameter $\vartheta = [\vartheta_1\ \vartheta_2]^T \in \mathbb{R}^2$, which provides the interpolation between the two vertices, is constant and satisfies

$$\vartheta \in \mathbf{\Theta} \triangleq \{\vartheta \in \mathbb{R}^2 : \vartheta_1 \geqslant 0, \vartheta_2 \geqslant 0 \text{ and } \vartheta_1 + \vartheta_2 = 1\} \tag{15}$$

Then, we have the following nonlinear polytopic model for aircraft with control surface impairment faults between 0% and 25%.

$$\dot{x} = f(x;\vartheta) + g(x;\vartheta)u_c \tag{16}$$

If we allow the control input to be generated by a polynomial state feedback controller

$$u_c = k(x) \in \mathcal{R}_3^1 \qquad \text{with } k(0) = 0 \tag{17}$$

the corresponding closed-loop system is described by the following stat-space equations

$$\dot{x} = f(x;\vartheta) + g(x;\vartheta)k(x) \tag{18}$$

4. STABILITY ANALYSIS AND STATE FEEDBACK CONTROLLER DESIGN

In order to reduce the conservativeness the following affine parameter-dependent Lyapunov function (instead of a fixed Lyapunov function) is adopted.

$$V(x;\vartheta) = V_1(x)\vartheta_1 + V_2(x)\vartheta_2,\ \vartheta \in \mathbf{\Theta} \tag{19}$$

Stabilization Problem[5]: Define two regions for a given parameter-dependent system of the form (18).

$$\mathcal{O}_\beta := \{x \in \mathbb{R}^n | p(x) \leq \beta\} \tag{20}$$

where $\beta > 0$ is the "radius" of $\mathcal{O}_\beta$, and $p(x) \in \mathcal{P}_n$ is known positive definite polynomial, independent of ϑ.

$$\mathbf{\Omega} := \{x \in \mathbb{R}^n | V(x;\vartheta) \leq 1, \forall\vartheta \in \mathbf{\Theta}\} \tag{21}$$

with $V(x;\vartheta)$ being an unknown Lyapunov function of the form (19).

Our objective is to search for a parameter-dependent Lyapunov function (19) and a polynomial state feedback controller (17) to maximize β such that,

$$V(x;\vartheta) > 0, \forall\ x \in \mathbb{R}^n \backslash \{0\} \text{ and } V(0) = 0, \forall\vartheta \in \mathbf{\Theta} \tag{22}$$

$$\mathcal{O}_\beta \subseteq \mathbf{\Omega} \tag{23}$$

$$\mathbf{\Omega}\backslash\{0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x;\vartheta) < 0, \forall\vartheta \in \mathbf{\Theta}\} \tag{24}$$

From the Lyapunov argument in [5], for each fixed $\bar{\vartheta} \in \mathbf{\Theta}$, the region $\mathbf{\Omega}_{\bar{\vartheta}} := \{x \in \mathbb{R}^n | V(x;\bar{\vartheta}) \leq 1\}$ is an invariant set and also a subset of region of attraction for the system (18) with that particular $\bar{\vartheta}$. Hence $\mathbf{\Omega} := \bigcap_{\bar{\vartheta}\in\mathbf{\Theta}} \mathbf{\Omega}_{\bar{\vartheta}}$ is a subset of region of attraction for all $\vartheta \in \mathbf{\Theta}$, and every point in the region $\mathcal{O}_\beta$ converges asymptotically to the origin point.

Lemma 1[5] Consider a parameter-dependent system (18) and a fixed positive definite function $p(x) \in \mathcal{P}_n$. If there exist positive definite polynomials $V_i(x) \in \mathcal{P}_n$ with $V_i(0) = 0$ for $i = 1, 2$, and $k(x) \in \mathcal{R}_n^m$ with $k(0) = 0$ such that

$$V_i(x) > 0, \forall\ x \in \mathbb{R}^n \backslash \{0\} \text{ and } V_i(0) = 0 \tag{25}$$

$$\{x \in \mathbb{R}^n | p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n | V_i(x) \leq 1\} \tag{26}$$

$$\{x \in \mathbb{R}^n | V_i(x) \leq 1\}\backslash\{0\} \subseteq \{x \in \mathbb{R}^n | M_1 < 0\} \tag{27}$$

$$\{x \in \mathbb{R}^n | V_i(x) \leq 1\}\backslash\{0\} \subseteq \{x \in \mathbb{R}^n | M_2 < 0\} \tag{28}$$

$$\{x \in \mathbb{R}^n | V_i(x) \leq 1\}\backslash\{0\} \subseteq \{x \in \mathbb{R}^n | M_3 < 0\} \tag{29}$$

where

$$M_1 = \nabla V_1 [f_1 + g_1 k] \tag{30}$$

$$M_2 = \nabla V_2 [f_2 + g_2 k] \tag{31}$$

$$M_3 = \nabla V_1 [f_2 + g_2 k] + \nabla V_2 [f_1 + g_1 k] \tag{32}$$

Then, the conditions (22)-(24) are satisfied. Hence, for all $\vartheta \in \mathbf{\Theta}$, the region $\mathbf{\Omega}$ is a subset of region of attraction for the system (18), and system (18) is asymptotically stable about the origin.

The proof of Lemma 1 can be referred to [5], it is omitted here.

With the Positivstellensatz and following the simplifying procedure given in [5], the stabilization problem can be formulated as the following optimization problem which can be solved via the semidefinite programming.

Sum of Squares (SOS) Problem [5]:

$$\max\ \beta \quad \text{over} \quad k(x) \in \mathcal{R}_n, V_1(x), V_2(x) \in \mathcal{P}_n \\ \text{and } V_1(0) = V_2(0) = 0, s_i(x) \in \mathbf{\Sigma}_n$$

subject to

$$V_1 - l_1 \text{ is SOS} \tag{33}$$
$$V_2 - l_2 \text{ is SOS} \tag{34}$$
$$-[s_1(\beta - p) + (V_1 - 1)] \text{ is SOS} \tag{35}$$
$$-[s_2(\beta - p) + (V_2 - 1)] \text{ is SOS} \tag{36}$$
$$-[s_3(1 - V_1) + s_9 M_1 + l_3] \text{ is SOS} \tag{37}$$
$$-[s_4(1 - V_1) + s_{10} M_2 + l_4] \text{ is SOS} \tag{38}$$
$$-[s_5(1 - V_1) + s_{11} M_3 + l_5] \text{ is SOS} \tag{39}$$
$$-[s_6(1 - V_2) + s_{12} M_1 + l_6] \text{ is SOS} \tag{40}$$
$$-[s_7(1 - V_2) + s_{13} M_2 + l_7] \text{ is SOS} \tag{41}$$
$$-[s_8(1 - V_2) + s_{14} M_3 + l_8] \text{ is SOS} \tag{42}$$

where M_1, M_2 and M_3 are as in (30)-(32), $l_j(x) \in \mathcal{P}_n \bigcap \mathbf{\Sigma}_n$ (i.e., positive definite in SOS form), $j = 1, \ldots, 8$, $l_j(0) = 0$ and $s_i(x) \in \mathbf{\Sigma}_n$, $i = 1, \ldots, 14$.

5. AIRCRAFT APPLICATION EXAMPLE

In this section, a design example for nonlinear F-8 aircraft model in Section 3 is presented to demonstrate the proposed controller design approach.

Finding the Lyapunov functions $V_1(x), V_2(x)$ and state feedback controller $k(x)$ that satisfy (33)-(42) is a nonconvex problem because of the nonlinear conditions in the constraints. An iterative algorithm [5] is used to solve the nonconvex problem. The stability constraints (33)-(42) are checked by SOS programming.

We supply feasible initial Lyapunov functions $V_1(x), V_2(x)$ and controller $k(x)$ in (43)-(45) over which the iterative algorithm improves the value of β to find optimal solutions.

$$V_1(x) = 0.14x_1^2 + 0.3286x_2^2 + 0.0016x_3^2 - 0.1538x_1x_2 \\ -0.0022x_1x_3 + 0.0048x_2x_3 \tag{43}$$

$$V_2(x) = 0.14x_1^2 + 0.3286x_2^2 + 0.0016x_3^2 - 0.1538x_1x_2 \\ -0.0022x_1x_3 + 0.0048x_2x_3 \tag{44}$$

$$k(x) = 1.7042x_1 + 7.4162x_2 + 7.4365x_3 \tag{45}$$

The regions of attraction are estimated by the variable sized region $\mathcal{O}_\beta := \{x \in \mathbb{R}^2 | p(x) \leq \beta\}$, where $p(x) = x^T P x, P = [4,\ -0.1,\ 0.03; -0.1,\ 0.59,\ 0; 0.03,\ 0,\ 0.05]$. The fixed positive definite polynomials $\{l_j(x)\}_{j=1}^8$ are chosen in the following form of $\varepsilon \Sigma_{i=1}^3 {x_i}^d$ with some small constant $\varepsilon > 0$ and d is the maximum degree of the corresponding polynomials $l_j(x)$.

By solving the SOS optimization problem, the maximum size of the region $\mathcal{O}_\beta$ is determined with the value of $\beta_1 = 1.7662$. The state feedback controller that guarantees the local stability with optimized ROA is

$$k_{nl1}(x) = -0.60357x_1 + 0.51918x_2 + 2.6414x_3 + 1.1853x_1^2 \\ -0.0039012x_2^2 + 0.15242x_3^2 - 0.2542x_1x_2 - 0.58084x_1x_3 \\ +0.15914x_2x_3 + 8.0966x_1^3 + 0.07795x_2^3 + 0.25808x_3^3 \\ -1.9417x_1^2x_2 - 4.2947x_1^2x_3 + 0.5879x_1x_2^2 + 0.34048x_1x_3^2 \\ +0.077199x_2^2x_3 - 0.13242x_2x_3^2 + 0.69357x_1x_2x_3 \tag{46}$$

and the corresponding Lyapunov functions are

$$V_{nl1}(x) = 2.2536x_1^2 + 0.26504x_2^2 + 0.01307x_3^2 \\ -0.058304x_1x_2 + 0.02543x_1x_3 + 0.01389x_2x_3 \tag{47}$$

$$V_{nl2}(x) = 2.2456x_1^2 + 0.28935x_2^2 + 0.017715x_3^2 \\ -0.16074x_1x_2 + 0.059581x_1x_3 + 0.021701x_2x_3 \tag{48}$$

The estimated regions of attraction $\{x \in \mathbb{R}^2 | V(x; \vartheta) \leq 1\}$ for various ϑ as computed using SOS optimization are shown in Fig. 1. Note that $\{x \in \mathbb{R}^2 | p(x) \leqslant \beta\}$ is contained in these regions of attraction.

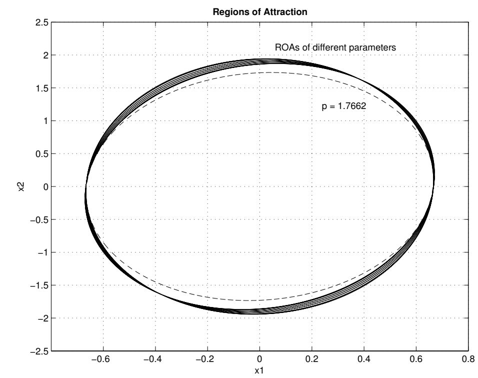

Fig. 1. Regions of attraction of the closed-loop systems with nonlinear feedback controller (46)

A plot of the response of the closed-loop F-8 aircraft model (18) with initial condition $x(0) = [0.5,\ 0.5,\ 0.5]^T$, which is within the estimated stability region, is shown in Fig. 2. In the simulation the uncertain parameter vector ϑ is assumed to be fixed as $[0.5,\ 0.5]^T$. In order to test the validity of the computed attraction regions, we also initialize the states of the nonlinear closed-loop F-8 aircraft model outside the estimated region of attraction. The simulation result is shown in Fig. 3. The initial conditions are simultaneously perturbed in all states. From Figs. 2 and 3, we can see that the nonlinear feedback controller (46) can stabilize the nonlinear aircraft model, even when the initial states are outside the estimated region of attraction by SOS optimization. Similar simulation results are obtained when the uncertain parameter vector ϑ is fixed as other values. The apparent conservativeness of the estimated region of attraction is due to the fact that Lemma 1 is only a sufficient condition, and that we are fitting a predetermined shape $p(x)$ to the ROA.

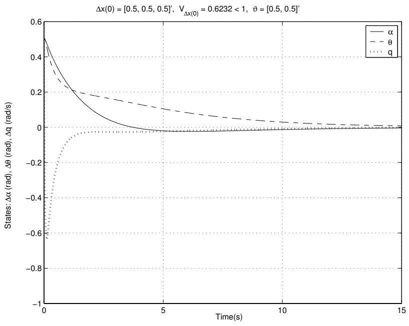

Fig. 2. Response of closed-loop system with states initialized within the estimated stability region, $\vartheta = [0.5,\ 0.5]^T$

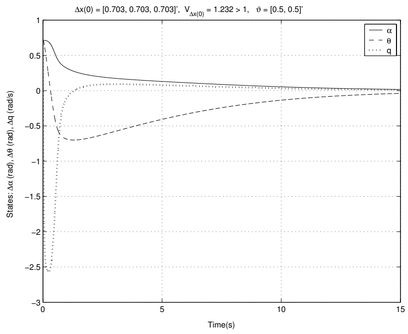

Fig. 3. Response of closed-loop system with states initialized outside the estimated stability region, $\vartheta = [0.5,\ 0.5]^T$

The closed-loop system responses of the nonlinear controller (46) derived from SOS optimization are compared with that of the third order controller in [16]. The responses of the states α, θ and q are shown in Fig. 4 for the third order controller in [16], and in Fig. 5-6 for the proposed nonlinear controller (46). Here the initial conditions for angle of attack are set to the largest value that stabilization can be achieved, while pitch angle and pitch rate are fixed as zero. The largest deviation in angle of attack that the third order controller in [16] can sustain is about 0.4727 rad (27.1 degree). While, the proposed nonlinear controller (46) can sustain larger deviations in angle of attack at about 0.721 rad (41.3 degree) and 0.7131 rad (40.8 degree) with parameter ϑ fixed as $[1,\ 0]^T$ and $[0,\ 1]^T$ respectively. In summary, it can be clearly seen that the proposed nonlinear controller (46) performs better than the third order controller in [16] in bringing the aircraft back to trim conditions.

To verify the robustness of our nonlinear stabilizing feedback control law derived from SOS optimization, simulations using the original nonlinear aircraft dynamics in (7)-(10) are performed, and the results are analyzed. Simulations are carried out for both small and large initial conditions in the angle of attack, which represent small and large deviations from the trim conditions due to disturbances. For nonlinear F-8 aircraft models in operation with 25% loss of control surface, the initial conditions for $\alpha(0)$ are selected as $7.6deg$ (5 deg above its operation point value) and $25.7deg$ (corresponding to the positive largest deviation that can be stabilized). The responses of the states α, θ, q and elevator δ_e are shown in Fig. 7-8, where

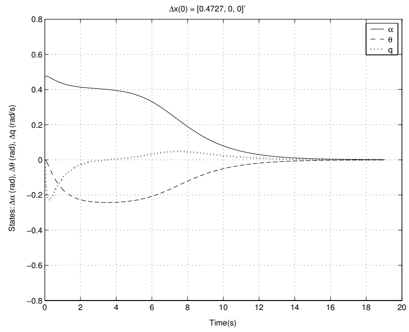

Fig. 4. Response of closed-loop system with the third order controller in [16]

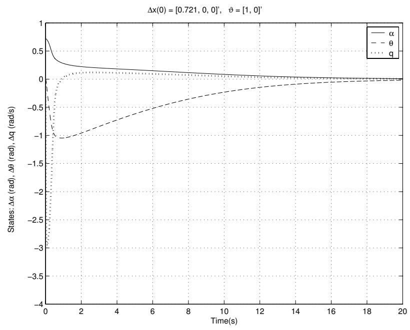

Fig. 5. Response of closed-loop system with proposed nonlinear controller (46), $\vartheta = [1,\ 0]^T$

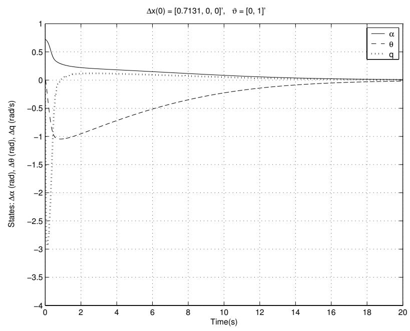

Fig. 6. Response of closed-loop system with proposed nonlinear controller (46), $\vartheta = [0,\ 1]^T$

input saturations are also considered. It is noticed that in the 3rd order nonlinear simulation (Fig. 6), the designed nonlinear controller (46) can sustain the deviation in angle of attack up to 0.7131 rad (40.8 deg). However, in the robustness simulation (Fig. 8) using the original nonlinear model (7)-(10), the largest deviation is only arrived at 25.7 deg. The main reason for this discrepancy is the fact that the proposed stabilizing controller is derived based on the simplified (3rd order) aircraft models in (11) and (12).

6. CONCLUSION

This paper studies the stability analysis and state feedback control design for polytopic parameter-dependent nonlinear systems with polynomial vector fields, which models aircraft dynamics with control surface impairment. Sufficient conditions to test and design for the stability of the closed-loop systems are presented based on the classical Lyapunov stability results. The feedback controller design

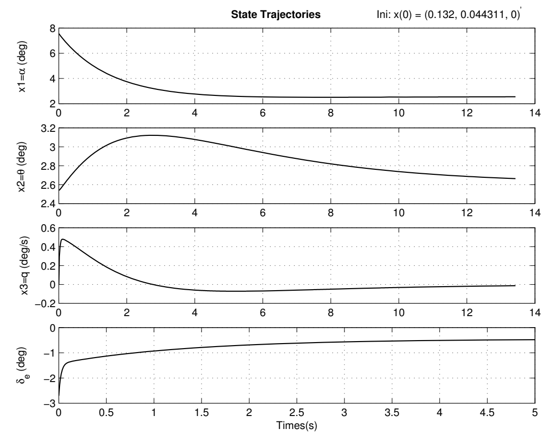

Fig. 7. State trajectories and elevator deflection, initial condition $\alpha(0) = 7.6deg$

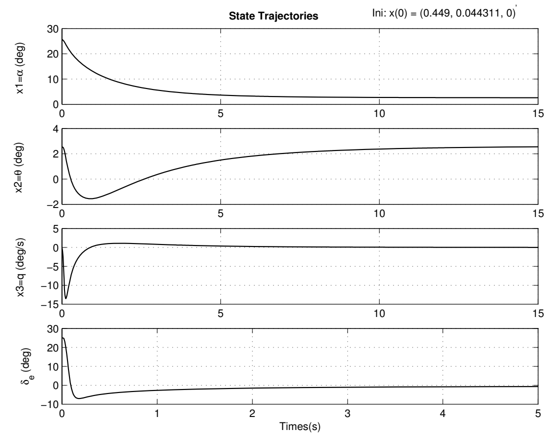

Fig. 8. State trajectories and elevator deflection, initial condition $\alpha(0) = 25.7deg$

method are applied to the longitudinal model of an F-8 aircraft. Robust stability can be guaranteed for the nonlinear aircraft model in normal operation and in the event of control surface faults. Simulation results show the effectiveness of the proposed method.

REFERENCES

[1] A. Isidori. *Nonlinear Control Systems.* Springer Verlag, New York, 1995.

[2] P. A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization.* California Institute of Technology, PhD thesis, 2000.

[3] W. H. Tan and A. Packard. Stability Region Analysis using Sum of Squares Programming. *In Proceedings of the American Control Conferrence*, Minneapolis, Minnesota, USA, 2006, pages 2297-2302.

[4] J. Bochnak, M. Coste and M.-F. Roy. *Real Algebraic Geometry.* Springer, 1998.

[5] Dan Zhao, Jian Liang Wang and Eng Kee Poh. Stability Analysis and Controller Synthesis for Parameter-Dependent Polynomial Nonlinear Systems. *In Proceedings of the American Control Conferrence*, New York, USA, 2007.

[6] C. Ebenbauer and T. Raff and F. Allgöwer. A Duality-based LPV Approach to Polynomial State Feedback Design. *In Proceedings of the American Control Conference*, Portland, OR, 2005, pp 703-708.

[7] H. Ichihara and E. Nobuyama. Minimax Polynomial Optimization by using Sum of Squares Relaxation and its Application to Robust Stability Analysis of Parameter-dependent Systems. *In Proceedings of the American Control Conferrence*, Portland, OR, 2005, pages 3433-3434.

[8] P. Kokotović and M. Arcak. Constructive Nonlinear Control: A Historical Perspective. *Automatica*, vol. 37, 2001, pages 637-662.

[9] K. Krishnaswamy, G. Papageorgiou, S. Glavaski and A. Papachristodoulou. Analysis of Aircraft Pitch Axis Stability Augmentation System using Sum of Squares Optimization, *In Proceedings of the American Control Conferrence*, Portland, OR, 2005, pages 1497-1502.

[10] A. Papachristodoulou. Analysis of Nonlinear Time Delay Systems using the Sum of Squares Decomposition. *In Proceedings of the American Control Conferrence*, Boston, Massachusetts, 2004, pages 4153-4158.

[11] S. Prajna and A. Papachristodoulou. Analysis of Switched and Hybrid Systems-beyond Piecewise Quadratic Methods. *In Proceedings of the American Control Conferrence*, Denver, Colorado, 2003, pages 2779-2784.

[12] S. Prajna, P. A. Parrilo and A. Rantzer. Nonlinear Control Synthesis by Convex Optimizaiton. *IEEE Transactions on Automatic Control*, vol. 49, pages 310-314, 2004

[13] S. Prajna, A. Papachristodoulou, P. Seiler and P. A. Parrilo. SOSTOOLS: Control Applications and New Developments. *IEEE International Symposium on Computer Aided Control Systems Design*, 2004, pages 315-320.

[14] S. Prajna, A. Papachristodoulou and F. Wu. Nonlinear Control Synthesis by Sum of Squares Optimization: A Lyapunov-based Approach. *In Proceedings of the Asian Control Conference*, 2004, pages 157-165.

[15] J. L. Wang and N. Sundararajan. A Nonlinear Flight Controller Design for Aircraft. *Control Eng. Practice*, Vol. 3, No. 6, pages 813-825, 1995

[16] William L. Garrard and John M. Jordan. Design of Nonlinear Automatic Flight Control Systems. *Automatica*, Vol. 13, pages 497-505, 1977