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Control of adaptive optics system: an H-infinite approach.

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Abstract: We apply robust control technics to an adaptive optics system including a dynamic model of the deformable mirror. The dynamic model of the mirror is a modification of the usual plate equation. We propose also a state-space approach to model the turbulent phase. A continuous time control of our model is suggested taking into account the frequential behavior of the turbulent phase. An H_∞ controller is designed in an infinite dimensional setting. Due to the multivariable nature of the control problem involved in adaptive optics systems, a significant improvement is obtained with respect to traditional single input single output methods.

Keywords: H_∞ control, Adaptive optics, Infinite-dimensional systems, Vibration and modal analysis

1. INTRODUCTION

The technological developments of the eighties have made possible the use of adaptive optics (AO) systems to actively correct the wave front distortions affecting an incident wavefront propagating through a turbulent medium. A particularly interesting application of this technique is in the field of astronomical ground-based imaging. The idea behind AO systems is to generate a corrected wavefront as close as possible to the genuine incident plane wavefront thanks to a deformable mirror. An AO system is principally based on a wavefront sensor measuring the resulting distortion of the wavefront after correction by the deformable mirror. Based on these measured signals, a control law is computed in order to reshape a deformable mirror through piezoelectric actuators. The active mirror system is made of a swing mirror dedicated to the correction of the tilts (first order modes) of the wavefront in two dimensions and a deformable mirror that is part of the control-loop for the correction of higher-order modes of aberrated wavefront. For additional details on basic principles of adaptive optics, the interested reader may have a look at the book Rodier [1999].

This paper is devoted to the design of specific control laws for an adaptive optics system formed by a bimorph mirror and a wavefront sensor (see Figure 1). Most often, the existing adaptive optics systems use static models and very basic control algorithms based on physical insights. Here, our goal is to consider the design of an adaptive optics system from a modern automatic control point of view as in Raynaud et al. [2006] and Frazier et al. [2004]. This means first that dynamics of the different elements involved in the control-loop have to be taken into account. In particular, a specific dynamic model for the deformable mirror is proposed for control purpose Miller et al. [1999]. Secondly, a state-space model of the turbulent phase, built from its frequency domain characteristics, is defined Conan et al. [1995].

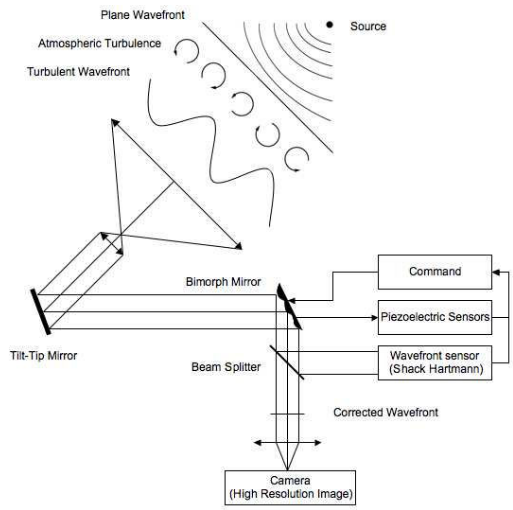

Fig. 1. An adaptive optics system

The main contribution concerns the infinite dimension setting introduced in this paper. More precisely, while in the literature, only finite dimensional models are considered, a model based on a particular partial differential equation is used for the deformable mirror.

In reference Lenczner et al. [2006], a thin elastic plate model of a deformable bimorph mirror is derived. The idea is then to elaborate a robust control strategy based on modern control tools developed during the last years and extended to the control of distributed parameter systems Van Keulen [1993]. Moreover, in contrast to Paschall et al. [1993] and Raynaud et al. [2006] we do not need to compute any interaction matrix corresponding to the relation between the input on the piezoelectric patches attached to the mirror and the output given by the Shack-Hartmann sensor.

In this article, the control problem is recast in an H_∞ control setting. The first motivation is that H_∞ control theory is now well established in the infinite dimensional case and H_∞ controller provides intrinsic properties of robustness while optimizing on the worst-case performance. Another motivation is the multivariable nature of the control problem involved in adaptive optics system design Frazier et al. [2004]. Current adaptive optics control systems use decoupling modal control to rewrite the original problem as many decoupled single input single output control problems. Because H_∞ control framework may easily handle a multivariable dynamic model of the bimorph deformable mirror in the synthesis process, the obtained robust controller outperforms usual static control approaches of the literature.

2. THE ADAPTIVE OPTICS MODEL

The bimorph mirror is composed of a purely elastic and reflective plate equipped with piezoelectric actuators (in order to deform the shape of the mirror) and piezoelectric sensors (to measure the effective deformation). A Shack-Hartmann sensor (that divides up the wavefront into separate facets, each focused by a micro-lens onto a subarray of CCD camera pixels - see Rodier [1999]) is used to analyzes the resulting phase ϕ_{res} of the wavefront, after reflection in the deformable mirror of the turbulent phase ϕ_{tur}.

Different types of disturbances have to be faced with: w_{mod} represents unstructured uncertainty (neglected dynamics) affecting the model, w_{piezo} and w_{SH} are noise signals respectively attached to piezoelectric and Shack-Hartmann sensors. Finally, ϕ_{tur} is the turbulent phase of the wavefront introduced by the atmospheric perturbation.

$e = e(r, \theta, t)$ denotes the transverse displacement of the circular mirror at point of polar coordinates (r, θ) and time t while λ is the wavelength of light. The corrected phase produced by e is then given by $\phi_{\text{cor}} = \frac{4\pi}{\lambda} e$ leading to a resulting phase:

$$\phi_{\text{res}} = -\frac{4\pi}{\lambda} e + \phi_{\text{tur}} \tag{1}$$

The optic sensor's output, computed with the Shack-Hartmann sensor, is given by:

$$y_{\text{SH}} = -\frac{4\pi}{\lambda} e + \phi_{\text{tur}} + c w_{\text{SH}} \tag{2}$$

where c is a modelling parameter of the perturbation.

Finally, we note that the control input is the voltage u applied to the piezoelectric actuators and the corresponding piezoelectric output is the voltage y_{pe} measured with the piezoelectric inclusions used as sensors (see equations (3) and (4) below). Indeed, in comparison with many other devices, where the only information used to compute the voltage u comes from the wavefront analyzer, the additional possibility of measuring the deflection of the mirror through a layer of piezoelectric sensors (see Figure 1) is considered here.

It is recalled that the goal of the adaptive optics control system is to minimize the resulting phase of the wavefront using Shack-Hartmann measurements.

Bimorph mirror model

To obtain the model of a bimorph mirror, we consider three different layers. One is purely elastic and reflective, the second one is equipped with piezoelectric inclusions used as actuators, the third one is equipped with piezoelectric inclusions used as sensors. The heterogeneities are periodically distributed. In reference Lenczner et al. [2006], the authors derive the following dynamical model of the mirror:

$$\rho\ \partial_{tt} e + Q_1 \Delta^2 e + Q_2 e = \widetilde{d}_{31} \Delta u + \rho b w_{\text{mod}} \tag{3}$$

with the initial conditions $e(r, \theta, t = 0) = e_0(r, \theta)$ and $\partial_t e(r, \theta, t = 0) = e_1(r, \theta)$. The voltage y_{pe} computed by the piezoelectric sensors is given by

$$y_{pe} = \tilde{e}_{31} \Delta e + d w_{\text{pe}}. \tag{4}$$

The following notations are defined:

- (r, θ) are the spatial coordinate of the point of the disc Ω of radius a and t is the time;
- Δ is the Laplacian operator and for $v(r, \theta)$
$$\Delta v = \frac{\partial v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \theta^2};$$
- u is the voltage applied to the inclusions of the actuator layer;
- ρ is the surface density, ν is the Poisson ratio of the mirror's material, Q_1 is the stiffness coefficient and Q_2 is a correction coefficient;
- $\tilde{e}_{31}$ and $\widetilde{d}_{31}$ are proportional to the piezoelectric tensor coefficient d_{31};
- b and d are linear applications on $L^2(\Omega)$;
- w_{mod} and w_{pe} are unknown perturbations modelling the model errors of the plate equation and the measurement noise of the piezoelectric output.

The boundary conditions are those of the free edges case (VLT and the experimental device SESAME, see Subsection 44.2):

$$\begin{aligned} \frac{\partial^2 e}{\partial r^2} + \nu \left(\frac{1}{r}\frac{\partial e}{\partial r} + \frac{1}{r^2}\frac{\partial^2 e}{\partial \theta^2} \right)\Bigg|_{r=a} &= 0 \\ \frac{\partial}{\partial r}(\Delta e) + \frac{1}{r}(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial e}{\partial \theta} \right)\Bigg|_{r=a} &= 0 \end{aligned} \tag{5}$$

Turbulent phase model

In order to complete our adaptive optics system model, we need to develop a model of the turbulence phase.

A usual representation of atmospheric phase distortion is made through the orthogonal basis of Zernike polynomials because the first Zernike modes correspond to the main optical aberrations. An infinite number of Zernike functions is required to characterize the wavefront, but a truncated basis is used in general for implementation purpose. Note that a 14-th order approximation contains 92% of the phase information, without taking into account the piston mode which represents the average phase distortion Paschall et al. [1993]. The tip/tilt modes are not part of our modelling of the turbulent phase because of their correction by a dedicated mirror. We will therefore work with the 12 first modes of Zernike given in reference Noll [1976] excluding the three first ones.

The turbulent phase $\phi_{\rm tur}$ is approximated as follows:

$$\phi_{\rm tur}(r,\theta,t) \approx \sum_{i=3}^{N_Z} \phi_i(t) Z_i(r,\theta)$$

where $N_Z \geq 14$. Z_i is the i-th Zernike function and for all i, $\phi_i(t)$ is a random time-varying coefficient corresponding to the projection of $\phi_{\rm tur}$ on Z_i.

To build a state-space representation of the turbulent phase, $\phi_{\rm tur}$ is modelled as the output of a linear shaping filter of the form :

$$\dot{\phi} = F\phi + Gw \tag{6}$$

where $\phi = (\phi_3, \cdots, \phi_{N_Z})$, $w = (w_3, \cdots, w_{N_Z})$, F and G are two time-invariant square matrices of $(N_Z - 2)$-dimension and w is a stationary zero-mean white gaussian noise. $\phi_{\rm tur}$ is therefore a stationary process.

In order to compute F and G, the results presented in Conan et al. [1995] and based on the Kolmogorov theory of turbulence and associated approximations in the frequency domain are used here. They confirm similar results proposed in Hogge et al. [1976] and complete the study of frequency domain behavior for each Zernike coefficient. Each Zernike function's spectrum are characterized by a cut-off frequency whose heuristic expression is given by:

$$f_{c_i} = 0.3(n_i + 1)\frac{V}{D} \tag{7}$$

where n_i is the radial order of the Zernike number i, V is the average wind-speed and D the diameter of the circular aperture.

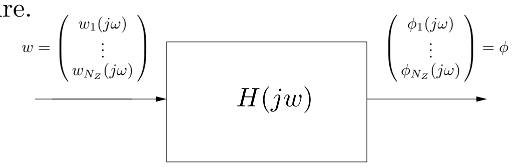

Fig. 2. Shaping filter generating ϕ

The random process ϕ is supposed to be composed of $N_Z - 2$ decoupled first-order Markov processes. For $i = 3 \cdots N_Z$, we have:

$$H_i(P) = \frac{\phi_i(p)}{\omega_i(p)} = \frac{1}{1+\tau_i p} \text{ with } \tau_i = \frac{1}{2\pi f_{c_i}} \tag{8}$$

In other words, $F = \text{diag}_i(-\frac{1}{\tau_i})$.

The matrix G is obtained from the steady-state Lyapunov equation verified by the correlation matrix $P_\phi(\infty)$:

$$GG' = -(FP_\phi(\infty) + P_\phi(\infty)F') \tag{9}$$

A closed-form expression for the spatial covariance matrix is given in Noll [1976].

$$\begin{aligned} P_\phi(\infty) &= \text{cov}(\phi_i,\phi_j) = E(\phi_i\phi_j) \\ &= 7.19 \times 10^{-3} \times (-1)^{(n_i+n_j-m_i-m_j)/2} \left(\frac{D}{r_0}\right)^{\frac{5}{3}} \\ &\times \sqrt{(n_i+1)(n_j+1)}\pi^{\frac{8}{3}} \\ &\times \frac{\Gamma(\frac{14}{3})\Gamma(\frac{n_i+n_j-\frac{5}{3}}{2})}{\Gamma(\frac{n_i-n_j+\frac{17}{3}}{2})\Gamma(\frac{n_j-n_i+\frac{17}{3}}{2})\Gamma(\frac{n_i+n_j+\frac{23}{3}}{2})} \end{aligned}$$

where Γ is the Gamma function and r_0 is the Fried parameter Rodier [1999]. Table 1 shows the non zero entries of the matrices F and G for $V = 9$ m.s^{-1} and $\frac{D}{r_0} = 8$ (as in Raynaud et al. [2006]).

i	j	$F_{i,j}$	$G_{i,j}$	i	j	$F_{i,j}$	$G_{i,j}$
1	1	-4.24	11.02	7	7	-8.48	1.73
1	6	0	-0.45	7	14	0	-0.40
1	13	0	0.02	8	3	0	-0.50
2	2	-4.24	9.72	8	8	-10.60	1.19
2	7	0	-0.45	9	9	-8.48	1.83
2	14	0	0.02	10	10	-8.48	1.83
3	3	-6.36	3.03	11	4	0	-0.50
3	8	0	-0.50	11	11	-10.60	1.19
4	4	-6.36	3.03	12	5	0	-0.50
4	11	0	-0.50	12	12	-10.60	1.19
5	5	-6.36	3.03	13	1	0	0.02
5	12	0	-0.50	13	6	0	-0.40
6	1	0	-0.45	13	13	-12.72	0.90
6	6	-8.487	1.73	14	2	0	0.02
6	13	0	-0.40	14	7	0	-0.40
7	2	0	-0.45	14	14	-12.72	0.90

Table 1. Atmospheric phase distortion state-space model

3. ROBUST CONTROL RESULTS

The point of this section is to prove that the new model we propose for AO systems is valid for an H_∞-control study. One of the difficulty comes from the infinite dimensional setting. For a survey of the H_∞-control theory for the infinite-dimensional case, the interested reader may have a look at Curtain et al. [1993] for the state-feedback case and Van Keulen [1993] for the output-feedback case. The main results are a generalization of results finite-dimensional regular H_∞-control problems (see for instance Doyle et al. [1996]). In particular, the solution will be given in terms of the solvability of two coupled Riccati equations.

The linear infinite-dimensional model derived from the partial differential equations presented in Section 2 has to fit in the following standard formalism of measurement-feedback control

$$\begin{cases} x' = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x + D_{21} w \end{cases} \tag{P}$$

where x is the state of the system, u is the control input, w is the disturbance input, y is the measured output and z is the controlled output.

Therefore, we introduce the following notations. The state vector is $x = (e, \partial_t e, \phi_{\rm tur})^T$ where e is the transverse displacement of the plate and $\phi_{\rm tur}$ is the projection of the turbulent phase on the N_z first Zernike modes. The exogenous disturbance inputs vector is $w = (w_{\rm mod}, w_{\rm SH}, w_{\rm tur}, w_{\rm pe})^T$ gathers the different perturbations The control vector u is defined as the voltage applied to piezoelectric patches. The measurement vector $y = (y_{\rm pe}, y_{\rm SH})$ is composed with the piezoelectric and the wavefront analyzer measured outputs. The controlled outputs vector $z = (\phi_{\rm res}, u)$ contains the resulting phase (see (1)) and the control input vector u.

The aim is to find a dynamic measurement-feedback controller K that exponentially stabilizes this system and ensures that the influence of w on z is smaller than some specific bound. The controller is assume to have the following form:

$$\begin{cases} p' &= Mp + Ny \\ u &= Lp + Ry \end{cases} \tag{K}$$

where M is the infinitesimal generator of a C_0-semigroup on a real separable Hilbert space and N, L and R are bounded linear operators. With this controller, the closed-loop system can easily be derived and defines a bounded linear map S_K such that $z(t) = (S_K w)(t)$. Its bound is denoted $||S_K||_\infty$.

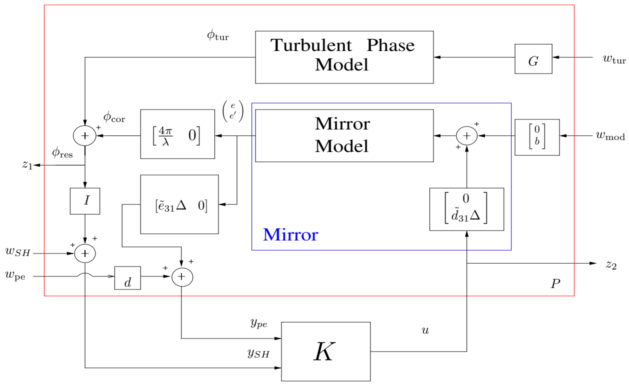

Fig. 3. Standard model for adaptive optics system control loop

The control loop defining the adaptive optics system is sketched in Figure 3. If we gather the different equations describing the system, we get:

$$\begin{aligned} &\partial_{tt} e + Q_1 \Delta^2 e + Q_2 e = \widetilde{d_{31}} \Delta u + b\rho w_{\text{mod}} \\ &\phi_{\text{res}} = \phi_{\text{tur}} - \frac{4\pi}{\lambda} e \\ &y_{pe} = \tilde{e}_{31} \Delta e + d w_{\text{pe}} \\ &y_{\text{SH}} = \phi_{\text{tur}} - \frac{4\pi}{\lambda} e + c w_{\text{SH}} \end{aligned} \tag{10}$$

In order to have a unified infinite dimensional modelling of the adaptive optics system's state, the following equation $\partial_t \phi_{\text{tur}} = \mathcal{F}\phi_{\text{tur}} + \mathcal{G} w_{\text{tur}}$ is added to (10).

Thus, the operators defining the standard form P are built from (10)

$$A = \begin{pmatrix} 0 & I & 0 \\ -\frac{Q_1}{\rho}\Delta^2 - \frac{Q_2}{\rho} I & 0 & 0 \\ 0 & 0 & \mathcal{F} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & \mathcal{G} & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 \\ \frac{\tilde{d}_{31}}{\rho}\Delta \\ 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -\frac{4\pi}{\lambda} I & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \tilde{e}_{31}\Delta & 0 & 0 \\ -\frac{4\pi}{\lambda} I & 0 & I \end{pmatrix}, \quad D_{21} = \begin{pmatrix} 0 & 0 & 0 & d \\ 0 & c & 0 & 0 \end{pmatrix}.$$

The appropriate functional spaces associated to the infinite-dimensional model are now precisely defined. With the boundary condition (5), we consider the state space (the mirror Ω is a disc of radius a)

$$\begin{aligned} X &= H^2_{\text{bc}}(\Omega) \times L^2(\Omega) \times L^2(\Omega) \\ &= \{e \in H^2(\Omega), e \text{ satisfying } (5)\} \times \left(L^2(\Omega)\right)^2 \end{aligned}$$

the input spaces $U = H^2(\Omega) \cap H^1_0(\Omega)$ and $W = \left(L^2(\Omega)\right)^4$ and the output spaces $Y = Z = \left(L^2(\Omega)\right)^2$, where L^2 is the Hilbert space of square integrable functions and H^1_0 and H^2 are the Sobolev spaces

$$H^1_0(\Omega) = \{f \in L^2(\Omega) / \ \forall i = 1,2,\ \partial_i f \in L^2(\Omega),\ f|_{\partial\Omega} = 0\}$$
$$H^2(\Omega) = \{f \in L^2(\Omega) / \ \forall i,j = 1,2,\ \partial_i f,\ \partial_i \partial_j f \in L^2(\Omega)\}$$

This model satisfies all the assumptions of the main theorem of reference Van Keulen [1993]. We give here a simplified version of this result:

Theorem 1. Van Keulen [1993] Let $\gamma > 0$. There exists an exponentially stabilizing dynamic output-feedback controller of the form (K) with $\|S_K\|_\infty < \gamma$ if and only if there exist two nonnegative definite operators P, $Q \in \mathcal{L}(X)$ satisfying the three conditions

(i) $\forall x \in D(A)$, $Px \in D(A^*)$,
$$\left(A^*P + PA + P(\gamma^{-2}B_1B_1^* - B_2B_2^*)P + C_1^*C_1\right)x = 0$$
and $A + (\gamma^{-2}B_1B_1^* - B_2B_2^*)P$ generates an exponentially stable semigroup,
(ii) $\forall x \in D(A^*)$, $Px \in D(A)$,
$$\left(AQ + QA^* + Q(\gamma^{-2}C_1^*C_1 - C_2^*C_2)Q + B_1B_1^*\right)x = 0$$
and $A^* + (\gamma^{-2}C_1^*C_1 - C_2^*C_2)Q$ generates an exponentially stable semigroup,
(iii)
$$r_\sigma(PQ) < \gamma^2,$$
where $r_\sigma(PQ)$ stands for the spectral radius of PQ.
In this case, the controller K given by (K) and

$$\begin{aligned} M &= A + (\gamma^{-2}B_1B_1^* - B_2B_2^*)P \\ &\quad -Q(I - \gamma^{-2}PQ)^{-1}C_2^*C_2 \\ N &= -Q(I - \gamma^{-2}PQ)^{-1}C_2^* \\ L &= B_2^*P \\ R &= 0 \end{aligned} \tag{11}$$

is exponentially stabilizing and guarantees that we have $\|S_K\|_\infty < \gamma$, *ie*

$$\|\phi_{\text{res}}\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \le \gamma \|w\|_{(L^2(\Omega))^4}.$$

Finally, if the solutions to the Riccati equations exists, then they are unique.

Upon additional assumptions that are not detailed here, it is necessary to prove that A is the infinitesimal generator of a C_0-semigroup on the real separable Hilbert space X. Actually, if we consider the unbounded linear operator

$$A_1 : \mathcal{D}(A_1) \to X$$
$$\begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ -\Delta^2 e_0 \\ 0 \end{pmatrix}$$

where

$$\mathcal{D}(A_1) = \{e_0 \in H^4(\Omega), e_0 \text{ st } (5)\} \times H^2(\Omega) \times L^2(\Omega),$$

then one can prove that A_1 is dissipative on X. Indeed, we prove that for all $x \in X$,

$$\langle A_1 x, x \rangle_X \le 0$$

using the following scalar product on $H^2_{\text{bc}}(\Omega)$ in cartesian coordinates $(x_1, x_2) \in \Omega$, as suggested in Lions et al. [1972]:

$$< u, v >_{H^2_{\rm bc}(\Omega)} = \int_\Omega \Delta u \Delta v - (1-\nu)\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right) + 2(1-\nu)\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\right) d\Omega.$$

Moreover, one can easily check that A_1 is also self-adjoint and onto. Therefore, from Lumer-Phillips' Theorem (see Pazy [1983], p. 15), A_1 generates a continuous semigroup of linear contractions acting on X. And finally, since A is the sum of A_1 and of a linear operator bounded on X (as $\mathcal{F}$ is assumed to be bounded, like F), the proof is complete (see Luo et al. [1999], p. 40).

Of course, from a numerical point of view, we need to get an appropriate finite dimensional model.

4. A TRUNCATED MODEL FOR PRACTICAL DESIGN

4.1 Truncation

The corresponding finite dimensional model can be presented as :

$$\begin{cases} x'_N = A_N x_N + B_{1N} w_N + B_{2N} u_N \\ z_N = C_{1N} x_N + D_{12N} u_N \\ y_N = C_{2N} x_N + D_{21N} w_N \end{cases} \qquad (12)$$

where the operators of system (P) have been replaced by real-valued matrices computed on truncated hermitian basis. We denote by N_B the number of eigenfunctions of operator Δ^2 we consider and by N_Z the number of Zernike modes used to describe $\phi_{\rm tur}$. Then, $x_N \in \mathbb{R}^{2N_B+N_Z}$ is the state vector, $w_N \in \mathbb{R}^{2N_B+2N_Z}$ is the exogenous perturbation vector, $u_N \in \mathbb{R}^{N_B}$ is the control vector, $z_N \in \mathbb{R}^{N_B+N_Z}$ is the controlled output vector and $y_N \in \mathbb{R}^{N_B+N_Z}$ is the measured output vector. The matrices A_N, B_{1N}, B_{2N}, C_{1N}, D_{12N}, C_{2N} and D_{21N} are of appropriate dimensions.

In order to compute these objects, we still consider the case of a circular bimorph mirror which is free at all the boundary (this is also the case of the mirror considered in Section 4.2 below). The eigenvectors of operator

$$-\frac{Q_1}{\rho}\Delta^2 - \frac{Q_2}{\rho} I$$

are given by, for all $(k,j) \in \mathbb{N}^2$,

$$L_{kj}(r,\theta) = a_{kj}\left(J_k\left(\frac{\lambda_{kj} r}{a}\right) + c_{kj} I_k\left(\frac{\lambda_{kj} r}{a}\right)\right)\cos(k\theta)$$

$$M_{kj}(r,\theta) = a_{kj}\left(J_k\left(\frac{\lambda_{kj} r}{a}\right) + c_{kj} I_k\left(\frac{\lambda_{kj} r}{a}\right)\right)\sin(k\theta)$$

where (r,θ) are the polar coordinates of $x \in \Omega$, J_k and I_k are, respectively, ordinary and modified Bessel function of first kind and order k, and $-\frac{Q_1}{\rho}\left(\frac{\lambda_{kj}}{a}\right)^4 - \frac{Q_2}{\rho}$ the corresponding eigenvalues. The family

$$\left\{L_{kj}, M_{kj}, (k,j) \in \mathbb{N}^2\right\}$$

is an Hilbertian basis of $H^2_{bc}(\Omega)$. The dimensionless coefficients λ_{kj} and c_{kj} depend on the boundary conditions while $a_{kj}$is computed using a normalization condition on the eigenvectors (see Amabili et al. [1995] for further details). In what follows, we consider the case of Poisson ratio $\nu = 0.2$ corresponding to the material the mirror is made of. Once a maximal azimuthal order is given (here $k_{\max} = 5$) the modes are classified according to increasing λ_{kj} and one has:

i	j	k	λ_{kj}	c_{kj}	a_{kj}
1	0	2	2.37805	0.18773	3.6157
2	1	0	2.96173	-0.092478	2.1984
3	0	3	3.60924	0.075982	4.4749
4	1	1	4.51025	-0.019949	3.8317
5	0	4	4.76934	0.034281	5.2453
6	0	5	5.89565	0.016333	5.9506
7	1	2	5.94302	-0.0056226	4.4178
8	0	2	6.18269	0.0032602	3.1394
9	1	3	7.30051	-0.0018233	4.9425
10	2	1	7.72338	0.0007269	4.9616

The sequence of functions L_{kj} and M_{kj} need to be reordered. They are now denoted by B_n and follow the increasing values of λ_{kj}, alternating cosine and sine and eliminating the null eigenvectors M_{0j}. Therefore,

$$\forall x \in X,\ x = \sum_{n\in\mathbb{N},\ i\geq 1} \alpha_i B_i(r,\theta)$$

where $(\alpha_n)_{n\geq 1}$ is a sequence of real numbers satisfying $\sum_{n\in\mathbb{N},\ n\geq 1} \alpha_n^2 < \infty$.

In reference Blevins [1979], one can find that this basis $(B_n)_{n\in\mathbb{N}}$ with free boundary conditions is not orthogonal in $L^2(\Omega)$. However, numerically, we can prove that this basis is nearly orthogonal, indeed lots of scalar products in $L^2(\Omega)$ are null and the others are small (10^{-6}) in comparison with unity. So, for more numerical facilities, we will use the scalar product in $L^2(\Omega)$ rather than in $H^2_{bc}(\Omega)$.

Given N_B and $N_Z \in \mathbb{N}$, we compute A_N, B_{1N}, B_{2N}, C_{1N}, C_{2N}, D_{12N} and D_{21N} using the "Bessel" truncated basis $\{B_0, B_1, \ldots, B_{N_B}\}$ and the Zernike one $\{Z_0, Z_1, \ldots, Z_{N_Z}\}$. We make analogous assumptions for b, c and d, i.e. $b = \mathrm{diag}_i(b_i)$, $c = \mathrm{diag}_i(c_i)$ and $d = \mathrm{diag}_i(d_i)$ where $(b_i)_{i\in\mathbb{N},\ i\geq 1}$, $(c_i)_{i\in\mathbb{N},\ i\geq 1}$ and $(d_i)_{i\in\mathbb{N},\ i\geq 1}$ are sequences of real numbers. Futhermore $\phi_{\rm res}$ is expressed on Besssel functions, so we need to estimate a projection matrix to define $\phi_{\rm tur}$ with Bessel spatial coordinates. We note Q this projection $N_B \times N_Z$-dimension matrix. Thus, the computed equation becomes:

$$\phi_{{\rm res},i} = -\frac{4\pi}{\lambda} e_i + \sum_{j=1}^{N_Z-2} Q_{ij}\phi_{j+2}$$

We denote by $\mathbf{0}$ each null matrix with the appropriate dimensions so that each following matrix makes sense. We get

$$A_N = \begin{bmatrix} \mathbf{0} & \mathbf{1}_{N_B} & \mathbf{0} \\ -\omega_i^2 \mathbf{1}_{N_B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F \end{bmatrix} \qquad B_{1N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G & \mathbf{0} \end{bmatrix}$$

$$B_{2N} = \begin{bmatrix} \mathbf{0} \\ \mathrm{block}_{ij}\left(\frac{\tilde{d}_{31}}{\rho}\langle \Delta\phi_i, \phi_j\rangle\right) \\ \mathbf{0} \end{bmatrix} \qquad C_{1N} = \begin{bmatrix} -\frac{4\pi}{\lambda}\mathbf{1}_{N_B} & \mathbf{0} & Q \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$C_{2N} = \begin{bmatrix} \mathrm{block}_{ij}\left(\tilde{e}_{31}\langle \Delta\phi_i, \phi_j\rangle\right) & \mathbf{0} & \mathbf{0} \\ -\frac{4\pi}{\lambda}\mathbf{1}_{N_B} & \mathbf{0} & Q \end{bmatrix}$$

$$D_{12N} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1}_{N_B} \end{bmatrix} \qquad D_{21N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & d \\ \mathbf{0} & c & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\omega_i^2 = \dfrac{Q_1}{\rho}\left(\dfrac{\lambda_i}{a}\right)^4 + \dfrac{Q_2}{\rho}$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $L^2(\Omega)$.

4.2 Numerical results

To get more realistic results to our numerical simuations, the physical background of the experimental device SESAME of the Observatoire de Paris is considered. This experimentation uses a bimorph mirror with a distribution of 31 piezoelectric actuators. The piezoelectric inclusions are PZT patches. We use the following physical constants:

- radius of the mirror: $a = 25 \times 10^{-3}$ m.
- stiffness coefficients: $\begin{array}{l} Q_1 = 84 \text{ Nm} \\ Q_2 = 11.25 \times 10^8 \text{ Nm}^{-3} \end{array}$.
- surfacic density: $\rho = 16.3$ kg.m^{-2}.
- piezoelectric modulus: $\tilde{d}_{31} = -0.0044$ NV^{-1}.
- wave length: $\lambda = 550$ nm.
- piezoelectric coefficient $\tilde{e}_{31} = -5.60 \times 10^3$Vm.

The performance of the control system is evaluated by considering the spatial norm $\|\phi_{\text{res}}\|_{L^2(\Omega)}$ compared to:

$$\|\phi_{\text{tur}}\|_{L^2(\Omega)} = \sum_{i=3}^{N_z} \phi_i(t)^2$$

For identical random initial conditions and tuning parameters chosen as $b_i = 0.001$, $c_i = 0.002$ and $d_i = 0.003$, $\forall i$, the Figure 4 depicts our results.

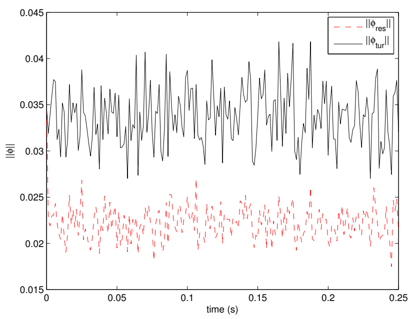

Fig. 4. $\|\phi_{\text{tur}}\|_{L^2(\Omega)}$ (-) and $\|\phi_{\text{res}}\|_{L^2(\Omega)}$ (–)

Using Monte Carlo simulations, the ratio between temporal average of $\|\phi_{\text{tur}}\|_{L^2(\Omega)}$ and $\|\phi_{\text{res}}\|_{L^2(\Omega)}$ is computing as 1.91 which represents a phase distortion attenuation of the reflected wavefront of 48%. These results are of the same order of magnitude as those presented in Paschall et al. [1993].

5. CONCLUSION

In this paper, a new framework to deal with the problem of adaptive optics is proposed. It is mainly based on an infinite-dimensional model of the deformable mirror associated with the definition of a standard model on which robust control techniques may be applied. The preliminary numerical experiments show a performance level comparable with the results of reference Paschall et al. [1993]. The main advantage of the approach suggested in this paper is that no interaction matrix is required to control the system. The authors are planing to take into account a model for the Shack-Hartmann wavefront sensor including a time delay associated with processing measurements. This will be covered in a next study.

REFERENCES

M. Amabili, A. Pasqualini and G. Dalpiaz Natural frequencies and modes of free-edge circular plates vibrating in vacuum or in contact with liquid, *J. Sound Vibration* 188 (1995), no. 5, 685-699.

R.D. Blevins, Formulas for natural frequency and mode shape, Kriger, 1979.

J.-M. Conan, G. Rousset et P.Y Madec Wavefront temporal spectra in high resolution imaging through turbulence, *J. Opt. Soc. Am. A*, Vol. 12, No. 7(1995).

R. Curtain A. M. Peters and B. Van Keulen, H_∞-control with state-feedback: the infinite-dimensional case, *J. Math. Syst. Estim. Control*, 3 (1993), 1, 1-39.

J. C. Doyle, K. Glover and K. Zhou Robust and Optimal Control, Prentice-Hall, 1996.

B.W. Frazier, R.K. Tyson, M. Smith and J. Roche, Theory and operation of a robust controller for a compact adaptive optics system, *Opt. Eng.* (2004) 43 (12), 2912-2920.

C. Hogge and R. Butts Frequency spectra for the geometric representation of wavefront distortions due to atmospheric turbulence, *IEEE transactions on antennas and propagation*, 1976.

Z.-H. Luo, B.-Z. Guo and O. Morgul Stability and stabilization of infinite dimensional systems with applications, *Communications and Control Engineering Series* Springer-Verlag, London, 1999.

M. Lenczner and C. Prieur, Asymptotic model of an active mirror, *13th IFAC Workshop on Control applications of Optimization*, Cachan, France, 2006.

J.-L. Lions and G. Duvaut, Les inéquations en Mécanique et en Physique, Dunod, 1972.

D.W. Miller and S.C.O. Grocott, Robust control of the multiple mirror telescope adaptive secondary mirror, *Opt. Eng.*, (1999) 38 (8), 1276-1287.

R. Noll, Zernike polynomials and atmospheric turbulence, *The Optical Society of America*, Vol 66 No 3, 1976.

R. Paschall and D. Anderson Linear quadratic Gaussian control of a deformable mirror adaptive optics system with time-delayed measurments, *Applied optics*, Vol.32 No 31, 1993.

M. Pazy Semigroups of linear operators and applications to partial differential equations, *Applied Mathematical Sciences*, 44 Springer-Verlag, New York, 1983.

H.-F. Raynaud, C. Kulcsár, C. Petit, J.-M. Conan, P. Viaris de Lesegno, Optimal control, observers and integrators in adaptive optics, *Optics express*, Vol. 14, No. 17, 2006.

F. Rodier, Adaptive optics in Astronomy, Cambridge University Press, 1999.

B. Van Keulen, H_∞-control with measurement-feedback for linear infinite-dimensional systems, *J. Math. Syst. Estim. Control* 3 (1993), 4, 373-411.