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State-space Approach to Pricing Design in OSNR Nash Game ⋆

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Abstract: The static nature of the noncooperative power control game model in optical networks makes it difficult to study and design an appropriate pricing scheme. In this paper, we derive a first-order best response dynamics from the game-theoretical model and formulate a general multi-input and multi-output (MIMO) state-space model. We use classical linear system theory to explain the controllability of the pricing and the observability of the power states. We use the output regulator theory to design a pricing policy for the network for a given optical signal-to-noise ratio (OSNR) target.

1. INTRODUCTION

Recent investigations of the dynamic and performance aspect of optical wavelength-division multiplexed (WDM) communication networks are inspired by the interest in an intelligent network management system that can maintain network stability and optical channel performance in an on-line fashion (Mukherjee [2007], B. Ramamurthy and Mukherjee [2007], Pavel [2004]). Channel performance is closely dependent on the optical signal-to-noise ratio (OSNR), dispersion and nonlinear effects, (Agrawal [2005]). In Zander [1992], Chraplyvy et al. [1992], some static approaches have been developed for a single link optimization. However, for a modern reconfigurable optical networks, where different channels can travel via different optical paths, it is desirable to implement a decentralized and iterative algorithm to intelligently control the network.

As an alternative to traditional system-wide optimization, non-cooperative game theory has been used to control and optimize network performances. In a large-scale networks, decisions are made independently with local network information, as it is difficult to gather real-time complete information for decision-making. Game theory's inherent property of distributedness and noncooperativeness makes itself an appropriate framework in the OSNR performance optimization.

Such non-cooperative model is considered in Pavel [2006], where an OSNR network model has been developed for decentralized optimization. Each user has a payoff function that is composed of utility and the cost calculated from network price. Under such framework, a closed-form solution of the Nash equilibrium (NE) is found and an iterative algorithm is designed to achieve the solution. However, the NE's static nature makes it difficult to further study and design the network pricing policy that affects each channel's utility function. Pricing of networks is one of the crucial control mechanisms. Proposed in Saraydar et al. [2002], pricing is introduced to provide incentives or a control signal to motivate users to adopt a social behavior, i.e., reach some social optimal solution. A pricing policy is needed to enforce a Nash equilibrium to attain a certain target solution.

In Pavel [2006], Srikant et al. [2002], a limited investigation has been on some special type of pricing schemes, such as uniform pricing and proportional pricing. In Saraydar et al. [2002], pricing algorithms are developed in a heuristic way without a rigorous demonstration of convergence and its uniform pricing policy doesn't fully motivate the service of differentiation. Therefore, it still remains a challenge to find an appropriate framework to study the pricing issue analytically.

In this regard, we develop a state-space model from each channel's best response dynamics and offer a different perspective towards the non-cooperative game in optical networks. In our model, we view pricing as a control signal determined by the network manager and channel power as a network state. Using the classical control theory, we are able to study the pricing controllability of our system and design a pricing scheme to drive the network to a desirable OSNR level. The systematic approach adopted in this paper allows us to investigate other interesting problems in networks. such as robustness and time-delay.

The main contribution of this paper is to connect the state-space control theory to the non-cooperative power control in networks, and build a novel framework to address the issue of pricing in optical networks. We construct a linear system model based on the relation of OSNR and OSND and give a closed-form non-uniform pricing policy for achieving given desired OSNR levels. This paper is organized as follows. In section II, we review OSNR Nash game and formulate the state-space model to design pricing mechanism in section III. In section IV, we study the effects of modeling uncertainty and time delay on the stability. We conclude and point out some future work in section V.

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2. STATE-SPACE MODEL OF OSNR GAME

2.1 OSNR Nash Game Model

We consider the same optical network model described in Pavel [2006]. We let $\mathcal{N}$ denote the set of of channels are transmitted and u_i be the $i-$th channel input optical power (at Tx), and $\mathbf{u} = [u_1, ..., u_N]^T$ the vector of all channels' input powers. The ith channel optical OSNR is thus given as

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j\in\mathcal{N}} \Gamma_{i,j} u_j}, i \in \mathcal{N} \tag{1}$$

where $\mathbf{\Gamma}$ is the full $n \times n$ system matrix which characterizes the coupling between channels. $n_{0,i}$ denotes the ith channel noise power at the transmitter. An OSNR game without constraints is defined by a triplet $\langle \mathcal{N}, (A_i), (J_i)\rangle$. $\mathcal{N}$ is the index set of players or channels; A_i is the strategy set $\{u_i \mid u_i \in [u_{i,\min}, u_{i,\max}]\}$; and, J_i is the cost function, chosen such that minimizing the cost is related to maximizing OSNR level. In Pavel [2006], J_i is defined as

$$J_i(u_i, u_{-i}) = \alpha_i u_i - \beta_i \ln\left(1 + a_i \frac{u_i}{X_{-i}}\right), i \in \mathcal{N} \tag{2}$$

where α_i, β_i are channel specific parameters, that quantify the willingness to pay the price and the desire to maximize its OSNR, respectively, a_i is a channel specific parameter, X_{-i} is defined as $X_{-i} = \sum_{j\neq i} \Gamma_{i,j} u_j + n_{0,i}$. Provided that $\sum_{j\neq i} \Gamma_{i,j} \le a_i$,the resulting NE solution is given in a closed form by

$$\widetilde{\mathbf{\Gamma}} \mathbf{u}^* = \widetilde{\mathbf{b}}, \tag{3}$$

where $\widetilde{\Gamma}_{i,j} = a_i$, for $j = i$; $\widetilde{\Gamma}_{i,j} = \Gamma_{i,j}$, for $j \neq i$ and $\widetilde{b}_i = \frac{a_i b_i}{\alpha_i} - n_{0,i}$.

2.2 State-Space Approach to OSNR Game

State-space method is a powerful tool to study dynamical systems. It provides a different viewpoint from the input-output frequency domain method and allows a way of systematic study of coupled systems. In this section, we use the features of state-space methods based on the derivation of best response dynamics of the OSNR game to design pricing schemes.

The static best response function for payoff functions in (2) can be derived by taking its first derivative with respect to u_i as follows.

$$u_i = BR(\mathbf{u}_{-i}) = \arg\min_{u_i} J_i(u_i, \mathbf{u}_{-i}) = \frac{\beta_i}{\alpha_i} - \frac{X_{-i}}{a_i}. \tag{4}$$

From (4), we can derive the first-order best response dynamics as in (5).

$$\dot{x}_i = \frac{a_i \beta_i}{\alpha_i} - X_{-i} - a_i x_i. \tag{5}$$

or equivalently,

$$\dot{x}_i = -a_i x_i - \sum_{j\neq i} \Gamma_{i,j} x_j + \frac{a_i \beta_i}{\alpha_i} - n_{0,i}, \forall i \in \mathcal{N} \tag{6}$$

where x_i is equal to u_i quantitatively but denotes the unconstrained state variable of the channel power i. It easy to observe that the static Nash equilibrium corresponds to the equilibrium from the dynamical coupled system in (6).

To keep the state-space in a linear form, we can define optical signal-noise difference (OSND), based on (1) as

$$OSND_i = u_i - m_i = (1 - \Gamma_{i,j}) u_i - \sum_{i\neq j} \Gamma_{i,j} u_j, \forall i \in \mathcal{N} \tag{7}$$

where m_i is defined from (1) as $m_i = \Gamma_{i,i} u_i + \sum_{j\neq i} \Gamma_{i,j} u_j + n_{0,i}$. OSND measures transmission quality just as OSNR does. Since OSND measures the difference of the optical power, we will use unit dBm for it. It is obvious that the higher the value of OSND, the better the transmission quality will be.

For the simplicity of notation, let $\gamma_i^d = OSND_i$ and $\gamma_i^r = OSNR_i, i \in \mathcal{N}$. Subsequently $\gamma^d = [\gamma_i^d] : \mathcal{R}^N \to \mathcal{R}^N$ and $\gamma^r = [\gamma_i^r] : \mathcal{R}^N \to \mathcal{R}^N$. Since $\gamma_i^d = u_i - m_i$ and $\gamma_i^r = u_i/m_i$, we can relate γ_i^d and γ_i^r by equation (8).

$$\gamma_i^r = \frac{u_i}{u_i - \gamma_i^d}, i \in \mathcal{N}. \tag{8}$$

Definition 1. An ONSR vector γ^r is feasible if there exists a power vector $\mathbf{u} \in \mathcal{R}^N$ such that $\gamma_i^r = u_i/m_i = OSNR_i(\mathbf{u}) = \frac{u_i}{\Gamma_{i,i} u_i + X_{-i}}, \forall i \in \mathcal{N}$.

Since not all given γ^r can be realized by a power vector $\mathbf{u} \in \mathcal{R}^N$, Definition 1 provides a concept on the feasibility of γ^r that can be chosen. With a given γ_i^d, we can calculate γ_i^r from (8) by a γ^r-feasible choice of signal $\mathbf{u}$. In the following Theorem 1, we are going to show that under certain conditions, the translation between γ^r and γ^d can actually become one-to-one.

Theorem 1. Let $\Theta : \mathcal{R}^N \to \mathcal{R}^N$ be a mapping from OSNR γ^r and OSNR γ^d,i.e., $\gamma^d = \Theta(\gamma^r)$. Θ is bijective provided that the following conditions hold:

- (C1) $\gamma_i^r < \frac{1}{\sum_{j\in\mathcal{N}} \Gamma_{ij}}, \forall i \in \mathcal{N}$.
- (C2) $\sum_{j\in\mathcal{N}} \Gamma_{ij} < 1, \forall i \in \mathcal{N}$.

Proof. Suppose a $\gamma^r \in \mathcal{R}^N$ is given. From (1), we can obtain

$$(1 - \gamma_i^r \Gamma_{ii}) u_i - \gamma_i^r \sum_{j\neq i} \Gamma_{ij} u_j = \gamma_i^r n_{0,i},$$

or in the matrix form as

$$\widehat{\mathbf{\Gamma}} \mathbf{u} = \widehat{\mathbf{b}}, \tag{9}$$

where $\hat{b}_i = \gamma_i^r n_{0,i}, \forall i \in \mathcal{N}$, and

$$\widehat{\mathbf{\Gamma}} = \begin{bmatrix} 1 - \gamma_1^r \Gamma_{11} & -\gamma_1^r \Gamma_{12} & \cdots & -\gamma_1^r \Gamma_{1N} \\ -\gamma_2^r \Gamma_{21} & 1 - \gamma_2^r \Gamma_{22} & \cdots & -\gamma_2^r \Gamma_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ -\gamma_N^r \Gamma_{N1} & \cdots & \cdots & 1 - \gamma_N^r \Gamma_{NN} \end{bmatrix}.$$

Provided that $\gamma_i^r < \frac{1}{\sum_{j\in\mathcal{N}} \Gamma_{ij}}$, $\widehat{\mathbf{\Gamma}}$ is strictly diagonal dominant and nonsingular; the mapping $\Theta_1 : \mathcal{R}^N \to \mathcal{R}^N$ from $\mathbf{u}$ to γ^r is a one-to-one bijective mapping such that $\mathbf{u} = \widehat{\mathbf{\Gamma}}^{-1} \widehat{\mathbf{b}}$. On the other hand, OSND γ^d and $\mathbf{u}$ are related in a linear fashion via (10)

$$\mathbf{C} \mathbf{u} = \gamma^d, \tag{10}$$

where $\mathbf{C} \in \mathcal{R}^{N\times N}$ is given by

$$\mathbf{C} = \begin{bmatrix} 1-\Gamma_{11} & -\Gamma_{12} & \cdots & -\Gamma_{1N} \\ -\Gamma_{21} & 1-\Gamma_{22} & \cdots & -\Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_{N1} & \cdots & \cdots & 1-\Gamma_{NN} \end{bmatrix}.$$

When $\sum_j \Gamma_{ij} < 1$, $\mathbf{C}$ is strictly diagonally dominant and there exists a unique solution $\mathbf{u} = \mathbf{C}^{-1}\mathbf{r}^d$. Thus, the mapping $\Theta_2 : \mathcal{R}^N \rightarrow \mathcal{R}^N$ from γ^d to $\mathbf{u}$ is bijective. Combining (9) and (10), we have

$$\gamma^d = \mathbf{C}\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}. \tag{11}$$

Under the conditions of $\gamma_i^r < \frac{1}{\sum_{j\in\mathcal{N}}\Gamma_{ij}}$ and $\sum_{j\in\mathcal{N}}\Gamma_{ij} < 1$, OSNR and OSND are one-to-one. Therefore, the mapping $\Theta = \Theta_1 \circ \Theta_2$ is bijective. ■

Corollary 1. An OSNR vector γ^r is feasible if and only if

$$\widehat{\mathbf{b}} \in \mathbf{R}(\widehat{\mathbf{\Gamma}})$$

The Corollary is an immediate result following (9) and provides a way to check feasibility of γ^r as defined in Definition 1.

Corollary 2. Suppose OSNR target is uniform among users, i.e., $\gamma^r = \gamma_0^r\mathbf{1}, \gamma_0^r \in \mathcal{R}$. An increase in γ_0^r will result in a lower $\|\gamma^d\|$, and vice versa, provided that $\|\widehat{\mathbf{\Gamma}}\| \leq \frac{1}{\gamma_0^r}$ and the conditions (C1), (C2) in Theorem 1 hold.

Proof. Let's take the norm on both sides of (11) and under uniform target γ_0^r we obtain

$$\|\gamma^d\| = \|\mathbf{C}\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}\| \leq \|\mathbf{C}\|\|(\mathbf{I}-\gamma_0^r\mathbf{\Gamma})^{-1}\|\|\widehat{\mathbf{b}}\|. \tag{12}$$

Since $\gamma_0^r\|\Gamma\| \leq 1$, using Lemma 2.3.3 in Golub and Loan [1996], we obtain

$$\|\gamma^d\| \leq \frac{\|\mathbf{C}\|\|\widehat{\mathbf{b}}\|}{1-\gamma_0^r\|\widehat{\mathbf{\Gamma}}\|}. \tag{13}$$

From (13), an increase in γ_0^r results in $\|\gamma^d\|$. ■

With (6) and (7), the state-space form of the best response dynamics is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} - \mathbf{n} \tag{14}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} - \mathbf{n} \tag{15}$$

where $\mathbf{x} \in \mathcal{R}^N$ is the state-vector physically modeling the evolution of the power vector $\mathbf{u}$ in optical networks; $\mathbf{v} \in \mathcal{R}^N$ is a vector of control variables relating to the pricing parameters component-wise by $v_i = 1/\alpha_i, \forall i$; $\mathbf{y}$ is the output vector that observes OSND. Matrices $\mathbf{A} \in \mathcal{R}^{N\times N}$, $\mathbf{B} \in \mathcal{R}^{N\times N}$ and vector $\mathbf{n} \in \mathcal{R}^N$ are given respectively as follows.

$$\mathbf{A} = \begin{bmatrix} -a_1 & -\Gamma_{12} & \cdots & -\Gamma_{1N} \\ -\Gamma_{21} & -a_2 & \cdots & -\Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_{N1} & \cdots & \cdots & -a_N \end{bmatrix},$$

$\mathbf{B} = \mathrm{diag}\{a_1\beta_1, a_2\beta_2, \cdots, a_N\beta_N\}$ and vector $\mathbf{n} \in \mathcal{R}^N = [n_{0,1}, n_{0,2}, \cdots, n_{0,N}]^T$.

State-space model (14) is a multi-input and multi-output (MIMO) system; however, the model is single-input and multi-output (SIMO) system, if $\mathbf{B}\mathbf{v}$ in (14) is replaced by $\overline{\mathbf{B}}v$, where $\overline{\mathbf{B}} \in \mathcal{R}^N$ is a vector given by $\overline{\mathbf{B}} = \mathbf{B}\mathbf{1} = [a_1\beta_1, \cdots, a_N\beta_N]^T$ and $v \in \mathcal{R}, v = 1/\alpha$ is a scalar pricing parameter. SIMO represents a uniformly priced Nash game, in which the network assigns a single network price to every user.

Due to the nonlinearity of OSNR expression, a direct OSNR output formulation will result in solving for a difficult nonlinear set of equations. Without losing generality, we next study a design of pricing to achieve desirable OSNR in the form of OSND as OSNR can be determined from OSND by (8) under (C1) and (C2). In this way, we are able to take the advantage of linearity of OSND and derive a closed form for the pricing scheme.

The state-space model (14) naturally allows us to examine the pricing design problems based on classical control theory by viewing it as controller. In the following development, we will ignore the term $\mathbf{n}$, since it is important to first develop some insightful results and then consider the model by viewing noise as a disturbance to the system. Furthermore, the term $\mathbf{n}$ in a typical network is usually on the magnitude of 1.0×10^{-4}mW, that is, less than $1\%-5\%$ of the common signal power.

3. PRICING CONTROL DESIGN

A common problem in OSNR Nash game is to design a pricing mechanism so that players can reach their OSNR targets at their steady-state, i.e., the best response dynamics $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}$ yields a solution $\mathbf{x}(t) = \mathbf{x}'$ at a sufficiently large t and for some given $\mathbf{x}'$ that corresponds to target OSNR from (1). In this section, instead of dealing directly with OSNR, we investigate the problem using OSND targets, as its linearity allows us to give some fundamental results in controllability and observability.

Based on classical control theory, it is obvious that (14) is pricing controllable if the controllability matrix has full rank. Since it is assumed that $a_i\beta_i \neq 0, \forall i$ in (14), diagonal matrix $\mathbf{B}$ is non-singular and therefore the OSNR game described in Section II-B is inherently pricing controllable.

In addition, we also can conclude that OSNR game is also power observable if $\|\mathbf{\Gamma}\| < 1$, since $\mathbf{C} = \mathbf{I}-\mathbf{\Gamma}$ is nonsingular from Lemma 2.3.3 in Golub and Loan [1996] and thus the observability matrix is full rank.

3.1 Constant Reference Signal Tracking

In this subsection, we study a regulator problem in which the output is desired to track given feasible OSNR levels. We will use classical regulator theory Wonham [1979] to develop insights into this pricing problem. Let's construct a dynamical system whose output is the given OSND $\overline{\gamma^d}$ obtained from OSNR $\overline{\gamma^r}$. Such a reference system is given by (16).

$$\dot{\mathbf{w}} = \mathbf{S}\mathbf{w} \tag{16}$$

$$\mathbf{y} = \mathbf{C}_d\mathbf{w} = \mathbf{y}_d \tag{17}$$

Since the given performance target OSND $\overline{\gamma^d}$ is a constant signal, we let $\mathbf{S} \in \mathcal{R}^{N\times N} = \mathbf{0}$, $\mathbf{C}_d \in \mathcal{R}^{N\times N} = \mathrm{diag}(\overline{\gamma_i^d})$,

$\mathbf{y}_d = \overline{\gamma^d}$ and $\mathbf{w} \in \mathcal{R}^N, \mathbf{w}(0) = \mathbf{1}$. It is obvious that (16) will yield $\mathbf{w}(t) = \mathbf{1}, \forall t$. We need to find a $\mathbf{v}$ such that $\mathbf{e}(t) = \mathbf{y} - \mathbf{y}_d$ will converge to $\mathbf{0}$.

Theorem 2. Suppose $\overline{\mathbf{A}} = \mathbf{A} + \mathbf{B}\mathbf{F}_1$ is Hurwitz and $\mathbf{F}_2 = \overline{\mathbf{F}}_2 - \mathbf{F}_1\mathbf{\Pi}$. A regulator (18)

$$\mathbf{v} = \overline{\mathbf{F}}_2\mathbf{w} + \mathbf{F}_1(\mathbf{x} - \mathbf{\Pi}\mathbf{w}) = \mathbf{F}_1\mathbf{x} + \mathbf{F}_2\mathbf{w} \tag{18}$$

exists if and only if there exists maps $\overline{\mathbf{F}}_2 \in \mathcal{R}^{N\times N}$ and $\mathbf{\Pi} : \mathcal{R}^N \to \mathcal{R}^N$ such that

$$0 = \mathbf{A}\mathbf{\Pi} + \mathbf{B}\overline{\mathbf{F}}_2 \tag{19}$$

$$\mathbf{C}\mathbf{\Pi} = \mathbf{C}_d \tag{20}$$

Equations (19) are called classic regulator equations or FBI equations named after Francis-Byrnes-Isidori Wonham [1979]. Solving this set of equations, we obtain

$$\overline{\mathbf{F}}_2 = -\mathbf{B}^{-1}\mathbf{A}\mathbf{C}^{-1}\mathbf{C}_d \tag{21}$$

$$\mathbf{\Pi} = \mathbf{C}^{-1}\mathbf{C}_d \tag{22}$$

under the assumption that

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \text{ is non-singular.} \tag{23}$$

and $\mathbf{B}, \mathbf{C} \in \mathcal{R}^{N\times N}$. Condition (23) is actually equivalent to $\mathbf{C}$ and $\mathbf{B}$ being non-singular. This result is summarized in the following proposition.

Theorem 2 gives a necessary and sufficient condition to design pricing values for an asymptotic tracking of given transmission performance OSND targets in OSNR Nash games. Different from an open-loop style pricing described in Srikant et al. [2002], Pavel [2006], the general controller described in Theorem 2 provides a feedback mechanism on the OSNR game dynamics and gives another degree of freedom in $\mathbf{F}_1$ to adjust the dynamical response in addition to the choice of a_i. It is desirable because the network is able to adjust the dynamics via $\mathbf{F}_1$ once a_i are set and submitted to the network. For instance, if we choose a_i such that $-\mathbf{A}$ is strictly diagonally dominant, i.e., $a_i > \sum_i \Gamma_{i,j}, \forall i$. $\overline{\mathbf{A}} = \mathbf{A}$ itself is already Hurwitz. Thus, we can simply let $\mathbf{F}_1 = 0$ and the pricing design can be simply found by

$$\mathbf{v} = \overline{\mathbf{F}}_2\mathbf{w} = -\mathbf{B}^{-1}\mathbf{A}\mathbf{C}^{-1}\overline{\gamma^d}, \tag{24}$$

and thus the pricing parameter in OSNR game is found as $\alpha_i = 1/v_i$. However, with the presence of network model uncertainties and network delay, we may need to further use $\mathbf{F}_1$ to ensure stability of the OSNR game. This issue will be elaborated in Section V.

Remark 1. Observing from (24), we can see that the norm of the pricing vector $\|\alpha\|$ increases as OSND $\|\overline{\gamma^d}\|$ decreases. We can use Corollary 2 to relate the pricing with OSNR: as the uniform OSNR target γ_0^r increases, we need to have on average smaller prices indicated by $\|\alpha\|$. This observation allows us to increase OSNR target levels in an allowable region by adjusting prices.

4. STABILITY UNDER MODELING UNCERTAINTY AND TIME-DELAY

In Section IV, we discussed pricing design to track a desired OSNR output. Since the stability of OSNR game is ensured by the assumption of the diagonal dominance, the first-order best response dynamics of the game is stable as an open loop system. However, with uncertainties in network conditions and modeling, it is necessary for us to ensure the stability using feedback in the pricing design. In this section, we take into account another two aspects of the state-space model. One is the parametrical uncertainty in OSNR network model, namely, uncertainties in variable $\Gamma_{i,j}$, which may deviate from its nominal value due to manufacturing error, temperature, and other network conditions. Another is delay in the network, mainly coming from power updates in each iteration at the interface with electronics.

4.1 Stability under Modeling Uncertainty

The OSNR network model described in Section II may be subject to parametrical uncertainties arising from device manufacturing, signal measurement, network condition, etc. In particular, parameters Γ_{ij} may change from its nominal value to some $\Gamma'_{i,j}$. Such uncertainties may be small but need to be taken into account to ensure the stability of the pricing control algorithm. One simple modification to the state-space model in (14) to study this problem is to use additive uncertainty, i.e.

$$\dot{\mathbf{x}} = (\mathbf{A} + \widetilde{\mathbf{A}})\mathbf{x} + \mathbf{B}\mathbf{v}. \tag{25}$$

where $\widetilde{A}_{i,j} = \Delta\Gamma_{i,j}$ for $i \neq j$ and $\widetilde{A}_{i,j} = \delta_{n,i}$ for $i = j$, in particular, to model the signal noise, which is usually less than 5% of the signal power.

Definition 2. (Stable Pricing Under Additive Uncertainty) A pricing scheme $\mathbf{v} = \mathbf{F}_1\mathbf{x} + \mathbf{F}_2\mathbf{w}$ is stable under uncertainties $\widetilde{\mathbf{A}}$ if the disturbed best response dynamics (25) is stable for any given

$$\widetilde{\mathbf{A}} \in \{\mathbf{\Delta} \in \mathcal{R}^{N\times N} \mid \|\mathbf{\Delta}\| < \infty\}.$$

Theorem 3. A pricing scheme $\mathbf{v}$ is stable under additive modeling uncertainty $\widetilde{\mathbf{A}}$ if and only if $\mathbf{B}\mathbf{F}_1$ is satisfies the following inequality

$$\|\mathbf{M}_a(s)\|_\infty < \frac{1}{\sqrt{\lambda_{\max}(\widetilde{\mathbf{A}}^T\widetilde{\mathbf{A}})}}, \tag{26}$$

where $\mathbf{M}_a(s) = (s\mathbf{I} - (\mathbf{A} + \mathbf{B}\mathbf{F}_1))^{-1}$, $\lambda_{\max}(\cdot)$ is the largest eigenvalue of matrix $(\cdot)$ and $\|F(s)\|_\infty = \sup_{\omega\in\mathcal{R}} \overline{\sigma}(F(j\omega))$

Proof. The stability study of (25) with controller $\mathbf{v} = \mathbf{F}_1\mathbf{x} + \mathbf{F}_2\mathbf{w}$ can be reduced to the study of internal stability, i.e., the stability of the following linear system:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{F}_1)\mathbf{x} + \widetilde{\mathbf{A}}\mathbf{x}. \tag{27}$$

Its representation of the best response dynamics is shown in Figure 1. It can be further reduced into a $\Delta - M_a$ configuration, where $\mathbf{M}_a(s) = (s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{F}_1)^{-1}$ and $\Delta = \widetilde{\mathbf{A}}$. Using the small-gain theorem Zhou et al. [2005], the system is stable if and only if $\|\mathbf{M}_a(s)\|_\infty \|\widetilde{\mathbf{A}}\|_2 < 1$. Thus, we complete the proof. ■

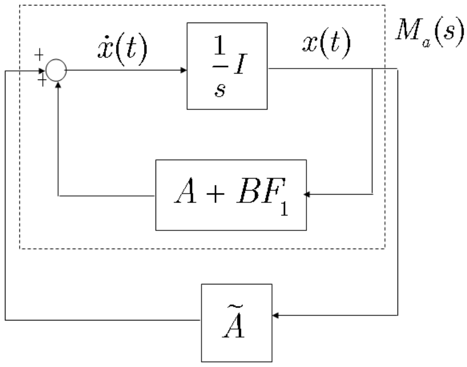

Fig. 1. Block Diagram Representation Under Additive Uncertainty

Theorem 3 indicates that for a given additive uncertainty matrix $\widetilde{\mathbf{A}}$, a stable pricing scheme is to design $\mathbf{BF}_1$ such that the inequality (26) holds. In other words, if the uncertainty in the modeling is bounded from above in norm, i.e., $\|\widetilde{\mathbf{A}}\|_2 \leq 1/\overline{\beta}, \overline{\beta} > 0$, then a pricing design needs to satisfy $\| (s\mathbf{I} - (\mathbf{A} + \mathbf{BF}_1))^{-1} \|_\infty < \overline{\beta}$.

Lemma 1. Let the transfer function $G(s) = (s\mathbf{I} - (\mathbf{A} + \mathbf{BF}_1))^{-1}$. $\|G(s)\|_\infty < \overline{\beta}$ if and only if $\mathbf{J}(\overline{\beta})$ has no eigenvalues on $j\omega$-axis, where

$$\mathbf{J}(\overline{\beta}) = \begin{bmatrix} \mathbf{A} + \mathbf{BF}_1 & \overline{\beta}^{-2}\mathbf{I} \\ \mathbf{I} & -\mathbf{A}^T - \mathbf{F}_1^T\mathbf{B}^T \end{bmatrix}$$

Lemma 1 is a direct application of Lemma 4.7 from Zhou et al. [2005]. It leads to the following design criterion on $\mathbf{F}_1$.

Theorem 4. Suppose $\mathbf{F}_1$ be a diagonal matrix given by $\mathbf{F}_1 = \mathrm{diag}\{f_i\}$, where f_i is a design parameter. The pricing scheme is stable under additive uncertainty $\widetilde{\mathbf{A}}$ such that $\|\widetilde{\mathbf{A}}\|_2 \leq 1/\overline{\beta}$, provided that $f_i < 1/\beta_i - \varphi_i$ or $f_i > 1/\beta_i + \varphi_i$, where

$$\varphi_i = \frac{\xi_i}{a_i\beta_i} \text{ and } \xi_i = \max\left(\sum_{j\neq i}\Gamma_{ij} + \overline{\beta}^{-2}, \sum_{j\neq i}\Gamma_{ji} + 1\right).$$

Proof. We use Gershgorin's Theorem to ensure that eigenvalues of $\mathbf{J}$ does not have eigenvalue onf $j\omega$-axis. Therefore, it requires that $|-a_i + a_i\beta_i f_i| > \sum_{j\neq i}\Gamma_{ij} + \overline{\beta}^{-2}$, and in addition, $|a_i - a_i\beta_i f_i| > \sum_{j\neq i}\Gamma_{ji} + 1$. From the above two inequalities, we arrive at $|1 - \beta_i f_i| > \frac{\xi_i}{a_i}$. And equivalently we have $f_i < 1/\beta_i - \varphi_i$, and $f_i > 1/\beta_i + \varphi_i$. ■

4.2 Stability under Time-Delay

From an algorithmic point of view, the best response dynamics in practice has delays in measuring the power from other channels. It has been justified in Stefanovic and Pavel [2007] that delays occur on the path from channel power u_j to the OSNR outputs $OSNR_i$ and from $OSNR_i$ to source u_i. For example, in an optical span about 100km, the optical signal travels between the receiver and transmitter for power regulation in approximately on the order of 1ms. The thousands of kilometer-long optical network may easily incur tens of milliseconds of delay, which is beyond negligence. We use $\tau_{i,j} \geq 0$ to represent an aggregate time-delay from source j to source i via $OSNR_i$. The best response with time-delay, therefore, is given by

$$\dot{x}_i(t) = -a_i x_i(t) - \sum_{j\neq i}\Gamma_{ij}x_j(t - \tau_{i,j}) + \frac{a_i\beta_i}{\alpha_i}.$$

Uniform Delay: In this section, we assume that time delays for each channel are uniform, i.e., $\tau_{i,j} = \tau, \forall i, j \in \mathcal{N}$. We can express the best response in the following standard form

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t - \tau) + \mathbf{B}\mathbf{v}(t) - \mathbf{n}.$$

Suppose the pricing controller is chosen as $\mathbf{v} = \mathbf{F}_1\mathbf{x} + \mathbf{F}_2\mathbf{w}$ as in section IV-B. Therefore the stability study of the affine system above is equivalent to the study of the linear system without constant term.

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_0 + \mathbf{BF}_1)\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t - \tau). \tag{28}$$

where matrix $\mathbf{A}_0 = \mathrm{diag}\{-a_1, -a_2, \cdots, -a_N\}$ and $\mathbf{A}_1$ are given by

$$\mathbf{A}_1 = \begin{bmatrix} 0 & -\Gamma_{12} & \cdots & -\Gamma_{1N} \\ -\Gamma_{21} & 0 & \cdots & -\Gamma_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ -\Gamma_{N1} & \cdots & -\Gamma_{N,N-1} & 0 \end{bmatrix}$$

Theorem 5. (Gu et al. [2003]) The system (28) is stable independent of delay if and only if

(1) $\mathbf{A}_0 + \mathbf{BF}_1$ is Hurwitz
(2) $\mathbf{A}_0 + \mathbf{BF}_1 + \mathbf{A}_1$ is Hurwitz, and,
(3) $$\rho((j\omega\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_1) < 1, \forall\omega > 1 \tag{29}$$
where $\rho(\cdot)$ denotes the spectral radius of a matrix.

In Theorem 5, $\mathbf{BF}_1$ provides a degree of freedom for design to ensure stability with the presence of delay. With the assumption of diagonal dominance on $-\mathbf{A}$, it can also be shown that conditions in Theorem 5 are satisfied with $\mathbf{F}_1 = 0$. It is obvious that $\mathbf{A}_0 + \mathbf{BF}_1$ is Hurwitz due to the fact that $a_i > 0$ without feedback pricing control, i.e., $\mathbf{F}_1 = 0$. In addition, when $\mathbf{F}_1 = 0$, $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$ has it eigenvalues on the open right half s-plane, from the assumption of the strict diagonal dominance. Thus, the second condition in Theorem 5 can also be easily verified to be true. It is not obvious to show that the third condition is obviously satisfied by system (28) when $\mathbf{F}_1 = 0$. This result is summarized in Theorem 6.

Theorem 6. Suppose $\mathbf{F}_1 = 0$. If $\widetilde{\mathbf{\Gamma}}$ is strictly diagonally dominant, i.e., $a_i > \sum_{j\neq i}\Gamma_{i,j}$, then the system (28) is stable independent of delay.

Proof. Let $\mathbf{H} = (j\omega\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{A}_1$ and r_i be the radius of the Gershgorin disc for $\mathbf{H}$ given by $r_i = \sum_{j\neq i}|H_{ij}|$. From the definition of $\mathbf{H}$, we obtain $r_i = \frac{\sum_{j\neq i}|\Gamma_{ij}|}{(j\omega + a_i)}$. Therefore, $|r_i| = \frac{\sum_{j\neq i}|\Gamma_{ij}|}{\sqrt{a_i^2 + \omega^2}} < 1, \forall\omega > 0$. Based on the Gershorgin Theorem, $\rho(\mathbf{H}) < 1$. It is obvious to verify that $\mathbf{A}_0$ and $\mathbf{A}_0 + \mathbf{A}_1$ are stable, under the condition of strict diagonal dominance. Using the sufficiency of Theorem 5, we complete the proof. ■

Non-uniform Delay: Theorem 6 yields results that coincide with Theorem 1 obtained in Stefanovic and Pavel

[2007] for uniform delays from a different point of view. In this section, we assume the time delay are not uniform in a way such that $\tau_{i,j} = \tau_i, \forall i, j \in \mathcal{N}$. The time-delayed best response dynamics is thus given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \sum_{k=1}^{N} \overline{\mathbf{A}}_k \mathbf{x}(t - \tau_k) + \mathbf{B}\mathbf{v}(t). \tag{30}$$

where $\overline{\mathbf{A}}_k^j = \mathbf{A}^k$, for $j = k$; $\overline{\mathbf{A}}_k^j = \mathbf{A}^k$, for $j \neq k$; and $\mathbf{A}^j$ denotes $j-$th row of matrix $\mathbf{A}^j$.

Theorem 7. (Gu et al. [2003]) Suppose that $\mathbf{A}_0 + \mathbf{B}\mathbf{F}_1$ is Hurwitz. Then the system (30) is stable independent of delay if

$$\|(s\mathbf{I} - (\mathbf{A}_0 + \mathbf{B}\mathbf{F}_1))^{-1} (\mathbf{A}_1, \cdots, \mathbf{A}_N)\|_\infty < \frac{1}{\sqrt{N}}.$$

Theorem 8. Suppose $\mathbf{F}_1 = 0$, $\sum_{j\neq i} \Gamma_{ij} < a_i/N$, then the system (30) is stable independent of delay.

Proof. Let $\overline{\mathbf{H}}(s) = (s\mathbf{I} - \mathbf{A}_0)^{-1} (\mathbf{A}_1, \cdots, \mathbf{A}_N)$.

$$\begin{aligned} \|\overline{\mathbf{H}}(s)\|_\infty &= \sup_\omega \overline{\sigma}(\overline{\mathbf{H}}(j\omega)) = \sup_\omega \|\overline{\mathbf{H}}(j\omega)\|_2 \qquad (31) \\ &\leq \sqrt{N} \sup_\omega \max_i \frac{\sum_j \Gamma_{ij}}{\sqrt{a_i^2 + \omega^2}} \\ &\leq \sqrt{N} \max_i \frac{\sum_j \Gamma_{ij}}{a_i}, . \end{aligned}$$

Suppose $\sum_{j\neq i} \Gamma_{ij} < a_i/N$, $\|\overline{\mathbf{H}}(s)\|_\infty < 1/\sqrt{N}$. From Theorem 7, we can conclude that the system (30) is stable independent of delay. ■

Theorem 8 shows that a stronger condition on diagonal dominance is required to ensure stability independent of delay without $\mathbf{F}_1$. However, we may relax this strong condition into strict diagonal dominance condition (as in Theorem 6) by using $\mathbf{F}_1$ as another degree of freedom.

Theorem 9. Let $\mathbf{R} = \mathbf{B}\mathbf{F}_1$. Suppose $\sum_j R_{ij} > a_i \left(1 - \frac{1}{N}\right) > 0$, and $\sum_{j\neq i} \Gamma_{ij} < a_i$, then the system (30) is stable independent of delay.

Proof. Following a similar argument in the proof of Theorem 8, we obtain

$$\begin{aligned} \|\overline{\mathbf{H}}(s)\|_\infty &\leq \sqrt{N} \sup_\omega \max_i \sum_j |\overline{H}_{ij}| \qquad (32) \\ &= \sqrt{N} \sup_\omega \max_i \frac{\sum_j |-\Gamma_{ij} + R_{ij}|}{\sqrt{a_i^2 + \omega^2}} \qquad (33) \\ &= \sqrt{N} \max_i \left(\frac{\sum_j \Gamma_{ij}}{a_i} - \frac{\sum_j R_{ij}}{a_i} \right) \leq \frac{1}{\sqrt{N}}. \qquad (34) \end{aligned}$$

Therefore, the results follow after applying Theorem 7. ■

Theorem 9 implies a way to design $\mathbf{F}_1$ to reach stability independent of delay. Moreover, since $\mathbf{B}$ is a diagonal matrix, a direct condition on $\mathbf{F}_1$ becomes $\sum_j (F_1)_{ij} > \frac{1}{\beta_i}\left(1 - \frac{1}{N}\right)$.

5. CONCLUSION

In this paper, we developed a state-space framework for the pricing design in the OSNR game. The classical control theory enables us to view pricing from a control design point of view and helps us develop insights into the pricing controllability of the network. As a result, we use regulator equations to find an analytical pricing policy so that the network is able to attain a given set of OSNR targets. We hope this study will initiate further investigations and extensions of this model in this area. In this paper, we did not take into account capacity constraints arising from a physical operating threshold in the network. It will be more realistic to find a control scheme that can only allow power states evolve within a simplex.

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