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Discrete Time Robust H^∞ Control of a Class of Nonlinear Uncertain Systems ⋆

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Abstract: This paper presents an approach to discrete time robust H^∞ control for a class of nonlinear uncertain systems based on the use of Sum Quadratic Constraints. The approach involves controllers which include copies of the system nonlinearities in the controller. The nonlinearities being considered are those which satisfy a certain global Lipschitz condition. The linear part of the controller is synthesized using linear robust H^∞ control theory and this leads to a nonlinear controller which gives an upper bound on the attainable disturbance attenuation level.

Keywords: Discrete-time systems, optimal control, Riccati equations, uncertain linear systems.

1. INTRODUCTION

This paper presents an approach to discrete-time robust H^∞ control for a class of nonlinear uncertain systems based on linear H^∞ control theory (see e.g. Basar and Bernhard (1995); Green and Limebeer (1995); Savkin and Petersen (1996)) and the use of Sum Quadratic Constraints (SQCs for short) which exploit repeated nonlinearities satisfying a global Lipschitz condition. The fundamental idea behind our approach is to modify the standard SQC approach to robust control by including a copy of the nonlinearity in the controller as shown in Figure 1. The idea of using a copy of the nonlinearity in nonlinear observer design was previously used in the paper Arcak and Kokotovic (2001). However, in contrast to Arcak and Kokotovic (2001), we construct the linear part of the controller in order to obtain a guaranteed level of disturbance attenuation. In our case, we combine both nonlinearities into the nonlinear system model and then use an SQC which exploits the repeated nonlinearity; see Figure 2. In this case, the nonlinear controller design problem is converted to a linear robust controller design problem. This approach enables us to use a discrete time version of the robust H^∞ control theory of Savkin and Petersen (1996) to construct the linear part of the controller and then the nonlinear controller is constructed by including a copy of the plant nonlinearity. The approach of this paper is related to that of the paper Petersen (2006) which considers a continuous time guaranteed cost control problem.

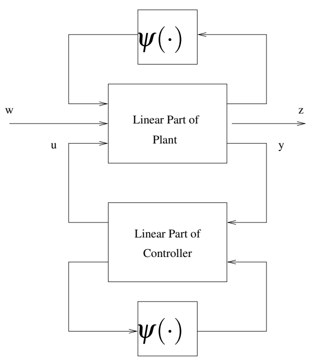

Fig. 1. Nonlinear system with nonlinear controller.

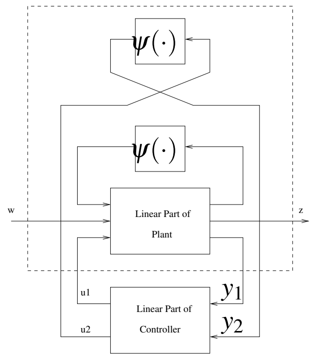

Fig. 2. Nonlinear system and controller redrawn with repeated nonlinearity.

2. PROBLEM STATEMENT

We consider a nonlinear uncertain system defined using a similar framework to that considered in Savkin and Petersen (1996); Moheimani et al. (1995). Let $k = 1, 2, \cdots, N$ be the finite time horizon. Then, we consider the state equations:

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$$
\begin{aligned}
x(k+1) &= A(k)x(k) \\
&\quad + \left[\sum_{i=1}^{f} \check{B}_{1,i}(k)\xi_i(k) + \sum_{j=1}^{g} \bar{B}_{1,j}(k)\mu_j(k)\right] \\
&\quad + B_1(k)w(k) + B_2(k)u(k); \quad x(1) = 0,
\end{aligned}
$$

$$
\begin{aligned}
\zeta_i(k) &= \check{C}_{1,i}(k)x(k) + \check{D}_{12,i}(k)u(k); \;\; i = 1,2,\ldots,f; \\
\nu_j(k) &= \bar{C}_{1,j}(k)x(k) + \bar{D}_{12,j}(k)u(k); \;\; j = 1,2,\ldots,g; \\
z(k) &= C_1^z(k)x(k) + D_{12}^z(k)u(k), \\
y(k) &= C_2(k)x(k) \\
&\quad + \left[\sum_{i=1}^{f} \check{D}_{21,i}(k)\xi_i(k) + \sum_{j=1}^{g} \bar{D}_{21,j}(k)\mu_j(k)\right] \\
&\quad + D_{21}(k)w(k); \qquad (1)
\end{aligned}
$$

where $x(k) \in \mathbf{R}^n$ is the *state*, $u(k) \in \mathbf{R}^m$ is the *control input*, $w(k) \in \mathbf{R}^p$ is the noise input, $\zeta_1(k) \in \mathbf{R}^{h_1}, \ldots, \zeta_f(k) \in \mathbf{R}^{h_f}$ are the *uncertainty outputs*, $\nu_1(k) \in \mathbf{R}, \ldots, \nu_g(k) \in \mathbf{R}$ are *nonlinearity outputs*, $\xi_1(k) \in \mathbf{R}^{r_1}, \ldots, \xi_f(k) \in \mathbf{R}^{r_f}$ are the *uncertainty inputs*, $\mu_1(k) \in \mathbf{R}, \ldots, \mu_g(k) \in \mathbf{R}$ are *nonlinearity inputs*, $z(k) \in \mathbf{R}^{n_z}$ is controlled output, and $y(k) \in \mathbf{R}^l$ is the *measured output*.

We assume, for simplicity, that

$$
\begin{aligned}
\check{D}_{21,i}(k) &\equiv 0 \;\; \forall \;\; i = 1,2,\ldots,\tilde{f}; \\
\bar{D}_{21,j}(k) &\equiv 0 \;\; \forall \;\; j = 1,2,\ldots,\tilde{g}; \\
\check{B}_{1,i}(k) &\equiv 0 \;\; \forall \;\; i = \tilde{f}+1, \tilde{f}+2, \ldots, f, \qquad (2) \\
\bar{B}_{1,j}(k) &\equiv 0 \;\; \forall \;\; j = \tilde{g}+1, \tilde{g}+2, \ldots, g.
\end{aligned}
$$

The nonlinearity inputs are related to the nonlinearity outputs by the following nonlinear relations

$$\mu_j(k) = \psi_j(\nu_j(k)) \;\; \forall j = 1,2,\ldots,g \qquad (3)$$

where the nonlinear functions $\psi_j(\cdot)$ are such that

$$\psi_j(0) = 0 \;\; \forall \;\; j = 1,2,\ldots,g \qquad (4)$$

and satisfy the following global Lipschitz conditions:

$$|\psi_j(\nu_1) - \psi_j(\nu_2)| \le \beta_j |\nu_1 - \nu_2| \qquad (5)$$

for all $\nu_1, \; \nu_2$ and for all $j = 1,2,\ldots,g$.

The uncertainty in the system is described by the following Sum Quadratic Constraints.

Definition 1. (Sum Quadratic Constraints; e.g., see Moheimani et al. (1995).) An uncertainty $\xi = (\xi(1), \cdots, \xi(N))$ of the form $\xi(k) = \phi(k,x,u,w)$ is admissible uncertainty for the system (1), (3) if, given any control input sequence $u(\cdot)$, noise input sequence $w(\cdot)$, and any corresponding solution to the system (1), (3), then

$$\sum_{k=1}^{N} \|\xi_i(k)\|^2 \le \sum_{k=1}^{N} \|\zeta_i(k)\|^2 + d_i, \qquad (6)$$

for all $i = 1,\ldots,f$. Here the d_i are given positive constants.

We wish to construct a nonlinear dynamic output feedback controller (say K_c) which leads to an upper bound on the associated worst-case disturbance attenuation level $\gamma_c \ge 0$ (see Savkin and Petersen (1996));

$$\gamma_c^2 := \sup_{w(\cdot)} \sup_{\xi(\cdot) \in \Xi} \frac{\sum_{k=1}^{N} \|z(k)\|^2 + x_{N+1}' Q_{N+1} x_{N+1} - \tau \sum_{i=1}^{f} d_i}{\sum_{k=1}^{N} \|w(k)\|^2}. \qquad (7)$$

Here τ is a positive constant, and $\tau \sum_{i=1}^{f} d_i$ is a measure of how much uncertainty we tolerate when $w = 0$.

The class of controllers considered are nonlinear output feedback controllers of the form

$$
\begin{aligned}
x_c(k+1) &= A_c(k)x_c(k) + \sum_{j=1}^{g} \bar{B}_{c,j}(k)\tilde{\mu}_j(k) \\
&\quad + B_c(k)y(k); \qquad x_c(0) = x_{c0} \\
\tilde{\nu}_j(k) &= \bar{C}_{c,j}(k)x_c(k); \;\; j = 1,2,\ldots,g \\
u(k) &= C_c(k)x_c(k) \qquad (8)
\end{aligned}
$$

where

$$\tilde{\mu}_j(k) = \psi_j(\tilde{\nu}_j(k)) \qquad (9)$$

for $j = 1,2,\ldots,g$. That is, we include copies of the nonlinearities (3) in the controller.

We first move the nonlinearities (9) into the plant description and introduce some new notations for the inputs and outputs of the system (8). That is, we introduce the notation:

$$
\tilde{y}(k) \triangleq \begin{bmatrix} y(k) \\ \tilde{\mu}_1(k) \\ \vdots \\ \tilde{\mu}_g(k) \end{bmatrix}; \;\; \tilde{u}(k) \triangleq \begin{bmatrix} u(k) \\ \tilde{\nu}_1(k) \\ \vdots \\ \tilde{\nu}_g(k) \end{bmatrix};
$$

$$
\tilde{B}_c \triangleq \begin{bmatrix} B_c & \bar{B}_{c,1} & \ldots & \bar{B}_{c,g} \end{bmatrix}; \tilde{C}_c \triangleq \begin{bmatrix} C_c \\ \bar{C}_{c,1} \\ \vdots \\ \bar{C}_{c,g} \end{bmatrix}. \qquad (10)
$$

Using this notation, the controller state equations (8) can be rewritten as

$$
\begin{aligned}
x_c(k+1) &= A_c(k)x_c(k) + \tilde{B}_c(k)\tilde{y}(k); \;\; x_c(0) = x_{c0} \\
\tilde{u}(k) &= \tilde{C}_c(k)x_c(k) \qquad (11)
\end{aligned}
$$

and the problem of controlling the nonlinear uncertain system (1), (3), (6) via the nonlinear controller (8), (9) is equivalent to the problem of controlling the nonlinear uncertain system (1), (3), (6), (9), (with repeated nonlinearities) via the linear controller (11).

Sum quadratic constraints for the repeated nonlinearities. The basis of our approach is to characterize the repeated nonlinearities (3), (9) by corresponding SQCs in order to allow a discrete-time analogue of the results of Savkin and Petersen (1996) to be applied. To obtain these SQCs, we observe that the conditions (5) imply

$$[\mu_j(k) - \tilde{\mu}_j(k)]^2 \le \beta_j^2 \left[\nu_j(k) - \tilde{\nu}_j(k)\right]^2 \qquad (12)$$

for $j = 1,2,\ldots,g$. Also, we have the conditions

$$
\begin{aligned}
[\mu_j(k)]^2 &\le \beta_j^2 \left[\nu_j(k)\right]^2 \\
[\tilde{\mu}_j(k)]^2 &\le \beta_j^2 \left[\tilde{\nu}_j(k)\right]^2
\end{aligned}
$$

for $j = 1,2,\ldots,g$. (Note that for specific nonlinearities, it may be possible to obtain tighter sector bounds than these which could be used to obtain less conservative results.) The above conditions imply that the following SQCs are satisfied:

$$\sum_{k=1}^{N} [\mu_j(k) - \tilde{\mu}_j(k)]^2 \le \sum_{k=1}^{N} \beta_j^2 \left[\nu_j(k) - \tilde{\nu}_j(k)\right]^2 + \bar{d}_{1j} \qquad (13)$$

$$\sum_{k=1}^{N}[\mu_j(k)]^2 \leq \sum_{k=1}^{N}\beta_j^2[\nu_j(k)]^2+\bar{d}_{2j} \quad (14)$$

$$\sum_{k=1}^{N}[\tilde{\mu}_j(k)]^2 \leq \sum_{k=1}^{N}\beta_j^2[\tilde{\nu}_j(k)]^2+\bar{d}_{3j} \quad (15)$$

for all $j=1,\ldots,g$. Here the $\bar{d}_{1j}$, $\bar{d}_{2j}$, $\bar{d}_{3j}$, are any positive constants.

Note that the SQC (13) is a critical SQC which exploits the repeated nonlinearities with one occurring in the plant and one occurring in the controller. The SQCs (14)-(15) are standard SQCs on the individual nonlinearities which are included to reduce the conservatism of the overall control system design.

We now introduce some additional notation to simplify the description of the system (1), together with the constraints (6), (13)-(15):

$$\tilde{\xi}^1(k)=\begin{bmatrix}\tilde{\xi}_1^1(k)\\ \vdots\\ \tilde{\xi}_{\tilde{f}+\tilde{g}}^1(k)\end{bmatrix} \triangleq \begin{bmatrix}\xi_1(k)\\ \vdots\\ \xi_{\tilde{f}}(k)\\ \mu_1(k)\\ \vdots\\ \mu_{\tilde{g}}(k)\end{bmatrix};$$

$$\tilde{\xi}^2(k)=\begin{bmatrix}\tilde{\xi}_1^2(k)\\ \vdots\\ \tilde{\xi}_{f-\tilde{f}+2g-\tilde{g}}^2(k)\end{bmatrix} \triangleq \begin{bmatrix}\xi_{\tilde{f}+1}(k)\\ \vdots\\ \xi_f(k)\\ \mu_{\tilde{g}+1}(k)\\ \vdots\\ \mu_g(k)\\ \tilde{\mu}_1(k)\\ \vdots\\ \tilde{\mu}_g(k)\end{bmatrix};$$

$$\tilde{\xi}(k)=\begin{bmatrix}\tilde{\xi}^1(k)\\ \tilde{\xi}^2(k)\end{bmatrix}=\begin{bmatrix}\tilde{\xi}_1(k)\\ \vdots\\ \tilde{\xi}_{f+2g}(k)\end{bmatrix} \quad (16)$$

$$\tilde{\zeta}(k)=\begin{bmatrix}\tilde{\zeta}_1(k)\\ \vdots\\ \tilde{\zeta}_{f+2g}(k)\end{bmatrix} \triangleq \begin{bmatrix}\zeta_1(k)\\ \vdots\\ \zeta_{\tilde{f}}(k)\\ \nu_1(k)\\ \vdots\\ \nu_{\tilde{g}}(k)\\ \zeta_{\tilde{f}+1}(k)\\ \vdots\\ \zeta_f(k)\\ \nu_{\tilde{g}+1}(k)\\ \vdots\\ \nu_g(k)\\ \tilde{\nu}_1(k)\\ \vdots\\ \tilde{\nu}_g(k)\end{bmatrix}; \quad (17)$$

$$\tilde{B}_2=[\,B_2\ \ 0_{n\times g}\,];$$
$$\tilde{B}_1=\left[\,\check{B}_{1,1}\ \ldots\ \check{B}_{1,\tilde{f}}\ \bar{B}_{1,1}\ \ldots\ \bar{B}_{1,\tilde{g}}\ \ 0_{n\times(2g+\sum_{i=\tilde{f}+1}^{f}r_i)}\,\right]; \quad (18)$$

$$\tilde{C}_1=\begin{bmatrix}\check{C}_{1,1}\\ \vdots\\ \check{C}_{1,\tilde{f}}\\ \bar{C}_{1,1}\\ \vdots\\ \bar{C}_{1,\tilde{g}}\\ \check{C}_{1,\tilde{f}+1}\\ \vdots\\ \check{C}_{1,f}\\ \bar{C}_{1,\tilde{g}+1}\\ \vdots\\ \bar{C}_{1,g}\\ 0_{g\times n}\end{bmatrix};\ \tilde{D}_{12}=\begin{bmatrix}\check{D}_{12,1} & 0_{h_1\times g}\\ \vdots & \vdots\\ \check{D}_{12,\tilde{f}} & 0_{h_{\tilde{f}}\times g}\\ \bar{D}_{12,1} & 0_{1\times g}\\ \vdots & \vdots\\ \bar{D}_{12,\tilde{g}} & 0_{1\times g}\\ \check{D}_{12,\tilde{f}+1} & 0_{h_{(\tilde{f}+1)}\times g}\\ \vdots & \vdots\\ \check{D}_{12,f} & 0_{h_f\times g}\\ \bar{D}_{12,\tilde{g}+1} & 0_{1\times g}\\ \vdots & \vdots\\ \bar{D}_{12,g} & 0_{1\times g}\\ 0_{g\times m} & I_{g\times g}\end{bmatrix};$$

$$\tilde{C}_2=\begin{bmatrix}C_2\\ 0_{g\times n}\end{bmatrix};\check{D}_{21}=\begin{bmatrix}D_{21}\\ 0_{g\times(l+g)}\end{bmatrix}; \quad (19)$$

$$\tilde{D}_{21}=\begin{bmatrix}0_{l\times(g+\sum_{i=1}^{\tilde{f}}r_i)} & \check{D}_{21,\tilde{f}+1} & \ldots & \check{D}_{21,f} & \bar{D}_{21,\tilde{g}+1} & \ldots & \bar{D}_{21,g} & 0_{l\times g}\\ 0_{g\times(g+\sum_{i=1}^{\tilde{f}}r_i)} & 0_{g\times r_{\tilde{g}+1}} & \ldots & 0_{g\times r_f} & 0_{g\times 1} & \ldots & 0_{g\times 1} & I_{g\times g}\end{bmatrix};$$

$$\tilde{D}_{12}^z=[\,D_{12}^z\ \ 0_{n_z\times g}\,]$$

where $h=\sum_{i=1}^{k}h_i$, $r=\sum_{i=1}^{k}r_i$. Using this notation and the notation (10), we can re-write the system (1) as follows:

$$\begin{aligned}x(k+1)&=\left[A(k)x(k)+\tilde{B}_1(k)\tilde{\xi}(k)+\tilde{B}_2(k)\tilde{u}(k)\right]\\ &\quad+B_1(k)w(k);\\ \tilde{\zeta}(k)&=\tilde{C}_1(k)x(k)+\tilde{D}_{12}(k)\tilde{u}(k);\\ z(k)&=C_1^z(k)x(k)+\tilde{D}_{12}^z(k)\tilde{u}(k),\\ \tilde{y}(k)&=\left[\tilde{C}_2(k)x(k)+\tilde{D}_{21}(k)\tilde{\xi}(k)\right]\\ &\quad+\check{D}_{21}(k)w(k).\end{aligned} \quad (20)$$

It is also straightforward to re-write the SQCs (6), (13)-(15) in the form

$$\sum_{k=0}^{N}\tilde{\xi}'(k)M_i\tilde{\xi}(k)\leq\left[\sum_{k=0}^{N}\tilde{\zeta}'(k)N_i\tilde{\zeta}(k)d+\tilde{d}_i\right] \quad (21)$$

for $i=1,2,\ldots,\tilde{h}$ where $\tilde{h}=f+3g$, $M_i\geq 0$, $N_i\geq 0$, and the $\tilde{d}_i$ are positive constants.

The set of all admissible uncertainty inputs $\tilde{\xi}(\cdot)$ for the uncertain system defined by the state equations (20) and the SQCs (21) is constructed as in Definition 1.

Remark 2. Since we have shown that the nonlinearities (3), (9) satisfy the SQCs (13) - (15), then it follows that if the linear uncertain system (20), (21) with the linear controller (11) leads to an upper bound on the worst-case disturbance attenuation level , then the nonlinear uncertain system (1), (6), (9), (3) with controller (11) will lead to the same upper bound on the associated worst-case disturbance attenuation level. Moreover, it then follows that the nonlinear uncertain system (1), (3), (6) with nonlinear controller (8), (9) will also lead to the same upper bound on the associated disturbance attenuation level. In the next section, we will use an extension of the results of Savkin and Petersen (1996) to construct a linear controller which leads to an upper bound on the worst-case attenuation level for the uncertain system (20), (21).

3. THE MAIN RESULTS

The main result of Savkin and Petersen (1996) solves a (continuous-time) robust H^∞ control problem for an uncertain linear system with uncertainty described by SQCs. In order to apply this result to the problem under consideration in this paper, we will show that the SQCs (6) lead to the satisfaction of such a SQC which is parameterized by a set of Lagrange multiplier parameters.

We first introduce some notation: Define a matrix valued function $\tilde{M}(\lambda) \triangleq \sum_{i=1}^{\tilde{h}} \lambda_i \tilde{M}_i$ where $\tilde{M}_i = \begin{bmatrix} -M_i & 0 \\ 0 & N_i \end{bmatrix}$, and $\lambda = [\,\lambda_1\ \lambda_2\ \ldots\ \lambda_{\tilde{h}}\,]' \in \mathbf{R}^{\tilde{h}}$. Then it follows from (21) that

$$\sum_{k=0}^{N} [\,\tilde{\xi}(k)'\ \tilde{\zeta}(k)'\,]\, \tilde{M}(\lambda) \begin{bmatrix} \tilde{\xi}(k) \\ \tilde{\zeta}(k) \end{bmatrix} + \tilde{d}(\lambda) \geq 0 \qquad (22)$$

for all $\lambda \in \Gamma$ where $\tilde{d}(\lambda) \triangleq \sum_{i=1}^{\tilde{h}} \lambda_i \tilde{d}_i$. Here,

$$\Gamma = \left\{ \lambda \in \mathbf{R}^{\tilde{h}} : \lambda_i \geq 0\ \forall i \right\}.$$

Now for any symmetric matrix $M \in \mathbf{R}^{(q+p)\times(q+p)}$ ($q = 2g + \sum_{i=1}^{f} r_i$, $p = 2g + \sum_{i=1}^{f} h_i$), let $\Pi(M)$ denote the number of strictly negative eigenvalues of the matrix M. Also, let $U(M)$ denote a matrix of orthonormal eigenvectors of M. Furthermore, if $\Pi(M) = q$ we partition $U(M)$ as

$$U(M) = \begin{bmatrix} U_{11}(M) & U_{12}(M) \\ U_{21}(M) & U_{22}(M) \end{bmatrix}$$

where $U_{11}(M) \in \mathbf{R}^{q\times q}$, $U_{12}(M) \in \mathbf{R}^{q\times p}$, $U_{21}(M) \in \mathbf{R}^{p\times q}$, $U_{22}(M) \in \mathbf{R}^{p\times p}$. Also, we assume the matrix $\begin{bmatrix} U_{11}(M) \\ U_{21}(M) \end{bmatrix}$ contains the eigenvectors corresponding to the strictly negative eigenvalues of M.

We will restrict attention to parameters $\lambda \in \Gamma$ such that the following conditions are satisfied:

$$\Pi(\tilde{M}(\lambda)) = q; \quad \det U_{11}(\tilde{M}(\lambda)) \neq 0. \qquad (23)$$

If these conditions are satisfied, it follows that there exists a nonsingular matrix T such that the matrix $T'\tilde{M}(\lambda)T$ is a diagonal matrix whose diagonal elements are in the set $\{-1, 1, 0\}$. As in Section 4.5 of Horn and Johnson (1985), this can be achieved with a matrix T of the form $T = U(\tilde{M}(\lambda))D^{-1}$ where $D = D(\tilde{M}(\lambda))$ is a diagonal matrix constructed from the eigenvalues of $\tilde{M}(\lambda)$ as follows: $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$. Here

$$D_1 = \begin{bmatrix} (-\bar{\mu}_1)^{1/2} & 0 & \ldots & 0 \\ 0 & (-\bar{\mu}_2)^{1/2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (-\bar{\mu}_q)^{1/2} \end{bmatrix};$$

$$D_2 = \begin{bmatrix} (\mu_1)^{1/2} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & (\mu_2)^{1/2} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & & & \vdots \\ 0 & 0 & \ldots & (\mu_{\bar{p}})^{1/2} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix};$$

where $\bar{\mu}_1,\ \bar{\mu}_2, \ldots, \bar{\mu}_q$ are the negative eigenvalues of the matrix $\tilde{M}(\lambda)$, $\mu_1,\ \mu_2, \ldots, \mu_{\bar{p}}$ are the positive eigenvalues of the matrix $\tilde{M}(\lambda)$, and the diagonal elements of the matrix D_2 corresponding to the zero eigenvalues of the matrix $\tilde{M}(\lambda)$, are replaced by ones.

With the matrix T defined as above, we obtain the following diagonalization of $\tilde{M}(\lambda)$:

$$T'\tilde{M}(\lambda)T = \begin{bmatrix} -I_{q\times q} & 0 & 0 \\ 0 & I_{\bar{p}\times\bar{p}} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (24)$$

Let

$$\tilde{T} = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix} := T^{-1} = DU(\tilde{M}(\lambda))'. \qquad (25)$$

Also, since $\det U_{11}(\tilde{M}(\lambda)) \neq 0$ and the matrix D is diagonal, it follows that $\det \tilde{T}_{11} \neq 0$. Now introduce a change of variables for the uncertainty inputs $\check{\xi}(k)$ and uncertainty outputs $\check{\zeta}(k)$ defined as follows

$$\begin{bmatrix} \tilde{\xi}(k) \\ \tilde{\zeta}(k) \end{bmatrix} = T \begin{bmatrix} \check{\xi}(k) \\ \check{\zeta}(k) \end{bmatrix}. \qquad (26)$$

Hence,

$$\begin{bmatrix} \check{\xi}(k) \\ \check{\zeta}(k) \end{bmatrix} = T^{-1} \begin{bmatrix} \tilde{\xi}(k) \\ \tilde{\zeta}(k) \end{bmatrix} = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\xi}(k) \\ \tilde{\zeta}(k) \end{bmatrix}. \qquad (27)$$

Also, it follows from (24) and (26) that the SQC (22) can be re-written as

$$\sum_{k=1}^{N} [\,\check{\xi}(k)'\ \check{\zeta}(k)'\,] \begin{bmatrix} -I & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} \check{\xi}(k) \\ \check{\zeta}(k) \end{bmatrix} + \tilde{d}(\lambda) \geq 0,$$

where $\Theta = \Theta^2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. This is equivalent to the SQC

$$\sum_{k=0}^{T} \|\check{\xi}(k)\|^2 - \|\bar{\zeta}(k)\|^2 - \tilde{d}(\lambda) \leq 0 \qquad (28)$$

where $\bar{\zeta}(k) \triangleq \Theta\check{\zeta}(k)$. It now follows from (27) that we can write

$$\tilde{\xi} = \tilde{T}_{11}^{-1}\left(\check{\xi} - \tilde{T}_{12}\tilde{\zeta}\right);$$
$$\check{\zeta} = \tilde{T}_{21}\tilde{T}_{11}^{-1}\left(\check{\xi} - \tilde{T}_{12}\tilde{\zeta}\right) + \tilde{T}_{22}\tilde{\zeta}.$$

Hence, we obtain

$$\begin{bmatrix} \bar{\xi} \\ \check{\zeta} \end{bmatrix} = \begin{bmatrix} \tilde{T}_{11}^{-1} & -\tilde{T}_{11}^{-1}\tilde{T}_{12} \\ \tilde{T}_{21}\tilde{T}_{11}^{-1} & \tilde{T}_{22} - \tilde{T}_{21}\tilde{T}_{11}^{-1}\tilde{T}_{12} \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{bmatrix}.$$

We now substitute these expressions into (20) to re-write this system as follows

$$\begin{aligned} x(k+1) &= \left[\bar{A}(k)x(k) + \bar{B}_1(k)\bar{\xi}(k) + \bar{B}_2(k)\tilde{u}(k)\right] \\ &\quad + B_1(k)w(k); \\ \bar{\zeta}(k) &= \bar{C}_1(k)x(k) + \bar{D}_{11}\bar{\xi}(k) + \bar{D}_{12}(k)\tilde{u}(k); \\ z(k) &= C_1^z(k)x(k) + \tilde{D}_{12}^z(k)\tilde{u}(k), \\ \tilde{y}(k) &= \left[\bar{C}_2(k)x(k) + \bar{D}_{21}(k)\bar{\xi}(k) + \bar{D}_{22}(k)\tilde{u}(k)\right] \\ &\quad + \check{D}_{21}(k)w(k) \end{aligned} \tag{29}$$

where

$$\begin{aligned} \bar{A}(k) &\triangleq A(k) - \tilde{B}_1(k)\tilde{T}_{11}^{-1}\tilde{T}_{12}\tilde{C}_1(k); \\ \bar{B}_1(k) &\triangleq \tilde{B}_1(k)\tilde{T}_{11}^{-1}; \\ \bar{B}_2(k) &\triangleq \tilde{B}_2(k) - \tilde{B}_1(k)\tilde{T}_{11}^{-1}\tilde{T}_{12}\tilde{D}_{12}(k); \\ \bar{C}_1(k) &\triangleq \Theta(\tilde{T}_{22} - \tilde{T}_{21}\tilde{T}_{11}^{-1}\tilde{T}_{12})\tilde{C}_1(k); \\ \bar{D}_{11} &\triangleq \Theta\tilde{T}_{21}\tilde{T}_{11}^{-1}; \\ \bar{D}_{12}(k) &\triangleq \Theta(\tilde{T}_{22} - \tilde{T}_{21}\tilde{T}_{11}^{-1}\tilde{T}_{12})\tilde{D}_{12}(k); \\ \bar{C}_2(k) &\triangleq \tilde{C}_2(k) - \tilde{D}_{21}(k)\tilde{T}_{11}^{-1}\tilde{T}_{12}\tilde{C}_1(k); \\ \bar{D}_{21}(k) &\triangleq \tilde{D}_{21}(k)\tilde{T}_{11}^{-1}; \\ \bar{D}_{22}(k) &\triangleq -\tilde{D}_{21}(k)\tilde{T}_{11}^{-1}\tilde{T}_{12}\tilde{D}_{12}(k). \end{aligned} \tag{30}$$

Note that the system (29) and the SQC (28) are dependent on the parameter $\lambda \in \Gamma$. Also, the approach of Savkin and Petersen (1996) cannot be directly applied to uncertain system (29), (28) due to the $\bar{D}_{11}$ and $\bar{D}_{22}$ terms in the state equations (29). We will deal with these terms using standard loop shifting ideas arising in H^∞ control theory; e.g., see Basar and Bernhard (1995).

The first step in this process is to remove the $\bar{D}_{11}$ term. In order to achieve this, we will require that

$$\bar{D}_{11}'\bar{D}_{11} < I. \tag{31}$$

Now assuming that this condition is satisfied, we can define

$$\Phi = I - \bar{D}_{11}'\bar{D}_{11} > 0; \;\; \bar{\Phi} = I - \bar{D}_{11}\bar{D}_{11}' > 0.$$

Also, define transformed uncertainty inputs and uncertainty outputs as

$$\begin{aligned} \hat{\xi} &\triangleq \Phi^{\frac{1}{2}}\bar{\xi} - \Phi^{-\frac{1}{2}}\bar{D}_{11}'\left[\bar{C}_1 x(k) + \bar{D}_{12}(k)\tilde{u}\right]; \\ \hat{\zeta} &\triangleq \bar{\Phi}^{-\frac{1}{2}}\left[\bar{C}_1(k)x + \bar{D}_{12}(k)\tilde{u}\right]. \end{aligned}$$

Hence,

$$\bar{\xi} = \Phi^{-\frac{1}{2}}\hat{\xi} + \Phi^{-1}\bar{D}_{11}'\left[\bar{C}_1(k)x + \bar{D}_{12}(k)\tilde{u}\right].$$

Now, using these definitions, it is straightforward to verify that

$$\|\bar{\xi}(k)\|^2 - \|\bar{\zeta}(k)\|^2 \equiv \|\hat{\xi}(k)\|^2 - \|\hat{\zeta}(k)\|^2.$$

Therefore, the SQC (28) can be re-written as

$$\sum_{k=0}^{N}[\|\hat{\xi}(k)\|^2 - \|\hat{\zeta}(k)\|^2] - \tilde{d}(\lambda) \leq 0. \tag{32}$$

Also, we can re-write the state equations (29) as

$$\begin{aligned} x(k+1) &= \left[\hat{A}(k)x(k) + \hat{B}_1(k)\hat{\xi}(k) + \hat{B}_2(k)\tilde{u}(k)\right] \\ &\quad + B_1(k)w(k); \\ \hat{\zeta}(k) &= \hat{C}_1(k)x(k) + \hat{D}_{12}(k)\tilde{u}(k); \\ z(k) &= C_1^z(k)x(k) + \tilde{D}_{12}^z(k)\tilde{u}(k), \\ \tilde{y}(k) &= \left[\hat{C}_2(k)x(k) + \hat{D}_{21}(k)\hat{\xi}(k) + \hat{D}_{22}(k)\tilde{u}(k)\right] \\ &\quad + \check{D}_{21}(k)w(k) \end{aligned} \tag{33}$$

where

$$\begin{aligned} \hat{A}(k) &\triangleq \bar{A}(k) + \bar{B}_1(k)\bar{D}_{11}'\bar{\Phi}^{-1}\bar{C}_1(k); \\ \hat{B}_1(k) &\triangleq \bar{B}_1(k)\Phi^{-\frac{1}{2}} \\ \hat{B}_2(k) &\triangleq \bar{B}_2(k) + \bar{B}_1(k)\bar{D}_{11}'\bar{\Phi}^{-1}\bar{D}_{12}(k); \\ \hat{C}_1(k) &\triangleq \bar{\Phi}^{-\frac{1}{2}}\bar{C}_1(k); \\ \hat{D}_{12}(k) &\triangleq \bar{\Phi}^{-\frac{1}{2}}\bar{D}_{12}(k); \\ \hat{C}_2(k) &\triangleq \bar{C}_2(k) + \bar{D}_{21}(k)\bar{D}_{11}'\bar{\Phi}^{-1}\bar{C}_1(k); \\ \hat{D}_{21}(k) &\triangleq \bar{D}_{21}(k)\Phi^{-\frac{1}{2}}; \\ \hat{D}_{22}(k) &\triangleq \bar{D}_{22}(k) + \bar{D}_{21}(k)\bar{D}_{11}'\bar{\Phi}^{-1}\bar{D}_{12}(k). \end{aligned} \tag{34}$$

Thus under the additional assumption that $\bar{D}_{11}'\bar{D}_{11} < I$, the uncertain system (29), (28) is equivalent to the uncertain system (33), (32) which has no D_{11} term in the state equations.

We now consider the removal of the $\hat{D}_{22}(k)$ term in the state equations (33). In order to achieve this, we define a transformed measured variable $\bar{y}(k)$ such that

$$\bar{y}(k) = \tilde{y}(k) - \hat{D}_{22}(k)\tilde{u}(k).$$

Hence, the state equations (33) can be re-written as

$$\begin{aligned} x(k+1) &= \left[\hat{A}(k)x(k) + \hat{B}_1(k)\hat{\xi}(k) + \hat{B}_2(k)\tilde{u}(k)\right] \\ &\quad + B_1(k)w(k); \\ \hat{\zeta}(k) &= \hat{C}_1(k)x(k) + \hat{D}_{12}(k)\tilde{u}(k); \\ \bar{y}(k) &= \left[\hat{C}_2(k)x(k) + \hat{D}_{21}(k)\hat{\xi}(k)\right] \\ &\quad + \check{D}_{21}(k)w(k) \end{aligned} \tag{35}$$

and applying a controller of the form

$$\begin{aligned} x_c(k+1) &= \bar{A}_c(k)x_c(k) + \tilde{B}_c(k)\bar{y}(k); \;\; x_c(0) = x_{c0} \\ \tilde{u}(k) &= \tilde{C}_c(k)x_c(k) \end{aligned} \tag{36}$$

to the uncertain system (35), (32) is equivalent to applying the original controller (11) to the uncertain system (33), (32) where

$$\bar{A}_c(k) = A_c(k) + \tilde{B}_c(k)\hat{D}_{22}(k)\tilde{C}_c(k). \tag{37}$$

Thus, the D_{11} and D_{22} terms have been removed and we can design a controller for the original uncertain system by designing a controller (36) for the uncertain system (35), (32) and then implement the controller (11), (37) on the original uncertain system.

We will apply results similar to that of Savkin and Petersen (1996) to the uncertain system defined by state equations (35) subject to the SQC (32). To this end, introduce the notations

$$
\begin{aligned}
\hat{\hat{B}}_1 &= \left[\frac{1}{\gamma} B_1 \ \ \frac{1}{\sqrt{\tau}} \hat{B}_1 \right], \quad \hat{\hat{D}}_{21} = \left[\frac{1}{\gamma} \check{D}_{21} \ \ \frac{1}{\sqrt{\tau}} \hat{D}_{21} \right], \\
\hat{\hat{C}}_1 &= \begin{bmatrix} C_1^z \\ \sqrt{\tau}\hat{C}_1 \end{bmatrix}, \quad \hat{\hat{D}}_{12} = \begin{bmatrix} D_{12}^z \\ \sqrt{\tau}\hat{D}_{12} \end{bmatrix}. \\
\bar{\bar{A}}(k) &:= \hat{A}(k) - \hat{B}_2(k) \left(\hat{\hat{D}}_{12}'(k) \hat{\hat{D}}_{12}(k) \right)^{-1} \hat{\hat{D}}_{12}'(k) \hat{\hat{C}}_1(k), \\
\bar{\bar{Q}}(k) &:= \hat{\hat{C}}_1(k)' \hat{\hat{C}}_1(k) - \hat{\hat{C}}_1(k)' \hat{\hat{D}}_{12}(k) \times \\
&\quad \left(\hat{\hat{D}}_{12}'(k) \hat{\hat{D}}_{12}(k) \right)^{-1} \hat{\hat{D}}_{12}'(k) \hat{C}_1(k), \\
\Gamma(k) &:= X(k+1)^{-1} + \hat{B}_2(k) \left(\hat{\hat{D}}_{12}'(k) \hat{\hat{D}}_{12}(k) \right)^{-1} \times \\
&\quad \hat{B}_2(k)' - \hat{\hat{B}}_1(k) \hat{\hat{B}}_1(k)', \\
\tilde{\tilde{A}}(k) &:= \hat{A}(k) - \hat{\hat{B}}_1(k) \hat{\hat{D}}_{21}(k)' \left(\hat{\hat{D}}_{21}(k) \hat{\hat{D}}_{21}(k)' \right)^{-1} \hat{C}_2(k), \\
\tilde{\tilde{\Sigma}}(k) &:= \hat{\hat{B}}_1(k) \hat{\hat{B}}_1(k)' - \hat{\hat{B}}_1(k) \hat{\hat{D}}_{21}(k)' \times \\
&\quad \left(\hat{\hat{D}}_{21}(k) \hat{\hat{D}}_{21}(k)' \right)^{-1} \hat{\hat{D}}_{21}(k) \hat{\hat{B}}_1(k)', \\
\Delta(k) &:= Y(k)^{-1} + \hat{C}_2(k)' \left(\hat{\hat{D}}_{21}(k) \hat{\hat{D}}_{21}(k)' \right)^{-1} \times \\
&\quad \hat{C}_2(k) - \hat{\hat{C}}_1(k)' \hat{\hat{C}}_1(k).
\end{aligned} \tag{38}
$$

The main result will be presented in terms of the following pair of Riccati recursions.

$$
\begin{aligned}
X(k) &= \bar{Q}(k) + \bar{A}(k)' \Gamma(k)^{-1} \bar{A}(k); \quad k = 1, \cdots, N. \\
X(N+1) &= Q_f.
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
Y(k+1) &= \tilde{\Sigma}(k) + \tilde{A}(k) \Delta(k)^{-1} \tilde{A}(k'); \quad k = 1, \cdots, N. \\
Y(1) &= Q_0^{-1}.
\end{aligned} \tag{40}
$$

We also require the matrices $\tilde{X}_k$, $\tilde{Y}_k$ defined respectively by

$$
\tilde{X}(k) = \bar{\bar{Q}}(k) + \bar{\bar{A}}(k)' \left(X(k+1)^{-1} - \hat{\hat{B}}_1(k) \hat{\hat{B}}_1(k)' \right)^{-1} \bar{\bar{A}}(k), \tag{41}
$$

and

$$
\tilde{Y}(k+1) = \tilde{\tilde{\Sigma}}(k) + \tilde{\tilde{A}}(k) \left(Y(k)^{-1} - \hat{\hat{C}}_1(k)' \hat{\hat{C}}_1(k) \right)^{-1} \tilde{\tilde{A}}(k)'. \tag{42}
$$

The solutions to these Riccati difference equations will be required to satisfy the following assumption:

Hypothesis 3. Given $\tau > 0$, $\gamma > 0$, and $\lambda \in \Gamma$ satisfying conditions (23), (31), there exist positive definite solutions, to Riccati recursions (40) and (39) , such that the spectral radius of the product matrix $\tilde{Y}_{k+1} X_{k+1}$ (or $Y_k \tilde{X}_k$) is strictly less than 1, for $k = 1, \cdots, N$.

If Hypothesis3 is satisfied, then we will construct a controller of the form (11) where matrices $\bar{A}_c$, $\tilde{B}_c$ and $\tilde{C}_c$ are defined as follows

$$
\begin{aligned}
\bar{A}_c(k) &= \hat{A}(k) + \hat{B}_2(k) C_c(k) + \tilde{\tilde{A}}(k) \Delta(k)^{-1} \hat{\hat{C}}_1' \times \\
&\quad \left(\hat{\hat{C}}_1 + \hat{\hat{D}}_{12} C_c(k) \right), \\
\tilde{B}_c(k) &= \left(\tilde{A}(k) \Delta(k)^{-1} C_2(k)' + D_{12}(k)' \right) \times \\
&\quad (D_{21}(k) D_{21}(k)')^{-1}, \\
\tilde{C}_c(k) &= -(D_{12}(k)' D_{12}(k)')^{-1} \left(B_2(k)' \Gamma(k+1)^{-1} \bar{A}(k) \right) \times \\
&\quad \left(I - \frac{1}{\gamma^2} Y(k) X(k) \right)^{-1},
\end{aligned} \tag{43}
$$

and $x_c(0) = 0$.

We now present the main results:

Theorem 4. Suppose there exist $\tau > 0$, $\gamma > 0$ and vector $\lambda \in \Gamma$ satisfying Hypothesis 3. Then
(i). the controller defined by (36), (43) has robust attenuation level γ when applied to the uncertain system defined by (35), (32).
(ii). If the controller defined by (11), (43), (34), (37) is applied to the uncertain system defined by (29), (28), then it can achieve the robust attenuation level γ.

Proof: Part (i) follows by the standard results of linear H^∞ theory (Savkin and Petersen (1996) and Chapter 6 in Basar and Bernhard (1995)). Part (ii) of the theorem follows from Part (i) of the theorem and the discussion above concerning the construction of the controller (11), (43), (37) together with the discussion above concerning the construction of the uncertain system (33), (32) from the uncertain system (29), (28). □

Using this theorem, we obtain the main result of this paper:

Theorem 5. Suppose there exist a constant $\tau > 0, \gamma > 0$ and vector $\lambda \in \Gamma$ satisfying Hypothesis 3. If the nonlinear controller defined by (8), (9), (10), (43), (30), (37) is applied to the nonlinear uncertain system defined by (1), (3), (6), then it has robust attenuation level γ.

Proof: This result follows directly from Part (ii) of the previous theorem and the construction of the system (29), the SQC (28) and the controller (11). □

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