Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Effect of Heterogeneity on Synchronization in Complex Network

Binbin Hao*, Qingshen Jing, Dan Wang*, Siying Zhang*, Yuanwei Jing***

**Institute of Information Science & Engineering,*
Northeastern University, Shenyang, Liaoning, 110004, P.R. of China
(e-mail: cdhbb525@126.com; ywjjing@ mail.neu.edu.cn).
***College of Mechanics, Peking University, Beijing, 100871, P.R. of China*
(e-mail: johnsonjing@126.com)

Abstract: Synchronization in classes of continuous-time dynamical networks with different topologies is investigated. A synchronization-optimal algorithm based on the synchronization criterion is proposed so as to get the appropriate topology such that complex network has the optimal synchronizability. It has been argued that heterogeneity suppresses synchronization in unweighted networks. However, it is shown in this work that synchronizability of Type I networks is independent of heterogeneity in the degree distribution, while synchronizability of Type II networks is dependent of both heterogeneity and scale of networks. It is presented the more heterogeneous and larger scale, the poorer synchronizability of Type II networks.

1. INTRODUCTION

Many social, biological, and communication systems can be properly described as complex networks with nodes representing individuals and edges mimicking the interactions among them (Newman, 2003). One of the ultimate goals of research on complex networks is to understand how the structure of complex networks affects the dynamical process taking place on them, such as traffic flow, epidemic spread (Boccaletti *et al.*, 2006). One of the most significant and interesting properties of a dynamical network is the synchronous output motion of its network nodes.

Synchronization in coupled dynamical networks and systems has been studied for many years within a common framework based on non-linear dynamical system theories. It has been observed, however, that most existing work have been concentrated on networks with completely regular topological structures (Li, 2005). It has been noticed that the topology of a network often plays a crucial role in determining its dynamical features. Especially, the discovery of the small-world effect and scale-free feature of most complex networks has led to a fascinating set of common problems concerning how a network structure facilitates and constrains the synchronization of complex networks (Wang *et al.,* 2002a, b; Hong *et al.,* 2002). Small-world networks are objects in between regular and random networks characterized by a small average distance between any two nodes, while keeping a relatively highly clustered structure (Watts *et al.*, 1998). Scale-free networks are characterized by highly heterogeneous distribution of degrees (number of links per node) and display a power-law distribution $p(k) \sim k^{-\gamma}$ in the node connectivity k (degree)(Barabási *et al.*, 1999). Compared with regular networks, the networks with small-world topology have better synchronizability (Wang *et al.*, 2002a). On the other hand, it has been shown that the more heterogeneous the network is, the more difficult it is to be synchronized (Takashi *et al.*, 2003).

Previous work has focused mainly on scale-free network (SFN)(Hong *et al.*, 2004; Fan *et al.*, 2005; Zhao *et al.*, 2006; Lu *et al.*, 2006), the different heterogeneity networks is obtained by adjusting power-law exponent γ. However, we observe the phenomenon: the number of edges of networks changes, as γ is decrease in random SFN (Takashi *et al.*, 2003; Motter *et al.*, 2005). So, the conclusion that heterogeneity of networks restrains the synchronizability can't exclude the effect of networks edges density. In this paper we investigate a class of tunable heterogeneity networks model (Jesús *et al.*, 2006), which can preserve the total number of final edges and nodes.

The aim of this work is to investigate the synchronizability classes of continuous-time dynamical networks with different topologies. Furthermore, based on the synchronization criterion, we propose a synchronization-optimal topology model. At last we investigate the relation between heterogeneity and network synchronizability based on tunable heterogeneity networks model.

2. NETWORK SYNCHRONIZATION

Consider a dynamical network consisting of N linearly coupled identical oscillators, with each oscillator being an $n-$dimensional dynamical system. The state equations of the network can be written as

$$\dot{x}_i = f(x) + c\sum_{j=1}^{N} G_{ij} h(x_j) \qquad i = 1, 2, \cdots, N \qquad (1)$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbf{R}^n$ are the state variables of node i, where $f = f(x)$ describes the dynamics of each individual oscillator, $h = h(x)$ is the inner coupling matrix, c is the overall coupling strength $(c > 0)$, and $G = (G_{ij})$ is the coupling matrix.

Dynamical network (1) is said to be (asymptotically) synchronized if

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t) \quad as\, t \to \infty \tag{2}$$

where $s(t)$ is a solution of an isolated node, which can be an equilibrium point, a periodic orbit, or a chaotic attractor, depending on the interest of study(Li *et al.,* 2003).

Suppose that all eigenvalues of the matrix G satisfy

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \cdots \geq \lambda_N \tag{3}$$

The variational equation governing the linear stability of the synchronous state $\{x_i(t) = s(t), \forall i\}$ of the network can be diagonalized into N blocks of the form

$$\dot{\eta}_i = [Jf(s) + \alpha Jh(s)]\eta_i \tag{4}$$

where J denotes the Jacobian operator, and $\alpha = c\lambda_i$. The largest Lyapunov exponent linked to $\alpha = c\lambda_i$, the so-called master stability function (MSF), fully determines the linear stability of the synchronous state. The region where the MSF $\Gamma(\alpha) = 0$ is denoted by $SR \in R$. Depending on dynamical function f, inner coupling matrix h, and synchronization state s, there are three possible types of networks (Barahona *et al.*, 2002).

Type I network: $SR \in (-\infty, \alpha_1)$, where $-\infty < \alpha_1 < 0$. For this type of networks, the synchronization state is stable if $c\lambda_2 < \alpha_1$ (Wang *et al.*, 2002b). This implies that synchronizability of Type I network can be characterized by the second largest eigenvalue λ_2 of the coupling matrix. Smaller λ_2 leads to better synchronizability.

Type II network: $SR \in (\alpha_2, \alpha_3)$, where $-\infty < \alpha_2 < \alpha_3 < 0$. For this type of networks, the synchronization state is stable if $c\lambda_N > \alpha_2$ and $c\lambda_2 < \alpha_3$, which leads to $\lambda_N / \lambda_2 < \alpha_2 / \alpha_3$. This implies that the synchronizability of Type II network can be characterized by the eigenratio $R = \lambda_N / \lambda_2$ of the coupling matrix. Smaller R leads to better synchronizability.

Type III network: $SR \in \Phi$. Synchronization cannot be achieved for this type of networks.

In this work, we investigate Type I and II networks. If dynamical network (1) satisfies (3), then has the synchronization criterion: the smaller λ_2 for Type I networks or $R = \lambda_N / \lambda_2$ for Type II networks leads to better synchronizability.

3. SYNCHRONIZATION-OPTIMAL NETWORK

Recent advances in complex network research have stimulated increasing interests in understanding the relationship between the topology and dynamics of complex networks. Recently, several works try to find the most synchronizable networks topological structure (Hong *et al.*, 2006; Fan *et al.*, 2005a, b). Fan and Wang give a synchronization-optimal growth networks. In that model, degree of nodes can only increase. This is the limitation if we want to get the model with the best synchronizability. Then, for a given number of nodes and edges networks, which is the most synchronizable structure? To get the structural for optimal synchronizability, we present random rewiring algorithm to adjust topology while keeping networks scale and edges unchanged. The algorithm is as follows:

A) ***Start with WS model:*** Construct an original network with N nodes and N_l edges by applying the WS small-world model.

B) ***Rewire edges:*** Rewire each edge of the network in turn. The criterion for rewiring the edges to which the new nodes connects is to optimize the synchronizability of the obtained network, equivalently, to minimize the second-largest eigenvalue of the corresponding coupling matrix Type I networks or the eigenratio R for Type II networks.

After $t \gg N_l$ time steps, we obtain a synchronization-optimal model if any rewiring can't decrease the second-largest eigenvalue λ_2 or eigenratio R of the corresponding coupling matrix. Rewiring within this context means shifting one end of the edge to a new node chosen in turn from the whole network, with the constraints that any two different nodes cannot have more than one edge between them, and no node can have an edge with itself.

For clarity, λ_2 is the second-largest eigenvalue, R is the eigenratio. The network has N nodes and $N_l = N * \mathrm{m}$ edges. Where m is determined by regular network. Regular network model is the nearest-neighbor coupled network (a lattice), which is a regular graph in which every node is connected to its first $2m$ neighbours (m on each side). In numerical computation, the eigenvalues are obtained by averaging the results of 10 runs. It is shown from fig.1 and fig.2 that the synchronization-optimal network can get the better synchronizability than WS, BA, ER random and regular networks for Type I network. And it is also shown from fig.3 and fig.4 that synchronization-optimal model has the better synchronizability for Type II network.

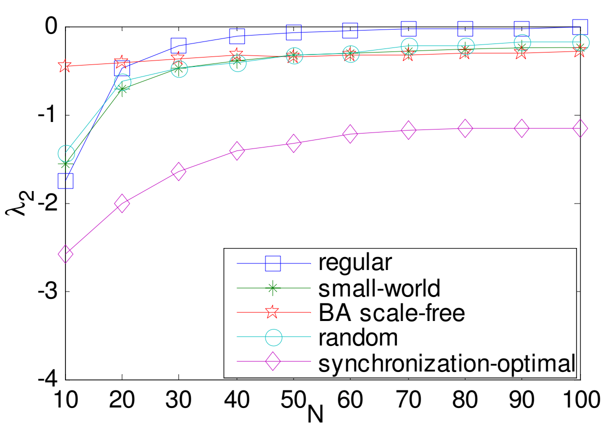

Fig.1. Changes in the second-largest eigenvalue of the coupling matrix of different topology when network nodes increasing. $m = 2, N_l = 2N$.

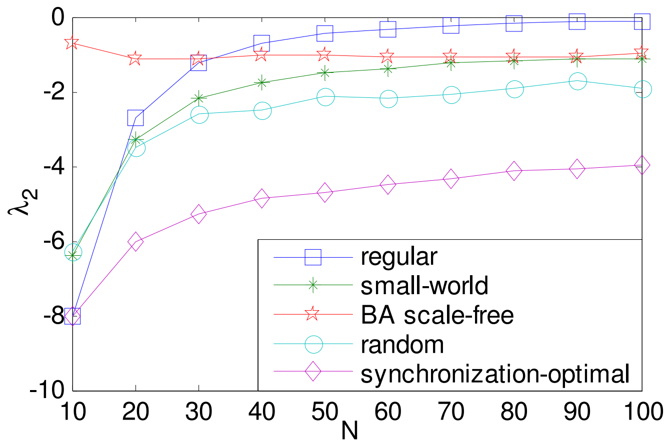

Fig.2. Changes in the second-largest eigenvalue of the coupling matrix of different topology when network nodes increasing. $m=4, N_l=4N$.

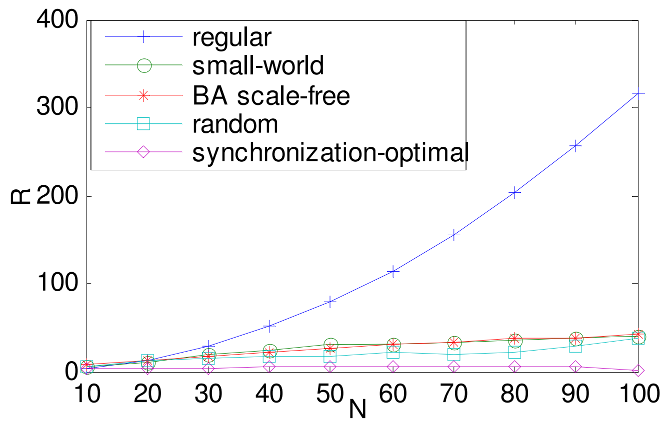

Fig.3. Changes in the eigenvalue R of the coupling matrix of different topology when network nodes increasing. $m=2, N_l=2N$.

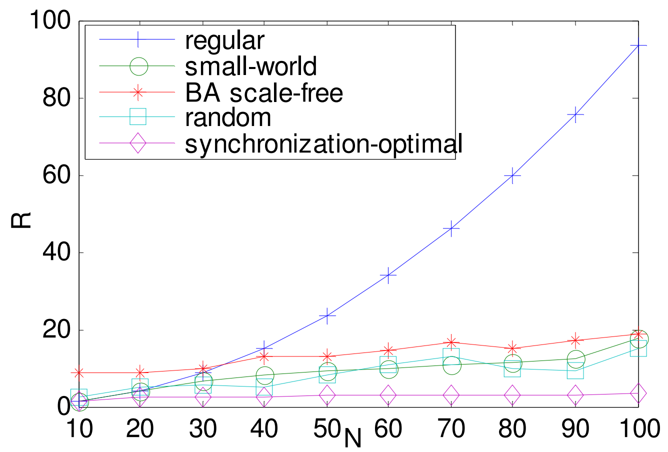

Fig.4. Changes in the eigenvalue R of the coupling matrix of different topology when network nodes increasing. $m=2, N_l=2N$.

It is noticed that star model as the synchronization-optimal model is got for Type I network when $m=1$. Star model is more heterogeneous than others, it seems contravene the result that heterogeneity suppresses the synchronization of network (Takashi *et al.*, 2003). The more homogeneous and random the Type II network structure is, the better synchronizability the network has. Which confirm the conclusion from scale-free networks (Takashi *et al.*, 2003). In next section, we investigate the relations between heterogeneity and synchronization in both types synchronization networks. Further more it is also observed that the edges density plays an important role in synchronization. Synchronization-optimal model has the better synchronizability when $m=4$ than $m=2$. So, it should be considered the effects of change of edges density with heterogeneous distribution of degrees on synchronizability.

4. A CLASS OF TUNABLE HETEROGENEITY NETWORKS

In section 3, we have compared the synchronizability of BA, regular, WS, and ER random networks with the synchronization-optimal model. The numerical computations show that the synchronization-optimal model has the better synchronizability than others for both types networks. In this section, we investigate the relations between heterogeneity and synchronization. For universality, we will not address a particular dynamical system such as scare-free networks, but concentrate on the family of networks generated with the model(Jesús *et al.*, 2006). This model generates a one-parameter family of networks labelled by $\alpha \in [0,1]$. The parameter α measures the degree of heterogeneity of the final networks so that $\alpha=0$ corresponds to the heterogeneous BA networks and $\alpha=1$ to homogeneous ER graphs. For intermediate values of α one obtains networks that have been grown combining both preferential attachment and homogeneous random linking so that each mechanism is chosen with probabilities $(1-\alpha)$ and α , respectively. It is worth stressing that the growth mechanism preserves the total number of edges N_l , and nodes N , for a proper comparison between different values of α . Thus, we can get networks with different heterogeneity but the same edges density. It is the one of main contributions that considering the effects of change of edges density with heterogeneous distribution of degrees on synchronization. A class of tunable heterogeneity networks is constructed as following:

Growth: A network of size N is generated starting from a fully connected core of m_0 nodes and a set $S(0)$ of $(N-m_0)$ unconnected nodes. At each time step, a new node (not selected before) is chosen from $S(0)$ and linked to m other nodes.

Chosen attachment: Each of the m edges is linked with probability α to a randomly chosen node (avoiding multiple and self-edges) from the whole set of $(N-1)$ remaining nodes and with probability $(1-\alpha)$ following a linear preferential attachment strategy.

After repeating the second process $(N-m_0)$ times, networks interpolating between the limiting cases of ER $(\alpha=1)$ and SF$(\alpha=0)$ topologies are generated. Furthermore, with this procedure, the degree of heterogeneity of the grown networks varies smoothly between the two limiting cases. Thus, we get a class of general networks with different topology and the same nodes and edges.

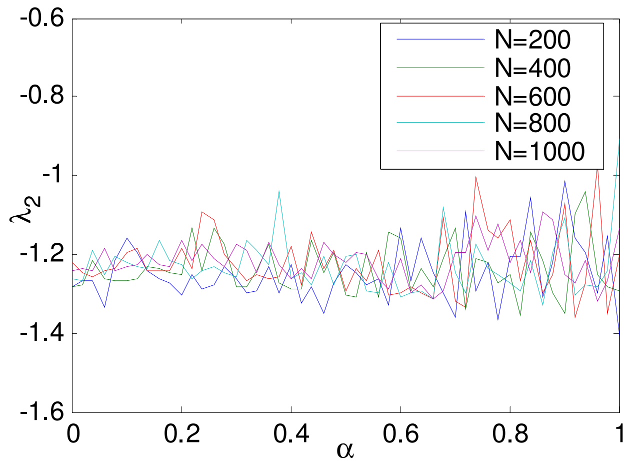

Fig.5. Synchronizability of Type I networks: λ_2 versus α with different network size N .

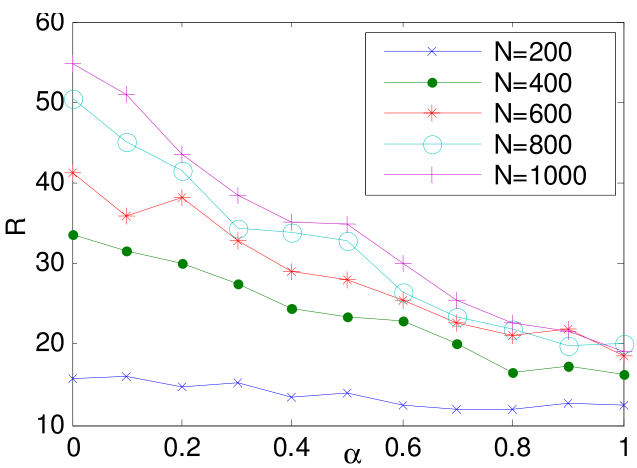

Fig.6. Synchronizability of Type II networks: $R=\lambda_N/\lambda_2$ versus α with different network size N .

Fig. 5 shows the synchronizability of Type I networks for different network size, while α ranges from 0 to 1, $\alpha=0$ corresponds to the heterogeneous BA networks and $\alpha=1$ to homogeneous ER graphs. It can be observed that the second-largest eigenvalue λ_2 is independent of α for the same network size. Because α measure the heterogeneity in degree distribution, this implies that the synchronizability of Type I is independent of heterogeneity. Fig. 6 shows the synchronizability of Type II networks for network size, while α ranges from 0 to 1. Obviously, eigenratio R decreases with the increase of α for the same N . Therefore, Type II networks are easier to synchronize with the increase of α for the same N . We also notice that for the same value of α , especially for $\alpha=0$, R increases with the increase of N . This implies that the Type II networks become harder to synchronize if network has larger size. For Type II networks, heterogeneity suppress the synchronization of networks,

5. CONCLUSIONS

We investigate the relation between network topology and synchronization, present a synchronization-optimal algorithm based on random rewired mechanism. Networks will have the better synchronizability by apply this algorithm. And we find that the edges density plays an important role in network synchronization, especially in the analyses synchronizability and heterogeneity. However, in previous works which is always neglected. In this paper, we investigate the synchronizability and heterogeneity while networks have the same nodes and edges. We find that synchronizability of Type I networks is independent of heterogeneity in the degree distribution, while synchronizability of Type II networks is dependent of both heterogeneity and scale of networks. It is presented the more heterogeneous and larger scale, the poorer synchronizability of Type II networks. Our research may be useful to understand the complete synchronization behaviors complex networks and design effective networks.

ACKNOWLEGEMENTS

This work is supported by the National Natural Science Foundation of China, under grant 60274009, and Specialized Research Fund for the Doctoral Program of Higher Education, under grant 20020145007.

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