Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A new criterion for synchronization in deterministic underdamped ratchets ⋆

Pingli Lu, Ying Yang, Lin Huang

State Key Lab for Turbulence and Complex Systems, Department of Mechanics and Aerospace Engineering, Peking University, Beijing, 100871, China (E-mail: pinglilu@pku.edu.cn, yy@mech.pku.edu.cn, hl35hj75@pku.edu.cn).

Abstract: This paper deals with the synchronization for deterministic underdamped ratchets. Using the technique derived from pendulum-like nonlinear analytic theory and Kalman-Yakubovich-Popov(KYP) lemma, simple linear matrix inequality (LMI) formulations are established to guarantee the stable synchronization between two periodically driven identical deterministic, underdamped ratchets. With this new efficient criterion, the direction of transporting particles can be controlled and the distance among particles also can be fixed on some values in order to separate them. Finally, simulation results verified the applicability and validity of the proposed approach.

1. INTRODUCTION

Recently, transport phenomena of nonlinear systems, which can extract usable work from unbiased nonequilibrium fluctuations, has been attracting considerable interest from many fields of scientific research. These so-called ratchets (or Brownian motors) can be modeled by a Brownian particle undergoing random walk on a periodic asymmetric potential and being acted upon by an external time-dependent force of zero average. An interesting feature of the ratchet model is the possibility to use diffusion to convert an ac driving signal (either a multiplicative (Magnasco [1993]) or an additive (Doering [1998]) fluctuating term) into a net dc current corresponding to the unidirectional motion of the particle through the system (ratchet effect). The ratchet effect has an extensive applications in rectifiers, pumps, particle separation devices, molecular switches, and transistors (see Astumian et al. [2002], Reimannet al. [2002] and the references therein). It is also of great interest in biology, since the working principles of molecular motors can be conveniently explained in terms of ratchet mechanisms (Julicher et al. [1997]). Moreover, it is possible to demonstrate quantum ratchet effects (Poletti et al. [2007], Robilliard et al. [1999], Schiavoni et al. [2003]) by using cold atoms. The phenomena of ratchet effect can be observed both in overdamped deterministic systems (Sarmiento et al. [1999], Chauwin et al. [1995]) and in underdamped chaotic ones (Jung et al. [1996]). This was recently stimulated some interest in the study of deterministic underdamped ratchets (Barbi et al. [2000], Mateos [2000]). These ratchets possess a classical chaotic dynamics that modifies significantly the transport properties (Jung et al. [1996], Barbi et al. [2001], Larrondo [2003]).

In 2003, Savel'ev et al. [2003] investigated the transport properties of interacting particles and proved that an attracting (repelling) interaction among identical particles can result in the amplification (inversion) of their net current. In addition, synchronized dynamics can be observed in the interacting or coupled ratchets when a threshold is reached. Accordingly, the study of synchronization in the interacting or coupled ratchets is certainly of wide concern. In particular, Vincent et al. [2004] studied the synchronized dynamics for unidirectionally coupled deterministic ratchets. Denisov et al. [2007] identified current reversals with synchronization/desynchronization transitions in the collective ratchet's dynamics. Anticipated synchronization is also considered for unidirectionally coupled ratchets with time-delayed feedback in Kostur et al. [2005], but only numerical results are obtained. More recently, Vincent et al. [2007a,b] researched the chaos synchronization and control of chaotic ratchets via active control and backstepping control in detail, but the stability of the error ratchet system is investigated in linear mechanism. In this paper, a novel condition for the stable synchronization between two identical underdamped ratchets is investigated with different initial conditions. One of the ratchet systems acts as the master system while the remaining one acts as the slave. The interaction between the two ratchets is unidirectional, meaning that the master affects the slave, but not vice versa. The error dynamics is similar to the typical pendulum-like nonlinear system with multiple equilibria (Leonov et al. [1996], Yang et al. [2004], Wang et al. [2004], Duan et al. [2005]). Furthermore, the stable synchronization of the two underdamped ratchets is equivalent to the global asymptotic stability of the corresponding error ratchet system. Then, using KYP lemma, the new simple and efficient criteria for synchronization in deterministic underdamped ratchets is established in terms of LMIs which are readily solvable by available numerical software. With this result, the complete synchronization (CS) of two deterministic ratchets with identical system parameters are guaranteed. In addition, we explore the phase synchronization (PS) between the master and the slave which is discussed in detail later, and the direction of particles also

⋆ This work is supported by the National Science Foundation of China under Grants 60334030, 60404007.

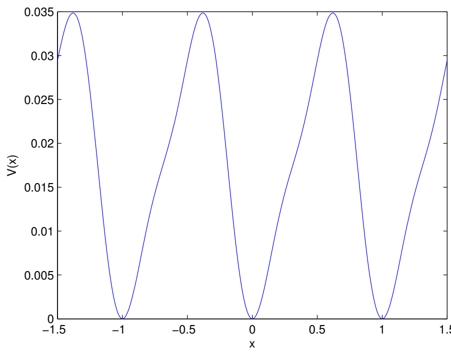


Fig. 1. The ratchet periodic potential with parameters $x_0 = -0.19, \delta \simeq 1.600$

can be controlled in virtue of the new criteria proposed in this paper.

The rest of this paper is organized as follows: Section 2 presents some basic results necessary for the successive development, while in Section 3, we give the new LMI-based criterion which guarantees the synchronization of the master and slave deterministic underdamped ratchets by using feedback control. Section 4 gives the simulation results in detail. Section 5 shows our conclusions of this paper.

We use the following notations in this paper. $\mathbb{R}^{n\times n}$ is the set of $n \times n$ real matrices. The superscript $*$ means conjugate transpose for complex matrices and T means transpose for real matrices. The matrix inequality $A < 0 (A \le 0)$ means that $A = A^T$ is negative (semi-)definite.

2. PRELIMINARIES

Consider the motion of a particle driven by a periodic time-dependent external force in a spatially periodic potential with an asymmetric profile. Here, the dynamics of the particle is deterministic due to the absence of the noise. The differential equation of motion, in dimensionless variables, is written as (Mateos [2000], Barbi et al. [2000], Kostur et al. [2005], Vincent et al. [2007a])

$$\ddot{x} + \alpha\dot{x} + \frac{dV(x)}{dx} = \beta\cos(\omega t) \quad (1)$$

where α is the friction coefficient. $V(x)$ is the asymmetric ratchet periodic potential, and β and ω are the amplitude and frequency of the driver, respectively. The dimensionless potential is given by

$$V(x) = C - \frac{1}{4\pi^2\delta}\left[\sin 2\pi(x - x_0) + \frac{1}{4}\sin 4\pi(x - x_0)\right] \quad (2)$$

where the constants C and x_0 are introduced in order to have the potential minimum $V(0) = 0$ (see Fig.1) in $x = 0$. Accordingly, the constants

$$C = -\frac{1}{4\pi^2\delta}\left[\sin 2\pi x_0 + \frac{1}{4}\sin 4\pi x_0\right]$$

It is obvious that the period of the ratchet potential $V(x)$ with regard to x is $T_V = 1$, which is also shown in Fig.1. In this paper, $\delta \simeq 1.600$, and $x_0 = -0.19$, see also the references Mateos [2000], Barbi et al. [2000], Kostur et al. [2005], Vincent et al. [2007a].

The nonlinear dynamics (1) can be embedded into a corresponding two-dimensional state space dynamics which is shown as below

$$\begin{cases} \dot{x}_1 = -\alpha x_1 - \dfrac{dV(x_2)}{dx_2} + \beta\cos(\omega t) \\ \dot{x}_2 = x_1 \end{cases} \quad (3)$$

with $x_2 = x, x_1 = \dot{x}$.

Lemma 1. (KYP Lemma Rantzer et al. [1996]). Given $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(i\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$ and (A, B) controllable, the following two statements are equivalent:

1° $\begin{bmatrix}(i\omega I - A)^{-1}B \\ I\end{bmatrix}^* M \begin{bmatrix}(i\omega I - A)^{-1}B \\ I\end{bmatrix} \le 0, \ \forall\omega \in \mathbb{R};$

2° there exists a matrix $P = P^T \in \mathbb{R}^{n\times n}$ such that

$$M + \begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} \le 0$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable.

3. SYNCHRONIZATION OF DETERMINISTIC RATCHETS USING FEEDBACK CONTROL

Consider the following master-slave scheme of two identical deterministic ratchets

$$M: \begin{cases} \dot{x}_1(t) = -\alpha x_1(t) - \dfrac{dV(x_2)}{dx_2} + \beta\cos(\omega t) \\ \dot{x}_2(t) = x_1(t) \end{cases} \quad (4)$$

and

$$S: \begin{cases} \dot{y}_1(t) = -\alpha y_1(t) - \dfrac{dV(y_2)}{dy_2} + \beta\cos(\omega t) + u_1(t) \\ \dot{y}_2(t) = y_1(t) + u_2(t) \end{cases} \quad (5)$$

with the controller

$$\begin{aligned} u_1(t) &= k(x_1(t) - y_1(t)) \\ u_2(t) &= \frac{dV(x_2)}{dx_2} - \frac{dV(y_2)}{dy_2} \end{aligned} \quad (6)$$

where the controller gain k is one way, that is, the slave is influenced by the master, but the latter is independent of the dynamics of the slave. Our aim is to design a gain scale k to synchronize the slave ratchet to the master ratchet with feedback control.

Defining the error variables

$$e_1 = x_1 - y_1, e_2 = x_2 - y_2$$

and from (6), we have the following error system between the master and the slave ratchets

$$\begin{cases} \dot{e}_1(t) = -(\alpha + k)e_1(t) - \varphi(e_2, y_2) \\ \dot{e}_2(t) = e_1(t) - \varphi(e_2, y_2) \end{cases} \quad (7)$$

where

$$\begin{aligned} \varphi(e_2, y_2) &= \frac{dV(e_2 + y_2)}{d(e_2 + y_2)} - \frac{dV(y_2)}{dy_2} \\ &= \frac{dV(x_2)}{dx_2} - \frac{dV(y_2)}{dy_2} \end{aligned}$$

The period of the nonlinear function $\varphi(e_2, y_2)$ about e_2(or y_2) is $T_\varphi = 1$ due to the periodic property of the potential V. In addition, the transfer function of the linear part of (7) from the input $\varphi(e_2, y_2)$ to the output $-\dot{e}_2$ is given by

$$G(s) = C(A - sI)^{-1}B - D = 1 + \frac{1}{s + \alpha + k}$$

where

$$A = -(\alpha + k), B = -1, C = 1, D = -1$$

We suppose that $\alpha + k \neq -1$, and the equilibrium of error ratchet system (7) is (e_{1eq}, e_{2eq}) which satisfies

$$-(\alpha + k)e_{1eq}(t) - \varphi(e_{2eq}, y_2) = 0$$
$$e_{1eq}(t) - \varphi(e_{2eq}, y_2) = 0$$

We have

$$\varphi(e_{2eq}, y_2) = 0$$

and

$$e_{1eq}(t) = 0$$

Since $\varphi(e_2, y_2)$ is T_φ periodic about e_2, the error ratchet system (7) has infinitely many isolated equilibria, namely, $e_{2eq} = \hat{e}_2 + n$, where $\varphi(\hat{e}_2, y_2) = 0, n = 0, \pm 1, \pm 2, \cdots$. In this case, the formulation of the error dynamics (7) is similar to that of pendulum-like nonlinear system with multiple equilibria (see Leonov et al. [1996]–Duan et al. [2005] and references therein).

Definition 2. (Leonov et al. [1996]). The error ratchet system (7) is global asymptotic stable if every its solution $(e_1(t), e_2(t)) \to (e_{1eq}(t), e_{2eq}(t))$ as $t \to \infty$, where $(e_{1eq}(t), e_{2eq}(t))$ is an equilibrium point of the system (7).

The aim of synchronization between the master and the slave ratchet is then to obtain the global asymptotic stability of solution $(e_1(t), e_2(t))$ for the error nonlinear system (7) as $t \to \infty$. The following result can be found in Leonov et al. [1996].

Lemma 3. Suppose $G(s)$ is stable and there exist scales κ, γ and ϵ with $\gamma > 0$ and $\epsilon > 0$ satisfying the following conditions:

(1) $\kappa \operatorname{Re} G(j\omega) - \epsilon G^*(j\omega)G(j\omega) - \gamma \geq 0$ for all $\omega \in \mathbb{R}$
(2) $4\epsilon\gamma > (\kappa\nu)^2$

where

$$\nu = \frac{\int_0^1 \int_0^1 \varphi(e_2, y_2) de_2 dy_2}{\int_0^1 \int_0^1 |\varphi(e_2, y_2)| de_2 dy_2} \tag{8}$$

then the error system (7) is global asymptotic stable.

Remark 4. We should point that the formulation of ν in (8) is a little different from that given in Leonov et al. [1996] due to the two variables e_2 and y_2. Particularly, we use the double integral here instead of the single integral in Leonov et al. [1996].

Our new results for ensuring the global asymptotic stability of the error ratchet system (7) is presented in the following theorem which also guarantees the synchronization of the master and the slave ratchet (4)-(6).

Theorem 5. If there exist scales $\kappa, p > 0, \gamma > 0, \epsilon > 0$ and f such that

$$\begin{bmatrix} \epsilon - 2\alpha p - f & \frac{1}{2}\kappa - p - \epsilon \\ \frac{1}{2}\kappa - p - \epsilon & \gamma + \epsilon - \kappa \end{bmatrix} \leq 0 \tag{9}$$

and

$$\begin{bmatrix} 2\epsilon & \kappa\nu \\ \kappa\nu & 2\gamma \end{bmatrix} > 0 \tag{10}$$

then the error ratchet system (7) is global asymptotic stable, i.e., the synchronization of the master-slave ratchets is guaranteed. In addition, the controller gain is obtained through $k = \frac{f}{2p}$.

Proof. In virtue of Lemma 1, if

$$M = \begin{bmatrix} \epsilon C^2 & C(\frac{1}{2}\kappa + \epsilon D) \\ C(\frac{1}{2}\kappa + \epsilon D) & \gamma + \epsilon D^2 + \kappa D \end{bmatrix}$$

then the inequality (9) is equivalent to the frequency domain inequality (1) of Lemma 3 take into account the foregoing concrete formulation of A, B, C, D. The LMI (10) can be directly obtained from the formulation (2) of Lemma 3 using by Schur's Lemma.

It is indicated from Theorem 5 that the common controller u_1 will be established for the master and slave ratchets if only the same parameter α. It is also shown that, with these results, the slave ratchet can follow the response of the master ratchet. In addition, the stability of the synchronized error $(e_1(t), e_2(t))$ is guaranteed theoretically in nonlinear mechanism. The phenomenon of coupled ratchets has been extensively vindicated experimentally in Mixtures of Brownian Particles using Brownian Motors (Savel'ev et al. [2003]) and in semiconductor laser running in the chaotic regime Tang et al. [2003]. In particular, it is proved from literature Savel'ev et al. [2003] that an attracting (repelling) interaction among identical particles can result in the amplification (inversion) of their net current. Accordingly, the phenomenon of synchronization of coupled ratchets is very significant to investigate the transfer properties of the ratchets in chemical, biological and physical systems.

4. NUMERICAL RESULTS

In this section, we investigate the synchronization between the master (4) and the slave ratchet (5) in virtue of the proposed method above. Let us consider the case where the both ratchets, the master and slave, are identical, i.e., they have the same parameters α, β, x_0 and ω. First, the controllers u_1 and u_2 are switched off. The transporting trajectories for the master and the slave ratchet are shown in Fig.2, when $\alpha = 0.1, \beta = 0.08, x_0 = -0.19$ and $\omega = 0.67$ (the same as Vincent et al. [2005, 2007a]). We notice that the master and the slave ratchets all display chaotic characteristics as exposed in Fig.2(a), and the corresponding error between x_2 and y_2 and the trajectories in phase space are illustrated in Fig.2(b) and (c) respectively, with two different sets of initial conditions $(x_1 = 0.25, x_2 = -0.1)$ and $(y_1 = -0.12, y_2 = 0.43)$. Then the controllers u_1 and u_2 are switched on, and the transport properties are denoted in Fig.3 using by Theorem 5. The corresponding parameters to be sought are as follows:

$$p = 50.3825, \quad \kappa = 161.2241$$
$$\gamma = 43.2917, \quad \epsilon = 39.5596, \quad k = 0.9778$$

It is depicted from Fig.3 that the two ratchets are completely synchronized (CS).

Now, let us consider the special transport properties of the ratchets presented in Mateos [2003], i.e., the reversal current can be obtained by choosing the appropriate initial condition, see Fig.4, for instance, the parameter $\beta = 0.156$, the initial conditions of the master and slave ratchets are selected as follows:

$$x_1(0) = 0.25, x_2(0) = -0.1, y_1(0) = -0.12, y_2(0) = 0.43$$

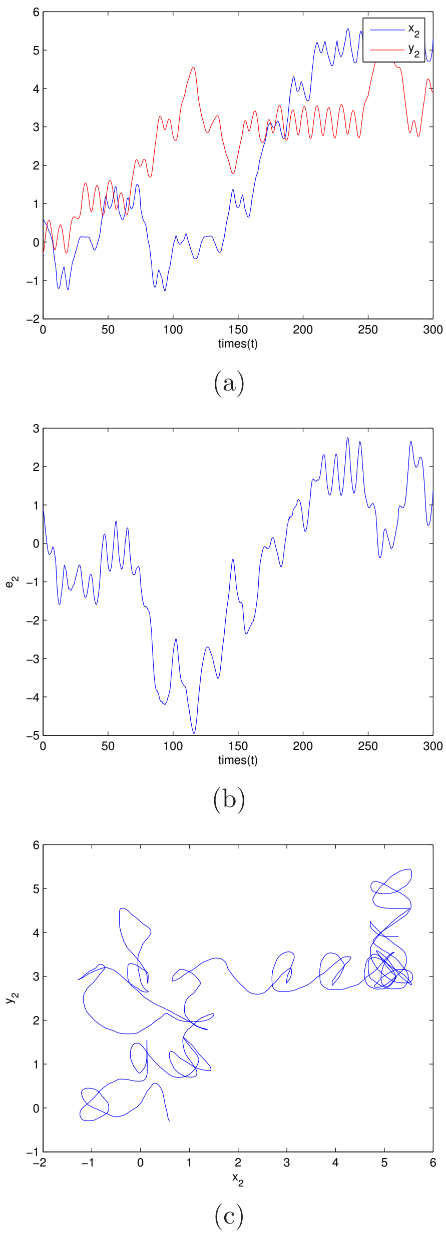


Fig. 2. Chaotic transport of the ratchets (4) and (5) with $u_1 = u_2 = 0$: (a) dynamics of x_2, y_2 (b) the error e_2 between x_2 and y_2 (c) the phase space with initials $(x_1 = 0.25, x_2 = -0.1)$ and $(y_1 = -0.12, y_2 = 0.43)$

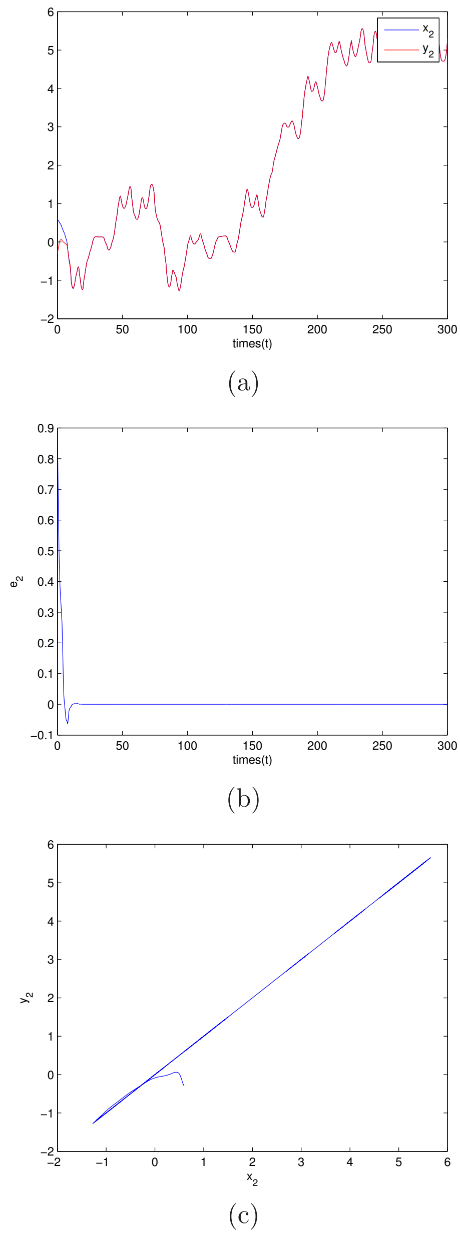


Fig. 3. Trajectories for the master and slave ratchet in completely synchronization under controllers u_1 and u_2: (a)dynamics of x_2, y_2 (b)the error e_2 between x_2 and y_2 (c) the phase space with the same initials above

It is exhibited from Fig.4 that the chaotic attractor (red line) generates a positive current and that the periodic attractor (blue line) generates a negative one which is already studied in Mateos [2003]. Here, we investigate the synchronization properties of the coexisting attractors and the performance of controllers u_1, u_2 acted on the ratchets. Applying Theorem 5 to the equations (4)-(6), the transporting trajectories of the master and slave ratchets are displayed in Fig.5. It is indicated that the complete synchronization(CS) between the master and slave ratchets is realized. It means that the slave ratchet can track the trajectory of the master ratchet and both of them transport particles in positive position which is very important for the transport properties of interacted particles and can result in the amplification of their net current(Savel'ev et al. [2003]).

Then, the following initial conditions

$$x_1(0) = 1.5, x_2(0) = -0.1, y_1(0) = -0.12, y_2(0) = 0.43$$

are made and the simulation results are shown in Fig.6. It is shown that the error $e_2 = x_2 - y_2 = 1$ and $e_1 = \dot{x}_2 - \dot{y}_2 = 0$ which means that the master and the slave ratchets transport particles in the same positive direction but always keep a fixed phase difference between them, called phase synchronization(PS). Such performance can achieve the repelling interaction among identical particles which can result in the inversion of net current(Savel'ev et al. [2003]). In addition, the fixed phase difference e_2 between the master and slave can be different when we

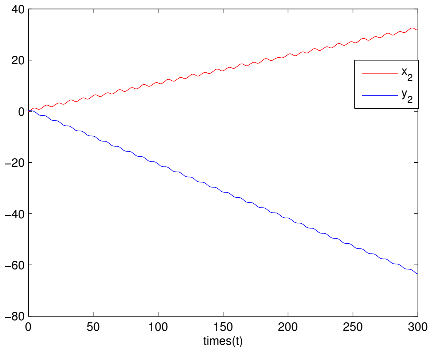
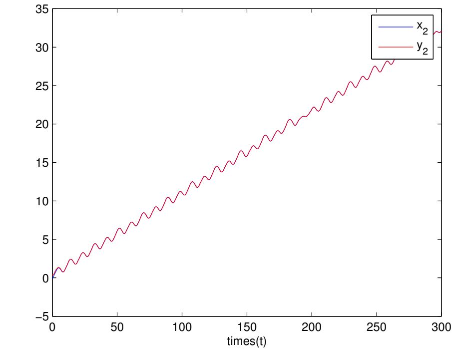


Fig. 4. Trajectories for the master and slave ratchet without controller u_1, u_2 with $\beta = 0.156$ and the other parameters are the same as before

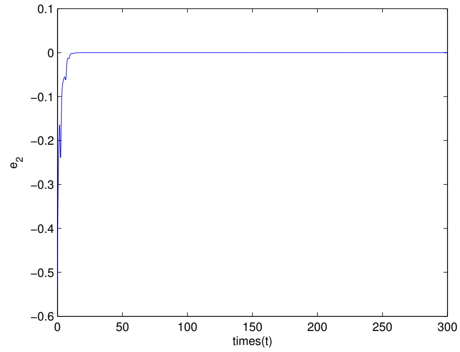
vary the initial conditions, and the value of e_2 can be only integers, i.e., $\pm 1, \pm 2, \cdots$, due to the periodicity of the nonlinear function φ in (7).

5. CONCLUSIONS

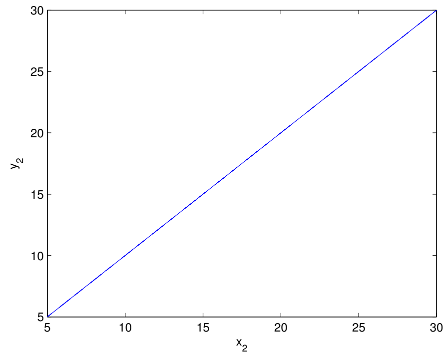
In this paper, we theoretically studied the synchronization of two deterministic ratchets by using pendulum-like analytical method. A new sufficient criterion has been established in terms of LMIs under which the stable synchronization can be obtained for the master and slave transporting ratchet trajectories, and the simulation results perfectly confirmed these conclusions. In particular, complete synchronization(CS) and phase synchronization(PS) are all achieved in the two deterministic ratchets with identical parameters. Moreover, with appropriate controllers, the direct of transporting particles can be dominated along expected one and the net current also can be amplified or reversed.



(a)



(b)



(c)

Fig. 5. Trajectories for the master and slave ratchet in completely synchronization under controllers u_1 and u_2: (a) dynamics of x_2, y_2 (b) the error e_2 (c) the phase plot

REFERENCES

M. O. Magnasco. Forced thermal ratchets. *Physical Review Letters*, 71:1477, 1993.

C. R. Doering. Stochastic ratchets. *Physica A*, 254:1–6, 1998.

R. D. Astumian and P. Hanggi. Brownian motors. *Phys. Today*, 55(11):33, 2002.

P. Reimann and P. Hanggi. Introduction to the physics of brownian motors. *Appl. Phys. A*, 75:169, 2002.

F.Julicher, A. Ajdari, and J. Prost. Modeling molecular motors. *Rev. Mod. Phys.*, 69:1269, 1997.

D. Poletti, G. G. Carlo, and B. Li. Current behavior of a quantum hamiltonian ratchet in resonance. *Physical Review E*, 75:011102, 2007.

C. Mennerat-Robilliard, D. Lucas, S. Guibal, J. Tabosa, C. Jurczak, J.Y. Courtois, and G. Grynberg. Ratchet for cold rubidium atoms: The asymmetric optical lattice. *Physical Review Letters*, 82:851, 1999.

M. Schiavoni, L. Sanchez-Palencia, F. Renzoni, and G. Grynberg. Phase control of directed diffusion in a symmetric optical lattice. *Physical Review Letters*, 90:094101, 2003.

A. Sarmiento and H. Larralde. Deterministic transport in ratchets. *Physical Review E*, 59(5):4878, 1999.

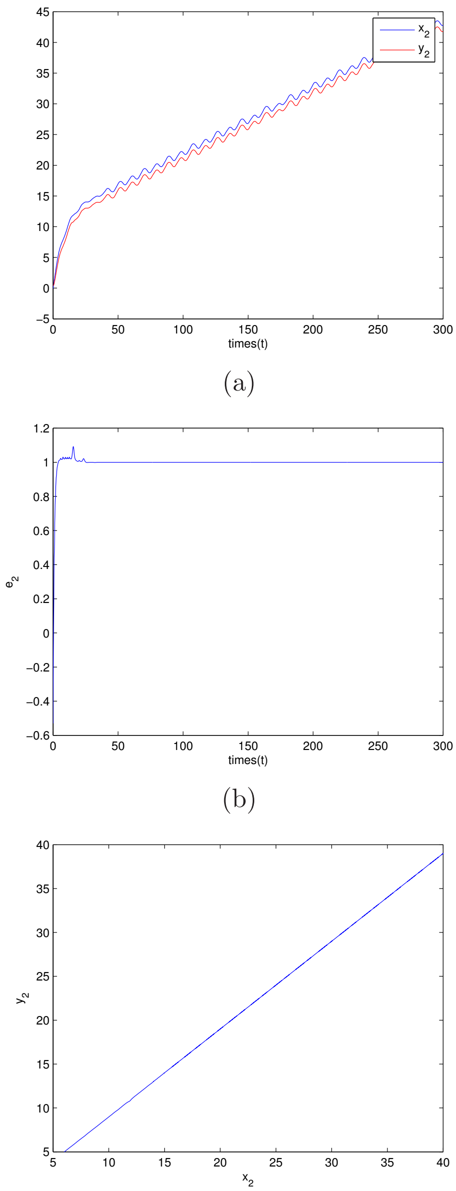
J. F. Chauwin, A. Ajdari, and J. Prost. Current reversal in asymmetric pumping. *Europhys. Lett.*, 32:373–378, 1995.

P. Jung, J. G. Kissner, and P. Hanggi. Regular and chaotic transport in asymmetric periodic potentials: Inertia ratchets. *Physical Review Letters*, 76:3436, 1996.

M. Barbi and M. Salemo. Phase locking effect and current reversals in deterministic underdamped ratchets. *Physical Review E*, 62:1988, 2000.

J. L. Mateos. Chaotic transport and current reversal in deterministic ratchets. *Physical Review Letters*, 84:258, 2000.

M. Barbi and M. Salemo. Stabilization of ratchet dynamics by weak periodic signals. *Physical Review E*, 63:066212, 2001.



(a) (b) (c)

Fig. 6. Trajectories for the master and slave ratchet in synchronization under controllers u_1 and u_2: (a) dynamics of x_2, y_2 (b) the error e_2 (c) the phase plot

H. A. Larrondo. Current basins of attraction in inertia ratchets. *Physica A*, 320:119, 2003.

S. Savel'ev, F. Marchesoni, and F. Nori. Controlling transport in mixtures of interacting particles using brownian motors. *Physical Review Letters*, 91:010601, 2003.

U. E. Vincent, A. N. Njah, and O. Akinlade. Phase synchronization in unidirectionally coupled chaotic ratchets. *Chaos*, 14:1018–1025, 2004.

S. Denisov. Collective current rectification. *Physica A*, 377:429–434, 2007.

M. Kostur, P. Hanggi, P. Talkner, and J. L. Mateos. Anticipated synchronization in coupled inertial ratchets with time-delayed feedback: A numerical study. *Physical Review E*, 72:036210, 2005.

U. E. Vincent and J. A. Laoye. Synchronization and control of directed transport in chaotic ratchets via active control. *Physics Letters A*, 363:91–95, 2007.

U. E. Vincent, A. N. Njahb, and J.A.Laoye. Controlling chaos and deterministic directed transport in inertia ratchets using backstepping control. *Physics D*, 231:130–136, 2007.

G. A. Leonov, D. V. Ponomarenko, and V. B. Smirnova. *Frequency-Domain Methods for Nonlinear Analysis*. World Scientific, Singapore, 1996.

Y. Yang, R. Fu, and L. Huang. Robust analysis and synthesis for a class of uncertain nonlinear systems with multiple equilibria. *Syst. Contr Lett.*, 53(1):89–105, 2004.

J. Z. Wang, L. Huang, and Z. S. Duan. Design of controller for a class of pendulum-like system guaranteeing dichotomy. *Automatica*, 40(6):1011–1016, 2004.

Z. S. Duan, J. Z. Wang, and L. Huang. Input and output coupled nonlinear systems. *IEEE Trans. on Circ. Sys. I.*, 52(3):567–575, 2005.

A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Syst. Contr Lett.*, 28(1):7–10, June 1996.

S. Tang and J. M. Liu. Experimental verification of anticipated and retarded synchronization in chaotic semiconductor lasers. *Physical Review Letters*, 90:194101, 2003.

U. E. Vincent, A. Kenfack, A. N.Njah, and O. Akinlade. Bifurcation and chaos in coupled ratchets exhibiting synchronized dynamics. *Physical Review E*, 72:056213, 2005.

J. L. Mateos. Current reversals in chaotic ratchets: the battle of the attractors. *Physica A*, 325:92, 2003.