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State and input observability for structured bilinear systems: A graph-theoretic approach

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Abstract: This paper deals with the state and input observability analysis for structured bilinear systems with unknown inputs. More precisely, we provide two groups of conditions, the first ones are necessary and the second ones are sufficient, to check whether or not a structured bilinear system is generically state and input observable. These conditions, which are far to be trivial, are expressed in quite simple graphic-terms. Moreover,the proposed method assumes only the knowledge of the system's structure and all the given conditions are easy to check because they deal with finding paths in a digraph. This makes the suggested approach particularly well suited to study large scale systems or systems with unknown parameters, as it may be the case during a conception stage. *Copyright© 2008 IFAC*

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1. INTRODUCTION

This paper deals with the characterization of the input and state observability of multi-input, multi-output structured bilinear systems. This particular class of nonlinear systems, whose dynamics is jointly linear in the state and the input variables was introduced in control theory in the 1960's. Many works have been focused on this kind of systems both for their applicative interest and their intrinsic simplicity. Indeed, industrial process control, economics and biology (switched circuits, mechanical brakes, controlled suspension systems, immunological systems, population growth, enzyme kinetics, ...) (Mohler [1991]) provide examples of bilinear systems. These systems are simpler and better understood than most other nonlinear systems. Furthermore, they are useful in designing control systems that must change their behaviour in purposeful ways.

The studies of this kind of systems are generally based on time-variant linear systems theory (D'Angelo [1970]) and matrix Lie groups (Mohler and Kolodziej [1980], Bruni et al. [1974]). Among these studies, the state observability of bilinear systems has been widely tackled since the results of (Williamson [1977], Grasselli and Isidori [1977]). The necessary and sufficient conditions to achieve this property are now very well known. These conditions have been established using geometric or algebraic tools. Nevertheless, the observability of bilinear systems with unknown inputs is still an open issue. In fact, the problem of estimating the state and the unknown input is of great interest mainly in control law synthesis, fault detection and isolation, fault tolerant control, supervision and so on. In this respect, many works are interested in the design of state observers for bilinear or more general nonlinear systems submitted to unknown inputs. Otherwise, in the context of fault detection and isolation, the issue of simultaneously observing at least a part of the state and the unknown input has been investigated in (Edelmayer et al. [2004], Ha and Trinh [2004], Jiang et al. [2004], Pillonetto and Saccomani [2006]) for general nonlinear systems.

In most cases, the studies on the state and input observability deal with algebraic and geometric tools. The use of such tools requires the exact knowledge of the state space matrices characterizing the system's model. However, in many modeling problems, these matrices have a number of fixed zero entries determined by the physical laws while the remaining entries are not precisely known. To study the properties of these systems in spite of the poor knowledge we have on them, the idea is that we only keep the zero/non-zero entries in the state space matrices. Thus, we consider models where the fixed zeros are conserved while the non-zero entries are replaced by free parameters. These models are useful to capture most of the structural available information from physical laws. Moreover, their study requires a low computational burden which allows one to deal with large scale systems. Many studies on structured systems are related to the graph-theoretic approach. However, until now, the graph approach has mainly been dedicated to the study of linear systems. The survey paper (Dion et al. [2003]) reviews the most significant results in this area. Among the studies which deal with nonlinear systems, (Svaricek [1993]) gives conditions to analyse the observability of bilinear systems. In (Bornard and Hammouri [2002]), the authors give graphical sufficient conditions for the uniform observability of nonlinear systems which are preliminarily put in a canonical form. More recently, (Boukhobza and Hamelin [2007]) provide necessary and sufficient graphical conditions ensuring the generic state observability of structured bilinear systems.

In this context, the present work aims to provide, using a graph approach, necessary and sufficient state and input observability conditions which have an intuitive interpretation and are quite simple to check. These features allow to obtain, with a low computational burden, a helpful characterization of the observability for large scale systems and for systems with uncertain parameters. More precisely, after subdividing the considered system into two subsystems, we provide two groups of conditions, the first ones are necessary and the second ones are sufficient, to check whether or not a structured bilinear system generically state and input observable.

The paper is organised as follows: after section 2, which is devoted to the problem formulation, a digraph representation of structured bilinear systems is defined in section 3. The main result is enounced in section 4. Finally, some concluding remarks are made.

2. PROBLEM STATEMENT

Consider the structured bilinear system (SBLS):

$$(\Sigma_\Lambda): \begin{cases} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i A_i x + Hw \\ y = Cx + Dw \end{cases} \qquad (1)$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$, $w = (w_1, \ldots, w_q)^T \in \mathbb{R}^q$ and $y = (y_1, \ldots, y_p)^T \in \mathbb{R}^p$ are respectively the state vector, the known input vector (control), the unknown input vector and the output vector. For $i = 0, \ldots, m$ A_i, H, C and D represent matrices of appropriate dimensions whose elements are either fixed to zero or assumed to be free non-zero parameters. The vector of these parameters is $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)^T$. If all the non-zeros parameters λ_i are fixed, we obtain an admissible realization of structured system (Σ_Λ). We say that a property is true generically if it is true for almost all realizations of the system or equivalently for almost all parameter values. Here "for almost all parameter values" is to be understood (Dion et al. [2003]) as "for all parameter values except for those in some proper algebraic variety in the parameter space". The proper algebraic variety for which the property is not true is the zero set of some nontrivial polynomial with real coefficients in the h parameters of the system.

In this paper, we study the generic observability of the state x and unknown input w. This study is mainly motivated by the fact that the first issue arising in state and input reconstruction is the evaluation of the state and input observability. We recall hereafter the definitions of the state and input observability, which are an extension to the structured systems of the ones given for non structured systems:

Definition 1. Consider structured system (Σ_Λ), we say that

- state $x(t)$ of system (Σ_Λ) is generically observable, if for almost all realizations of (Σ_Λ), for almost all the control input values $u(t)$, $y(t) = 0$ for $t \geq 0$ implies $x(t) = 0$ for $t \geq 0$.
- input $w(t)$ of system (Σ_Λ) is generically observable, if for almost all realizations of (Σ_Λ), for almost all the control input values $u(t)$, $y(t) = 0$ for $t \geq 0$ implies $w(t) = 0$ for $t > 0$. □

Roughly speaking, generic state and input observability means that a change in unknown input or of initial state can reflect itself in a change of measurements. This property is also equivalent to the fact that the state and the input can be expressed in function of input u, output y and their derivatives. The generic state and input observability is an important property because we can prove that there exists generically an observer which achieves the state and input reconstruction only if the system is generically state and input observable.

3. DIGRAPH REPRESENTATION OF A STRUCTURED BILINEAR SYSTEM

This section is devoted in a first stage to the directed graph (or digraph) which is used to represent structured bilinear system (Σ_Λ). Next, we will give some helpful notations and definitions.

3.1 Digraph definition for structured bilinear system

The digraph associated to (Σ_Λ) is noted $\mathcal{G}(\Sigma_\Lambda)$ and is constituted by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$: $\mathcal{G}(\Sigma_\Lambda) = (\mathcal{V}, \mathcal{E})$. The vertices are associated to the state, unknown input and output components of (Σ_Λ) and the directed edges represent the existence of dynamic or static relations between these variables. More precisely, $\mathcal{V} = \mathbf{X} \cup \mathbf{Y} \cup \mathbf{W}$, where $\mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ is the set of state vertices, $\mathbf{Y} = \{\mathbf{y_1}, \ldots, \mathbf{y_p}\}$ is the set of output vertices and $\mathbf{W} = \{\mathbf{w_1}, \ldots, \mathbf{w_q}\}$ is the set of unknown input vertices.

The edge set is $\mathcal{E} = \bigcup_{k=0}^{m} A_k$-edges $\cup$ C-edges $\cup$ D-edges $\cup$ H-edges, where

for $k = 0, \ldots, m$, A_k-edges $= \{(\mathbf{x_i}, \mathbf{x_j}) \mid A_k(j,i) \neq 0\}$,
C-edges $= \{(\mathbf{x_i}, \mathbf{y_j}) \mid C(j,i) \neq 0\}$,
D-edges $= \{(\mathbf{w_i}, \mathbf{y_j}) \mid D(j,i) \neq 0\}$ and
H-edges $= \{(\mathbf{w_i}, \mathbf{x_j}) \mid H(j,i) \neq 0\}$.

Here, $(\mathbf{v_1}, \mathbf{v_2})$ denotes a directed edge from vertex $\mathbf{v_1} \in \mathcal{V}$ to vertex $\mathbf{v_2} \in \mathcal{V}$. Moreover, we take the following notation: $\bar{A}_0$-*edges* $= A_0$-*edges* $\cup$ C-edges $\cup$ D-edges $\cup$ H-edges and for $i = 1, \ldots, m$, $\bar{A}_i$-edges $= A_i$-edges. In order to preserve the information about the belonging of the edges in the digraph representation, we indicate symbol u_i under each A_i-edges and u_0 under $\bar{A}_0$-edges.

Hereafter, we illustrate the proposed digraph representation.

Example 1. Consider the structured system defined by:

$$A_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&\lambda_1&0&0&0&\lambda_2&0&0&0&0&0 \\ 0&\lambda_3&0&0&0&0&\lambda_4&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&\lambda_5&\lambda_6&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, H = \begin{pmatrix} 0&0 \\ \lambda_{14}&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&\lambda_{15} \\ 0&\lambda_{16} \end{pmatrix}, D = 0,$$

$$A_1 = \begin{pmatrix} \lambda_7&0&0&0&0&0&0&0&0&0&0 \\ \lambda_8&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&\lambda_9&0&0&0&0&0&0&0 \\ 0&0&0&0&\lambda_{10}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\lambda_{11}&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$ A_2 is such that only $A_2(7,3)$ and $A_2(11,8)$ are non-zero *i.e.* $A_2(7,3) = \lambda_{12}$ and $A_2(11,8) = \lambda_{13}$ and $C = \begin{pmatrix} 0&0&0&0&0&0&0&0&\lambda_{17}&0&0 \\ 0&0&0&0&0&0&0&0&\lambda_{18}&\lambda_{19}&0 \\ 0&0&0&0&0&0&0&0&0&0&\lambda_{20} \end{pmatrix}$. This model is associated to the digraph of Figure 1.

3.2 Definitions and notations

- Two edges $e_1 = (\mathbf{v_1}, \mathbf{v'_1})$ and $e_2 = (\mathbf{v_2}, \mathbf{v'_2})$ are v-disjoint if $\mathbf{v_1} \neq \mathbf{v_2}$ and $\mathbf{v'_1} \neq \mathbf{v'_2}$.
- $P = \mathbf{v_0} \xrightarrow{u_{i_1}} \mathbf{v_1} \xrightarrow{u_{i_2}} \ldots \xrightarrow{u_{i_s}} \mathbf{v_s}$ denotes a path P which contains vertices $\mathbf{v_0}$, $\mathbf{v_1}$, $\ldots$, $\mathbf{v_s}$ and where for $j = 1, \ldots, s$, $(\mathbf{v_{j-1}}, \mathbf{v_j}) \in \bar{A}_{i_j}$-edges. A path is simple when every vertex occurs only once in this path. When $\mathbf{v_0} = \mathbf{v_s}$ and when vertices $\mathbf{v_{r_0}}, \mathbf{v_{r_1}}, \ldots, \mathbf{v_{r_i}}$ are distinct, path $\mathbf{P}$ is called a *cycle*. Path P is an $\mathbf{Y}$-*topped* path if its end vertex is an element of $\mathbf{Y}$.
- Some paths are disjoint if they have no common vertex.

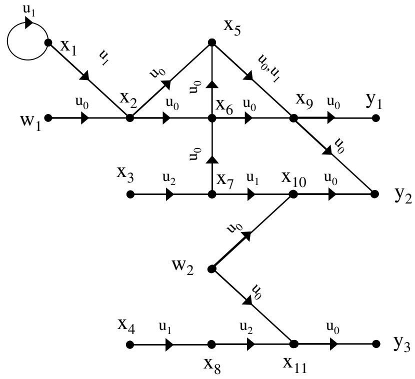

Figure 1. Digraph representing system of Example 1

In the sequel, let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote two vertex subsets.

• A path P is said a $\mathcal{V}_1$-$\mathcal{V}_2$ path if its begin vertex belongs to $\mathcal{V}_1$ and its end vertex belongs to $\mathcal{V}_2$.

• A set of ℓ disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths is called a $\mathcal{V}_1$-$\mathcal{V}_2$ linking of size ℓ. The linkings which consist of a maximal number of disjoint paths are called maximal $\mathcal{V}_1$-$\mathcal{V}_2$ linkings. We define $\rho\big[\mathcal{V}_1, \mathcal{V}_2\big]$ as the maximal number of disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths. Finally, we denote by $\mu\left[\mathcal{V}_1, \mathcal{V}_2\right]$ the minimal number of vertices of $\mathbf{X} \cup \mathbf{Y}$ belonging to a maximal $\mathcal{V}_1$-$\mathcal{V}_2$ linking.

In example 1, $\rho\big[\mathbf{W}, \mathbf{Y}\big] = 2$ and $\mu\big[\mathbf{W}, \mathbf{Y}\big] = 6$.

• $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) \overset{def}{=} \big\{\mathbf{v} \in \mathcal{V} \mid \mathbf{v}$ is included in every maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linking$\big\}$. $V_{ess}(\mathcal{V}_1, \mathcal{V}_2)$ denotes the set of all essential vertices (van der Woude [2000]), which correspond by definition to vertices present in all the maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings.

• $\mathbf{S} \subseteq \mathcal{V}$ is a separator between sets $\mathcal{V}_1$ and $\mathcal{V}_2$, if every path from $\mathcal{V}_1$ to $\mathcal{V}_2$ contains at least one vertex in $\mathbf{S}$. We call minimum (size) separators between $\mathcal{V}_1$ and $\mathcal{V}_2$ any separators having the smallest size, which is equal to $\rho\big[\mathcal{V}_1, \mathcal{V}_2\big]$.

• There is an uniquely determined minimum separator between $\mathcal{V}_1$ and $\mathcal{V}_2$ noted $\mathbf{S^o}(\mathcal{V}_1, \mathcal{V}_2)$, called minimum output separator van der Woude [2000] and which is the set of begin vertices of all direct $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) - \mathcal{V}_2$ paths, where $V_{ess}(\mathcal{V}_1, \mathcal{V}_2) \cap \mathcal{V}_2$ is considered, in the present definition, as input vertices.

• Edges $e_1 = \mathbf{v_0^1} \xrightarrow{u_{i_1}} \mathbf{v_1^1}$ and $e_2 = \mathbf{v_0^2} \xrightarrow{u_{i_2}} \mathbf{v_1^2}$ are A-disjoint iff the following conditions hold

Cond1- $\mathbf{v_0^1} \neq \mathbf{v_0^2}$;

Cond2- Either $\mathbf{v_1^1} \neq \mathbf{v_1^2}$ or $i_1 \neq i_2$ which is equivalent to say that either they have distinct end vertices or are associated to distinct indices.

• Some edges are A-disjoint if they are mutually A-disjoint.

• $\theta\big[\mathcal{V}_1, \mathcal{V}_2\big]$ is the maximal number of v-disjoint edges from $\mathcal{V}_1$ to $\mathcal{V}_2$. Similarly, $\theta_A\big[\mathcal{V}_1, \mathcal{V}_2\big]$ is the maximal number of A-disjoint edges from $\mathcal{V}_1$ to $\mathcal{V}_2$.

• A subgraph $\mathcal{S}_\mathcal{G} = (\mathcal{S}_\mathcal{V}, \mathcal{S}_\mathcal{E})$ of $\mathcal{G}(\Sigma_\Lambda)$ is defined by an edge subset $\mathcal{S}_\mathcal{E} \subseteq \mathcal{E}$ and a vertex subset $\mathcal{S}_\mathcal{V} \subseteq \mathcal{V}$ including the begin and the end vertices of the elements of $\mathcal{S}_\mathcal{E}$.

$\mathcal{S}_\mathcal{G}$ is an A-disjoint (resp. v-disjoint) subgraph if all its edges are A-disjoint (resp. v-disjoint). $\mathcal{S}_\mathcal{G}$ covers a vertex $\mathbf{s}$ if there exists an edge $e \in \mathcal{S}_\mathcal{E}$ such that $\mathbf{s}$ is the begin vertex of e.

4. MAIN RESULTS

4.1 Preliminary results

Before presenting the main proposition of the paper, we first recall some results on the generic observability of bilinear systems without unknown inputs (Boukhobza and Hamelin [2007]).

Theorem 1. A structured bilinear system

$$(\Sigma_B) : \begin{cases} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i A_i x \\ y = Cx \end{cases}$$

is generically observable iff in its associated digraph $\mathcal{G}(\Sigma_B)$

i. every state vertex is the begin vertex of an $\mathbf{Y}$-*topped* path;

ii. there exist an A-disjoint subgraph $\mathcal{G}(\Sigma_B)$ which covers all the state vertices.

We can immediately deduce from Theorem 1 that a necessary condition to the generic observability of structured bilinear system with unknown input:

Corollary 1. A structured bilinear system (Σ_Λ) is generically observable only if in its associated digraph $\mathcal{G}(\Sigma_\Lambda)$

i. every state vertex is the begin vertex of an $\mathbf{Y}$-*topped* path;

ii. there exist an A-disjoint subgraph $\mathcal{S}_\mathcal{G}$ which covers all the state vertices.

Proof:

It is obvious that (Σ_Λ) is generically observable only if the following extended bilinear system without unknown input

$$(\Sigma_e) : \begin{cases} \dot{x} = A_0 x + \sum_{i=1}^{m} u_i A_i x + Hw \\ \dot{w} = 0 \\ y = Cx + Dw \end{cases} \tag{2}$$

is generically observable. Note that (Σ_Λ) and (Σ_e) are associated to the same digraph $\mathcal{G}(\Sigma_\Lambda)$. Applying Theorem 1 to system (Σ_e), we obtain conditions of Corollary 1. △

Corollary 1 provide necessary conditions to the state and input observability of system (Σ_Λ). In order to refine this necessary conditions for obtaining more accurate analysis of the generic state and input observability of (Σ_Λ), we proceed like in Boukhobza et al. [2007] by carrying out a specific subdivision of the graph associated to (Σ_Λ).

4.2 Canonical subdivision of system (Σ_Λ)

We define a subdivision of structured bilinear system (Σ_Λ). This subdivision is presented and commented in (Boukhobza et al. [2007]):

Definition 2. For structured system (Σ_Λ) represented by digraph $\mathcal{G}(\Sigma_\Lambda)$, we define the vertex subsets:

$\mathbf{\Delta_0} \overset{def}{=} \big\{\mathbf{x_i} \,|\, \rho\big[\mathbf{W} \cup \{\mathbf{x_i}\}, \mathbf{Y}\big] = \rho\big[\mathbf{W}, \mathbf{Y}\big]\big\}$;

$\mathbf{X_1} \overset{def}{=} \big\{\mathbf{x_i} \,|\, \rho\big[\mathbf{W} \cup \{\mathbf{x_i}\}, \mathbf{Y}\big] > \rho\big[\mathbf{W}, \mathbf{Y}\big]\big\} = \mathbf{X} \setminus \mathbf{\Delta_0}$;

$\mathbf{Y_0} \overset{def}{=} \big\{\mathbf{y_i} \,|\, \rho\big[\mathbf{W}, \mathbf{Y}\big] > \rho\big[\mathbf{W}, \mathbf{Y} \setminus \{\mathbf{y_i}\}\big]\big\} = \mathbf{Y} \cap V_{ess}(\mathbf{W}, \mathbf{Y})$;

$\mathbf{Y_1} \overset{def}{=} \mathbf{Y} \setminus \mathbf{Y_0}$;

$\mathbf{W_0} \overset{def}{=} \big\{\mathbf{u_i} \,|\, \theta\big[\{\mathbf{u_i}\}, \mathbf{X_1} \cup \mathbf{Y_1}\big] = 0\big\}$; $\mathbf{W_1} \overset{def}{=} \mathbf{W} \setminus \mathbf{W_0}$;

$\mathbf{X_s} \overset{def}{=} \mathbf{S^o}(\mathbf{W_0}, \mathbf{Y}) \cap \mathbf{X}$ and $\mathbf{X_0} \overset{def}{=} \mathbf{\Delta_0} \setminus \mathbf{X_s}$.

Furthermore, we denote $n_0 = \text{card}\,(\mathbf{X_0})$, $n_s = \text{card}\,(\mathbf{X_s})$, $n_1 = \text{card}\,(\mathbf{X_1})$, $q_0 = \text{card}\,(\mathbf{W_0})$, $q_1 = \text{card}\,(\mathbf{W_1})$, $p_0 = \text{card}\,(\mathbf{Y_0})$ and $p_1 = \text{card}\,(\mathbf{Y_1})$.

Let us illustrate this definition on the system described in Example 1. Yet, we have already mentioned that $\rho\big[\mathbf{W}, \mathbf{Y}\big] = 2$. Since $\mathbf{Y} \cap V_{ess}(\mathbf{W}, \mathbf{Y}) = \emptyset$, we have that $\mathbf{Y_0} = \emptyset$, $\mathbf{Y_1} = \mathbf{Y}$.

Moreover, $\mathbf{\Delta_0} = \{\mathbf{x_1},\, \mathbf{x_2},\, \mathbf{x_5},\, \mathbf{x_6},\, \mathbf{x_9}\}$, $\mathbf{X_1} = \{\mathbf{x_3},\, \mathbf{x_4},\, \mathbf{x_7},\, \mathbf{x_8},\, \mathbf{x_{10}},\, \mathbf{x_{11}}\}$. Furthermore, contrary to $\mathbf{w_2}$, $\mathbf{w_1}$ cannot be linked with an edge to an element of $\mathbf{X_1} \cup \mathbf{Y_1}$, so $\mathbf{W_0} = \{\mathbf{w_1}\}$ and $\mathbf{W_1} = \{\mathbf{w_2}\}$. Finally, $V_{ess}(\mathbf{W_0}, \mathbf{Y}) = \{\mathbf{w_1}, \mathbf{x_2}, \mathbf{x_9}\}$ and so $\mathbf{S^o}(\mathbf{W_0}, \mathbf{Y}) = \{\mathbf{x_9}\}$. Thus, $\mathbf{X_s} = \{\mathbf{x_9}\}$ and $\mathbf{X_0} = \mathbf{\Delta_0} \setminus \mathbf{X_s} = \{\mathbf{x_1},\, \mathbf{x_2},\, \mathbf{x_5},\, \mathbf{x_6}\}$.

Note that, for a simple interpretation of the vertex subsets defined above, $\mathbf{X_0} \cup \mathbf{X_s}$ merges all the state vertices which are not connected to $\mathbf{Y}$, the state vertices belonging to $V_{ess}(\mathbf{U}, \mathbf{Y})$ and the state vertices from which all $\mathbf{Y}$-*topped* paths lead to $V_{ess}(\mathbf{U}, \mathbf{Y})$. In Boukhobza et al. [2007], the subdivision of the system described above is introduced. Mainly, if the system is left invertible *i.e.* $\rho\big[\mathbf{W}, \mathbf{Y}\big] = \text{card}\,(\mathbf{W})$, then $V_{ess}(\mathbf{W}, \mathbf{Y}) = V_{ess}(\mathbf{W_0}, \mathbf{Y}) \cup \mathbf{W_1}$; $\theta\big[\mathbf{X_s}, \mathbf{X_1} \cup \mathbf{Y_1}\big] = n_s$; $\mathbf{S^o}(\mathbf{W}, \mathbf{Y}) = \mathbf{S^o}(\mathbf{W_0}, \mathbf{Y}) \cup \mathbf{W_1} = \mathbf{S^o}(\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}) \cup \mathbf{W_1} = \mathbf{X_s} \cup \mathbf{Y_0} \cup \mathbf{W_1}$ and $\theta\big[\mathbf{X_0} \cup \mathbf{W_0}, \mathbf{X_1} \cup \mathbf{Y_1}\big] = 0$ (*i.e.* there is no edge from $\mathbf{X_0} \cup \mathbf{W_0}$ to $\mathbf{X_1} \cup \mathbf{Y_1}$).

Note that all the elements of $\mathbf{X_1}$ are output connected as well as all elements of $\mathbf{X_s}$ and $\mathbf{W_1}$.

Using definitions of $\mathbf{X_0}$, $\mathbf{X_s}$, $\mathbf{W_0}$, $\mathbf{W_1}$, $\mathbf{Y_0}$, $\mathbf{Y_1}$ and the properties of the subdivision given above, we can write system (Σ_Λ) as :

$$\left\{\begin{array}{rcl} \dot{X}_0(t) &=& A_{0,0}(u)X_0(t) + A_{0,s}(u)X_s(t) + A_{0,1}(u)X_1(t) + \\ && G_{0,0}W_0(t) + G_{0,1}W_1(t) \\ \dot{X}_s(t) &=& A_{s,0}(u)X_0(t) + A_{s,s}(u)X_s(t) + A_{s,1}(u)X_1(t) + \\ && G_{s,0}W_0(t) + G_{s,1}W_1(t) \\ \dot{X}_1(t) &=& A_{1,s}(u)X_s(t) + A_{1,1}(u)X_1(t) + G_{1,1}(u)W_1(t) \\ Y_0(t) &=& C_{0,0}X_0(t) + C_{0,s}X_s(t) + C_{0,1}X_1(t) + D_{0,0}W_0(t) + \\ && D_{0,1}W_1(t) \\ Y_1(t) &=& C_{1,s}X_s(t) + C_{1,1}X_1(t) + D_{1,1}W_1(t) \end{array}\right. \tag{3}$$

where X_0, X_s, W_0, W_1, Y_0 and Y_1 represent the state, unknown input and output associated to vertex subsets $\mathbf{X_0}$, $\mathbf{X_s}$, $\mathbf{W_0}$, $\mathbf{W_1}$, $\mathbf{Y_0}$ and $\mathbf{Y_1}$ respectively and for the simplicity of the notations, for $i,\, j \in \{0,\, 1,\, s\}$ and $k \in \{0,\, 1,\, \ldots,\, m\}$, $A_{i,j}(u)$ is in the form $A_{i,j}(u) = A_{i,j,0} + \sum_{k=1}^{m} u_k A_{i,j,k}$.

Starting from form 3, let us define the two following systems:

$$(\Sigma_0)\left\{\begin{array}{rcl} \dot{X}_0(t) &=& A_{0,0}(u)X_0(t) + A_{0,s}(u)X_s(t) + A_{0,1}(u)X_1(t) + \\ && G_{0,0}W_0(t) + G_{0,1}W_1(t) \\ \dot{X}_s(t) &=& A_{s,0}(u)X_0(t) + A_{s,s}(u)X_s(t) + A_{s,1}(u)X_1(t) + \\ && G_{s,0}W_0(t) + G_{s,1}W_1(t) \\ Y_0(t) &=& C_{0,0}X_0(t) + C_{0,s}X_s(t) + C_{0,1}X_1(t) + D_{0,0}W_0(t) + \\ && D_{0,1}W_1(t) \\ Y_s(t) &=& X_s(t) \\ Y_{x_1}(t) &=& X_1(t) \\ Y_w(t) &=& W_1(t) \end{array}\right.$$

$$(\Sigma_1)\left\{\begin{array}{rcl} \dot{X}_1(t) &=& A_{1,s}(u)X_s(t) + A_{1,1}(u)X_1(t) + G_{1,1}(u)W_1(t) \\ Y_1(t) &=& C_{1,s}X_s(t) + C_{1,1}X_1(t) + D_{1,1}W_1(t) \end{array}\right. \tag{4}$$

Roughly speaking, system (Σ_0) is defined by input W_0, state X_0 and output constituted by X_s and Y_0. For system (Σ_0), the entries X_1 and W_1 are assumed to be measured. System (Σ_1) is defined by input W_1 and X_s, state X_1 and output Y_1.

The following Lemma links the observability of (Σ_Λ) to the observability of both (Σ_0) and (Σ_1):

Lemma 1. Structured bilinear system (Σ_Λ) is generically state and input observable iff both structured systems (Σ_0) and (Σ_1) are generically state and input observable.

Proof:

Sufficiency: On the one hand, state variables and unknown inputs of structured bilinear (Σ_1) are generically observable means that X_1, W_1 and X_s can be expressed in function of Y_1, u and their derivatives. On the other hand, state variables and unknown inputs of structured bilinear (Σ_0) are generically observable mean that in (Σ_Λ), variables X_0, W_0 can be expressed in function of Y_1, u, X_s, X_1, W_1 and their derivatives and so according to the observability of (Σ_1), in function of Y_0, Y_1, u and their derivatives. Consequently, the fact that for both structured systems (Σ_0) and (Σ_1) state variables and unknown inputs are generically observable implies that state variables and unknown inputs of structured bilinear system (Σ_Λ) are generically observable.

Necessity: If state variables X_0 and unknown input W_0 of system (Σ_0) are not generically observable, then, since these variables are not present in (Σ_1) they can not be observable for structured system (Σ_Λ). Otherwise, because of $\rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] = \rho\big[\mathbf{W_0} \cup \mathbf{X_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] = \rho\big[\mathbf{W_0} \cup \mathbf{X}, \mathbf{Y_0} \cup \mathbf{X_s}\big] = \text{card}\,(\mathbf{W_0})$ the number of equations useful to the observation of unknown variables is equal to $\mu\big[\mathbf{W_0}, \mathbf{X_s} \cup \mathbf{Y_0}\big] - \rho\big[\mathbf{W_0}, \mathbf{X_s} \cup \mathbf{Y_0}\big]$. Indeed, every supplementary equation depends on $\dot{W}$.

Moreover, since, the number of elements of $\mathbf{W_0} \cup \mathbf{X_0}$ is at least equal to $\mu\big[\mathbf{W_0}, \mathbf{X_s} \cup \mathbf{Y_0}\big] - \rho\big[\mathbf{W_0}, \mathbf{X_s} \cup \mathbf{Y_0}\big]$, it is not possible to observe X_1, X_s or W_1. Therefore, if these variables are not generically observable from equations of subsystem (Σ_1) they can not be observable for structured system (Σ_Λ).

Hence, if one of systems (Σ_0) and (Σ_1) is not generically input and state observable, then (Σ_Λ) is not input and state observable. This ends the proof of Lemma 1. △

According to the previous lemma, to establish the generic observability of unknown variables of SBLS (Σ_Λ), we study, hereafter, the generic observability of unknown variables of both (Σ_0) and (Σ_1).

4.3 Generic observability of the square subsystem (Σ_0)

The necessary and sufficient condition which ensures the generic state and input observability of (Σ_0) is:

Proposition 1. Structured bilinear system (Σ_0) is generically state and input observable iff in digraph $\mathcal{G}(\Sigma_\Lambda)$ associated to (Σ_Λ), $\mathbf{X_0} \cup \mathbf{W_0} \subseteq V_{ess}(\mathbf{W}, \mathbf{Y})$.

Proof:

Sufficiency: Consider structured linear system constructed from (Σ_0) by putting inputs $u = u^* = cst$. Knowing that $dim(W_0) = dim(Y_0) + dim(Y_s)$ and that $\rho\big[\mathbf{W_0} \cup \mathbf{X_s} \cup \mathbf{X_0}, \mathbf{Y_0} \cup \mathbf{Y_s}\big] = \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{Y_s}\big] = \text{card}\,(\mathbf{W_0})$. The unknown variables associated to $\mathbf{X_0} \cup \mathbf{X_s} \cup \mathbf{W_0}$ of this system are observable iff (Hou and Müller [1999]) matrix $P_0(s) \overset{def}{=} \begin{pmatrix} A^\lambda_{0,0}(u^*) - sI_{n_0} & A^\lambda_{0,s}(u^*) & G^\lambda_{0,0} \\ A^\lambda_{s,0}(u^*) & A^\lambda_{s,s}(u^*) - sI_{n_s} & G^\lambda_{s,0} \\ C^\lambda_{0,0} & C^\lambda_{0,s} & D^\lambda_{0,0} \\ 0 & I_{n_s} & 0 \end{pmatrix}$ has generically a full column rank $\forall\, s \in \mathbb{C}$. We can prove (Boukhobza et al. [2007]) using results of Theorem 5.1 of (van der Woude [2000]) that the degree of the determinant of $P_0(s)$ is generically equal to $dim(X_0) + dim(W_0) - \Big(\mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W}, \mathbf{Y_0} \cup \mathbf{X_s}\big]\Big)$.

On the other hand , since $V_{ess}(\mathbf{W}, \mathbf{Y}) = V_{ess}(\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}) \cup \mathbf{W_1}$, we have that $\mathbf{W_0} \cup \mathbf{X_0} \subseteq V_{ess}(\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s})$ implies that $\mathbf{W_0} \cup \mathbf{X_0} \subseteq V_{ess}(\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s})$. This involves that $\text{card}\,(\mathbf{X_0} \cup \mathbf{W_0}) = \mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big]$. In this case the degree of $det(P_0(s))$ is equal to $dim(X_0) + dim(W_0) - (dim(X_0) + dim(W_0) + dim(Y_0) + dim(X_s)) - (dim(Y_0) + dim(X_s)) = 0$. Moreover, since the the existence of maximum size linking which covers all vertices of $\mathbf{W_0} \cup \mathbf{X_0}$ implies also that $\theta\big[\mathbf{W_0} \cup \mathbf{X_0}, \mathbf{X_0} \cup \mathbf{Y_0} \cup \mathbf{X_s}\big] = dim(\mathbf{X_0}) + dim(\mathbf{W_0})$, and because of $g\_rank(P_0(0)) = \theta\big[\mathbf{W_0} \cup \mathbf{X_0}, \mathbf{X_0} \cup \mathbf{Y_0} \cup \mathbf{X_s}\big]$ then $g\_rank(P_0(0)) = dim(X_0) + dim(W_0)$. Consequently, $\mathbf{W_0} \cup \mathbf{X_0} \subseteq V_{ess}(\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s})$ ensures that almost all linear system constituted by system (Σ_0) with constant inputs u is generically input and state observable and so also that SBLS (Σ_0) is generically observable.

Necessity: The state variables of system (Σ_0) are X_0 and X_s. X_s is observable since we have the output $Y_s = X_s$. The unknown input variables are W_0, W_1 and X_1. W_1 and X_1 are measured through Y_w and Y_{x_1}. Thus, these two output vectors cannot be used to observe W_0 or X_0. Furthermore, since the system is square with $\rho\big[W_0, Y_0 \cup X_s\big] = card(W_0)$, the number of equations which can be used to observe the system (*i.e.* the equation where the unknown input derivatives do not intervene) is equal to $\mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big]$. To observe all the elements of $\mathbf{W_0} \cup \mathbf{X_0}$ the number of these elements must be less or equal to the number of equations. Thus, a necessary condition to observe elements of $\mathbf{W_0} \cup \mathbf{X_0}$ is that

$$\text{card}\,(\mathbf{X_0} \cup \mathbf{W_0}) \leq \mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] \quad (5)$$

Yet, all the vertices in a maximal size $\mathbf{W_0} - \mathbf{X_s} \cup \mathbf{Y_0}$ linking are included in $\mathbf{W_0} \cup \mathbf{X_0} \cup \mathbf{Y_0} \cup \mathbf{X_s}$. Thus, we have always that $\text{card}\,(\mathbf{X_0} \cup \mathbf{W_0}) \geq \mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big]$. So, condition (5) can be written as $\text{card}\,(\mathbf{X_0} \cup \mathbf{W_0}) = \mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big]$. Moreover, the minimal number of vertices in a maximal size $\mathbf{W_0} - \mathbf{X_s} \cup \mathbf{Y_0}$ linking is equal by definition to $\mu\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big] - \rho\big[\mathbf{W_0}, \mathbf{Y_0} \cup \mathbf{X_s}\big]$ and so to $\text{card}\,(\mathbf{X_0} \cup \mathbf{W_0})$. Then any maximal linking $\mathbf{W_0} - \mathbf{X_s} \cup \mathbf{Y_0}$ covers all the vertices of $\mathbf{X_0} \cup \mathbf{W_0}$. So, $\mathbf{X_0} \cup \mathbf{W_0} = \mathbf{V_{ess}}(\mathbf{W_0}, \mathbf{X_s} \cup \mathbf{Y_0}) \subseteq V_{ess}(\mathbf{W}, \mathbf{Y})$. △

For the system considered in Example 1, we have $\mathbf{Y_0} = \emptyset$, $\mathbf{W_0} = \{\mathbf{w_1}\}$, $\mathbf{X_0} = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_5}, \mathbf{x_6}, \}$ and $\mathbf{X_s} = \{\mathbf{x_9}\}$. Moreover, $V_{ess}(\mathbf{W}, \mathbf{Y}) = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{x_2}, \mathbf{x_9}\}$ and since $\mathbf{X_0} \cup \mathbf{W_0} \not\subseteq V_{ess}(\mathbf{W}, \mathbf{Y})$, subsystem (Σ_0) is not generically input and state observable. Therefore, from Lemma 1 system of Example 1 is also not generically observable.

4.4 Generic observability of subsystem (Σ_1)

For the observability of subsystem (Σ_1), we do not have a necessary and sufficient conditions. However, we provide two groups of conditions. The first conditions are necessary and the second ones are sufficient. Starting from Corollary 1, we have:

Proposition 2. Structured bilinear system (Σ_1), is generically observable only if in digraph $\mathcal{G}(\Sigma_\Lambda)$, there exists an A-disjoint subgraph $\mathcal{S_G}$ which covers all the vertices included in $\mathbf{X_1} \cup \mathbf{X_s} \cup \mathbf{W_1}$ and such that all the edges of $\mathcal{S_G}$ end in $\mathbf{X_1} \cup \mathbf{Y_1}$.

We can also state the following sufficient condition :

Proposition 3. Structured bilinear system (Σ_1), is generically observable if in digraph $\mathcal{G}(\Sigma_\Lambda)$, there exists a disjoint union of
- a maximal $\mathbf{W_1} \cup \mathbf{X_s}$-$\mathbf{Y_1}$ linking having a minimum length and
- an A-disjoint subgraph $\mathcal{S_G}$ which covers all the vertices included in $\mathbf{X_1} \cup \mathbf{X_s} \cup \mathbf{W_1}$ with the constraint that all the edges of $\mathcal{S_G}$ end in $\mathbf{X_1} \cup \mathbf{Y_1}$.

Sketch of the Proof:
Let us assume that conditions of Proposition 3 are satisfied and let us denote by $\mathbf{X_W}$ and $\mathbf{Y_W}$ the state and output vertices covered by the maximal $\mathbf{W_1} \cup \mathbf{X_s}$-$\mathbf{Y_1}$ linking having a minimum length. Similarly, we denote $\mathbf{X_r} = \mathbf{X_1} \setminus \mathbf{X_W}$ and $\mathbf{Y_r} = \mathbf{Y_1} \setminus \mathbf{Y_W}$. The first condition of Proposition 3 implies that we can express W_1, X_s and X_W in function of input u and output components constituting Y_r, their derivatives and X_r. This implies that the dynamics of X_r can be written only in function of X_r, Y_1 and input u and their derivatives. By considering Y_W as a known variable which does not influe on the observability of X_r, This is equivalent to say that the subsystem defined by state X_r and output Y_r is described by a graph equivalent to the one representing (Σ_1) restricted to state vertices $\mathbf{X_r}$ and output vertices $\mathbf{Y_r}$ plus some additional edges from $\mathbf{X_r}$ to $\mathbf{X_r}$. Knowing that these additional edges are related to free parameters in the matrices representing the dynamics of X_r, the second condition of Proposition 3 implies, from Theorem 1, that state X_r is observable and equivalently can be expressed in function of u, Y_1 and their derivatives. Thus, substituting X_r by this expression, we have that W_1, X_s and X_W are also expressed in function of u, Y_1 and their derivatives. Therefore, all the state and input components X_1, X_s, W_1 are strongly observable and so system (Σ_1) is generically input and state observable. △

In the case of the system considered in Example 1, for subsystem (Σ_1) defined by input $\mathbf{W_1} \cup \mathbf{X_s} = \{\mathbf{w_2}\} \cup \{\mathbf{x_9}\}$, state $\mathbf{X_1} = \{\mathbf{x_3}, \mathbf{x_4}, \mathbf{x_7}, \mathbf{x_8}, \mathbf{x_{10}}, \mathbf{x_{11}}\}$ and the output $\mathbf{Y_1} = \mathbf{Y}$, the necessary condition is satisfied. Indeed, Figure 2 displays an A-disjoint subgraph $\mathcal{S_G}$ which covers all the vertices included in $\mathbf{X_1} \cup \mathbf{X_s} \cup \mathbf{W_1}$ and where all the edges constituting $\mathcal{S_G}$ end in $\mathbf{X_1} \cup \mathbf{Y_1}$.

We can exhibit now two groups of conditions to characterize

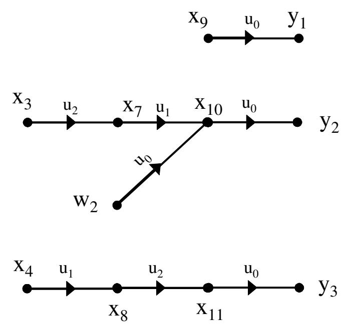

Figure 2. A-disjoint subgraph $\mathcal{S_G}$ which covers all the vertices included in $\mathbf{X_1} \cup \mathbf{X_s} \cup \mathbf{W_1}$ for Example 1

the state and input observability of SBLS (Σ_Λ):

Necessary conditions : Structured bilinear system (Σ_Λ) is generically state and input observable only if
- $\mathbf{X_0} \cup \mathbf{W_0} \subseteq V_{ess}(\mathbf{W}, \mathbf{Y})$
- in digraph $\mathcal{G}(\Sigma_\Lambda)$, there exists an A-disjoint subgraph $\mathcal{S_G}$ which covers all the vertices included in $\mathbf{X_1} \cup \mathbf{X_s} \cup \mathbf{W_1}$ and such that all the edges of $\mathcal{S_G}$ end in $\mathbf{X_1} \cup \mathbf{Y_1}$.

Sufficient conditions : Structured bilinear system (Σ_Λ) is generically state and input observable if
- $\mathbf{X_0} \cup \mathbf{W_0} \subseteq V_{ess}(\mathbf{W}, \mathbf{Y})$
- in digraph $\mathcal{G}(\Sigma_\Lambda)$, there exists a disjoint union of a maximal $\mathbf{W_1} \cup \mathbf{X_s}$-$\mathbf{Y_1}$ linking having a minimum length and an A-disjoint subgraph $\mathcal{S_G}$ which covers all the vertices included in $\mathbf{X_1} \cup \mathbf{X_s} \cup \mathbf{W_1}$ with the constraint that all the edges of $\mathcal{S_G}$ end in $\mathbf{X_1} \cup \mathbf{Y_1}$.

5. CONCLUSION

In this paper, we propose an analysis tool to study the generic state and input observability of structured bilinear systems. Using a graphic representation dedicated to this class of nonlinear systems, some necessary and/or sufficient conditions are provided and expressed in graphic terms. More precisely, we subdivide the considered system into two particular subsystems named (Σ_0) and (Σ_1), such that the generic state and input observability of the original system is equivalent to the generic state and input observability of (Σ_0) and (Σ_1) simultaneously. Then, we enounce necessary and sufficient graphical condition to the generic state and input observability of (Σ_0). Finally we provide, for the generic state and input observability of (Σ_1), some necessary and other sufficient conditions.

All the presented conditions are far to be trivial. Furthermore they need few information about the system and are very easy to check by means of well-known combinatorial techniques or simply by hand for small systems. That makes our approach particularly suited for large-scale and sparse systems as it is free from numerical difficulties. In fact, the proposed analysis is based on three steps. First, we have subdivide the system into two subsystems. Then, we check the observability of (Σ_0) by computing a maximal input-output linking size and the set of essential vertices in an input-output maximal linkings, which is also based on a calculation of maximal input-output linking size. Finally, we analyse the observability of (Σ_1) by computing the maximal A-disjoint matching for the necessary condition and by searching the minimal length of input-output maximal linking for the sufficient condition.

More precisely, the subdivision of the system requires $n+p+1$ computations of maximal linking size and at most $n+q$ computations of maximal matching size. Using a transformation of the problem into a Max-Flow one, the computation of the maximal linking size requires algorithms which have a complexity order $O(N^2\sqrt{M})$, where M is the number of edges in the digraph and $N = n + p + q$ the number of vertices. For our digraphs, in the worst case $M = (m + 1) \cdot n^2 + n \cdot p + n \cdot q + q \cdot p$. To compute the maximal matching and so the maximal A-disjoint matching, we use the Bipmatch method (Micali and Vazirani [1980]). The complexity order of algorithms using this method is, in the worst case, $O(M \cdot N^{0.5})$.

For subsystem (Σ_0), the necessary and sufficient condition of Proposition 1 necessitates only the test that all vertices of $\mathbf{X_0} \cup \mathbf{W_0}$ are essential. This is easily done with also a computation of a maximal linking size.

For subsystem (Σ_1), the first necessary condition can be checked using depth search algorithms. These algorithms have a complexity order $O(M \cdot N)$. Thus, the complexity of these algorithms, in our case, is $O(m \cdot n^3)$, (assuming without loss of generality that $p \leq n$ and $q < n$). To verify the second necessary condition, we compute the maximal matching in a bipartite graph (Boukhobza and Hamelin [2007]). So, for checking the second necessary condition, we can use algorithms which have complexity order $O(m^{3/2} \cdot n^{5/2})$. The sufficient condition requires, in addition to the computation of maximal A-disjoint matching discussed previously, the characterization of the maximal input-output linking of minimal length. This can be done with an algorithm which complexity order equals $O(N^3 \times M^{0.5})$ using an algorithm similar to the primal-dual one presented in (Hovelaque et al. [1996]).

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