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Robust Output-Feedback MPC with Soft State Constraints

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Abstract: In this paper, we present a robust output-feedback model predictive control (MPC) design for a class of open-loop stable systems with hard input- and soft state constraints. The proposed output-feedback design is based on a linear state estimator and a novel parameterization of the soft state constraints that has the advantage of leading to optimization problems of prescribable size. Robustness against unstructured model uncertainty is obtained by choosing the cost function parameters so as to satisfy a linear matrix inequality condition. The robust output-feedback design incorporates a novel state-feedback design, which may be seen as a generalization of a previous proposal.

1. INTRODUCTION

Model predictive control (MPC) policies are optimization based control policies that calculate the current control input by solving an optimization problem parameterized by the current system state. The optimization problem in MPC is typically an open-loop optimal control problem which incorporates a dynamic model of the system in order to calculate predictions from the current state over a future horizon. A major advantage of the MPC methodology is that constraints on system inputs and states can be handled by imposing constraints on the predictions. However, due to model uncertainty, unmeasured disturbances, etc., constrained MPC policies may face infeasibility problems, that is, they may encounter a system state for which the constraints on the associated predictions can not be met.

To avoid the occurrence of infeasibility problems, it is often beneficial to make use of *soft constraints* (de Oliveira and Biegler, 1994; Zheng and Morari, 1995; Scokaert and Rawlings, 1999; Choi and Kwon, 2003). In an MPC policy with soft constraints, constraint violations are allowed but (near) constraint satisfaction and good performance can nevertheless be achieved, provided the slack variables that parameterize the constraint violations are suitably penalized in the cost function. Zheng and Morari (1995) suggest to penalize the maximum constraint violation over the horizon. This approach, however, may lead to poor performance and be difficult to tune, especially for non-minimum phase systems, as shown by Scokaert and Rawlings (1999). To overcome such difficulties, Scokaert and Rawlings (1999) proposed an MPC policy which penalizes the sum of the norm of the constraint violations.

A computational issue inherent in the approach of both Zheng and Morari (1995) and Scokaert and Rawlings (1999) is that the optimization problem includes an infinite number of soft constraints. Hence, to solve the problem accurately using finite dimensional optimization it is necessary to remove (or finitely parameterize) all constraints after a suitably long horizon $\bar{N}$. However, the horizon $\bar{N}$ generally depends on, and may increase indefinitely with, the size of the current state (Choi and Kwon, 2003).

In this paper, we propose a soft-constrained MPC scheme which only requires the solution of a single quadratic programme (QP) of prescribable size. The proposed design generalizes the design of Scokaert and Rawlings (1999) in the case of quadratic penalty functions. That is, if the terminal constraint set and the horizons are chosen to be sufficiently large, then the proposed state-feedback MPC scheme yields equivalent results to that of Scokaert and Rawlings (1999). A major advantage of the new design, however, is that it is not necessary to choose the horizons sufficiently large, since the algorithm is globally exponentially stable for any horizon length and is based on a finite dimensional QP whose size is independent of the current state.

Another, perhaps more significant, advantage of the proposed state-feedback design is that it extends naturally to a robust output-feedback design, for a class of open-loop stable systems with unstructured model uncertainty, when combined with a linear state estimator. In particular, we show that global ℓ_2-stability can be ensured by choosing the cost function parameters so as to satisfy a linear matrix inequality (LMI) condition. The robust output-feedback design described here may be seen as a generalization of the robust input-constrained design proposed in Løvaas, Seron, and Goodwin (2008). The present approach to stability analysis is an application of the "dissipativity approach" proposed in Løvaas, Seron, and Goodwin (2007a).

The paper outline is as follows: Section 2 describes the uncertain open-loop system. Section 3 proposes a state-feedback MPC policy for the nominal component of the system. Section 4 extends the results of Section 3 and proposes a robust output-feedback MPC policy for the uncertain system. Throughout we use the following notation: $\|x\|_P^2$ denotes $x^{\mathrm{T}}Px$, $[a, \cdots, c]$ denotes $[a^{\mathrm{T}} \cdots c^{\mathrm{T}}]^{\mathrm{T}}$ and I_q denotes the $q \times q$ identity matrix.

2. SYSTEM DESCRIPTION

We consider a discrete-time system whose evolution from time $k = 0$ onwards is described by the feedback interconnection shown in Fig. 1, where

$$G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} \qquad (1)$$

is a known rational transfer function, and where Δ is a causal operator satisfying

$$\|\Delta\|_\infty \leq 1. \qquad (2)$$

We will impose strict causality and open-loop stability for any causal Δ satisfying (2). To this end, we require the following assumption:

Assumption 2.1. $G(z)$ is stable and causal,

$$\|G_{22}(z)\|_\infty < 1, \qquad (3)$$

and $G_{11}(z)$ and $G_{12}(z)$ are strictly causal.

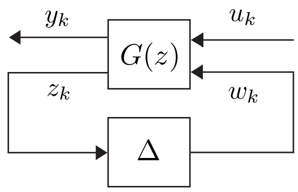

Fig. 1. Uncertain system.

Subject to Assumption 2.1, a state-space representation of the system, such that

$$\begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} = \begin{bmatrix} C \\ C_z \end{bmatrix} (zI - A)^{-1} [B \ B_w] + \begin{bmatrix} 0 & 0 \\ D & D_w \end{bmatrix},$$

has the form

$$x_{k+1} = Ax_k + Bu_k + B_w w_k, \quad x_0 = x, \qquad (4a)$$
$$y_k = Cx_k, \qquad (4b)$$
$$z_k = C_z x_k + Du_k + D_w w_k, \qquad (4c)$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state, and where the auxiliary system input $w_k \in \mathbb{R}^{n_w}$ is the response of Δ to the auxiliary system output $z_k \in \mathbb{R}^{n_z}$. Here we assume that Δ has a finite initial condition so that, in view of (2), its input and output satisfy the following sum quadratic constraint (SQC):

$$\sum_{k=0}^{L} \begin{bmatrix} z_k \\ w_k \end{bmatrix}^{\mathrm{T}} M_1 \begin{bmatrix} z_k \\ w_k \end{bmatrix} + \beta_1 \geq 0, \quad M_1 \triangleq \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad (5)$$

for any integer $L \geq 0$ and some scalar β_1.

The system is subject to the following constraints

$$Vu_k \leq v, \ \forall k \geq 0, \qquad (6)$$
$$Hx_k \leq h, \ \forall k \geq 0, \qquad (7)$$

where $V \in \mathbb{R}^{n_v \times n_u}$, $v \geq 0$, $H \in \mathbb{R}^{n_h \times n_x}$ and $h \geq 0$ are matrices of appropriate dimension. The input constraints (6) are "hard" and must be respected at all time, whereas the state constraints (7) are "soft" and will be treated by penalizing constraint violations in the MPC cost function.

Remark 2.1. The input-output map, $\bar{G}$, of the system in Fig. 1 is given by the linear fractional transformation

$$\bar{G} = G_{11} + G_{12}\Delta (I - G_{22}\Delta)^{-1} G_{21}.$$

Hence, we interpret $G_{11}(z)$ as the nominal model of the system, whereas inequality (2) and $G_{12}(z)$, $G_{21}(z)$, $G_{22}(z)$ constitute a description of the model uncertainty associated with $G_{11}(z)$.

To simplify the presentation, we assume that the maximal output admissible set O_∞ associated with the state constraints (7) is finitely determined with $0 \in \mathrm{int}(O_\infty)$, that is, we have

$$\begin{aligned} O_\infty &\triangleq \{x \,|\, HA^k x \leq h, \ \forall k \geq 0\} \\ &= \{x \,|\, T_\infty x \leq t_\infty\}, \end{aligned} \qquad (8)$$

for some matrix T_∞ and vector t_∞ with positive elements. A sufficient condition for (8) to hold is that $h > 0$, $H = [\bar{C}, -\bar{C}]$, where the pair $\bar{C}$, A is observable (see, e.g., Gilbert and Tan (1991)).

3. NOMINAL CASE WITH STATE-FEEDBACK

In this section, we propose a soft-constrained state-feedback MPC policy for the nominal system obtained by use of $B_w = 0$ in (4), that is,

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 = x. \qquad (9)$$

The proposed state-feedback policy will be subsequently used in Section 4 to construct a robust output-feedback policy for the uncertain system (4).

3.1 Existing State-Feedback MPC

For clarity, we shall present the state-feedback design as a generalization of the MPC scheme proposed by Scokaert and Rawlings (1999). At each time step k, the latter MPC scheme solves the following optimal control problem, using the current state of the system (9) as a parameter $x = x_k$:

$$[P]: \quad \Psi = \min_{\{u_i\}_{i=0}^{N_u-1}, \{\epsilon_i\}_{i=0}^{\infty}} \sum_{i=0}^{\infty} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 + \|\epsilon_i\|_S^2\right)$$

$$\text{subject to} \begin{cases} x_0 & = x \\ x_{i+1} & = Ax_i + Bu_i \\ Vu_i & \leq v, \ \forall i \in \{0, \cdots, N_u - 1\} \\ u_i & = 0, \ \forall i \geq N_u \\ Hx_i & \leq h + \epsilon_i, \ \forall i \geq 0 \end{cases}$$

Here, we require $N_u \geq 1$, $Q \geq 0$, $R > 0$ and $S > 0$.

Remark 3.1. Unlike "Problem 2: Soft Constraint MPC" in Scokaert and Rawlings (1999), the present problem $[P]$ does not include constraints of the type $\epsilon_i \geq 0$ and stage cost terms of the type $s^{\mathrm{T}}\epsilon_i$, for some vector $s \geq 0$. However, our treatment corresponds precisely to choosing $s = 0$ in Scokaert and Rawlings (1999), since the constraints $\epsilon_i \geq 0$ will be satisfied naturally in problem $[P]$ whenever $S > 0$ is diagonal, as assumed/recommended by Scokaert and Rawlings (1999). For brevity and consistency with Section 4, we shall not address the case $s \neq 0$, although generalizations to non-quadratic penalty functions are possible (see Remark 3.4).

Problem $[P]$ is of infinite dimension. However, the restriction that $\epsilon_i = 0$ for all $i \geq N$, where N is suitably large, is not suboptimal (Scokaert and Rawlings, 1999) and may be used to formulate an equivalent finite dimensional problem. Although Scokaert and Rawlings (1999) did not consider any specific finite dimensional optimization suited for solving $[P]$, it is convenient in view of Lemma 1 below and related results in Chmielewski and Manousiouthakis

(1996); Scokaert and Rawlings (1998) to consider the following:

$$[P^N]: \quad \Psi^N = \min_{\{u_i\}_{i=0}^{N_u-1},\,\{\epsilon_i\}_{i=0}^{\infty}} \sum_{i=0}^{\infty} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 + \|\epsilon_i\|_S^2\right)$$

$$\text{subject to} \begin{cases} x_0 &= x \\ x_{i+1} &= Ax_i + Bu_i \\ Vu_i &\le v,\ \forall i \in \{0,\cdots,N_u-1\} \\ u_i &= 0,\ \forall i \ge N_u \\ Hx_i &\le h + \epsilon_i,\ \forall i \in \{0,\cdots,N-1\} \\ \epsilon_i &= 0,\ \forall i \ge N \end{cases}$$

Here, we assume, for simplicity, $N \ge N_u$. We denote by $\mathbf{u}^N = \{u_i^N\}_{i=0}^{N_u-1}$ and $\epsilon^N = \{\epsilon_i^N\}_{i=0}^{\infty}$ the optimal sequences of $[P^N]$ and by $\{x_i^N\}$ the associated state trajectory. The following lemma provides a necessary and sufficient condition for $[P^N]$ to be equivalent to $[P]$. The lemma has been adapted from Lemma 2 in Chmielewski and Manousiouthakis (1996).

Lemma 1. (i) $\Psi^\infty \triangleq \lim_{N\to\infty} \Psi^N$ exists.
(ii) $\Psi^N = \Psi^\infty \Leftrightarrow \Psi^N = \Psi^{N+k};\ \forall k \ge 0 \Leftrightarrow (\mathbf{u}^N, \epsilon^N) = (\mathbf{u}^{N+k}, \epsilon^{N+k});\ \forall k \ge 0 \Leftrightarrow x_N^N \in O_\infty$.
(iii) If there exists N such that $x_N^N \in O_\infty$ then $\Psi^N = \Psi^\infty = \Psi$.
(iv) There exists N such that $x_N^N \in O_\infty$.

It follows from Lemma 1 that to solve $[P]$ it suffices to solve $[P^N]$, $N \ge \bar{N}(x)$, where

$$\bar{N}(x) \triangleq \min\{N \mid N \ge N_u,\ x_N^N \in O_\infty\}. \tag{10}$$

Furthermore, the integer $\bar{N}(x)$ can be identified by solving $[P^N]$ for increasing values of N until the condition $x_N^N \in O_\infty$ is met. The MPC scheme of Scokaert and Rawlings (1999) can thus be described as follows:

Algorithm 1. Off-line: (i) Choose any integer $N_u \ge 1$. (ii) Choose any matrices $Q \ge 0$, $R > 0$, $S > 0$. *On-line*: At each time step $k \ge 0$, solve $[P^N]$, using $N \ge \bar{N}(x_k)$ and $x = x_k$, then apply $u_k = u_0^N$ to (9).

Algorithm 1 can be shown to be exponentially stable (Scokaert and Rawlings, 1999). However, if Algorithm 1 is implemented incorrectly so that the condition $N \ge \bar{N}(x_k)$ is violated, then the closed-loop system may be unstable.

3.2 Proposed State-Feedback MPC

The following optimization problem generalizes $[P^N]$ and leads to an MPC scheme which is closed-loop stable regardless of our choice of, for example, $N \ge N_u \ge 1$:

$$[P^{N,N_\epsilon}]: \quad J^*(x) = \min_{U,\varepsilon,e} J(x,U,\varepsilon,e)$$

$$\text{subject to} \begin{cases} x_0 &= x \\ x_{i+1} &= Ax_i + Bu_i \\ Vu_i &\le v,\ \forall i \in \{0,\cdots,N_u-1\} \\ u_i &= 0,\ \forall i \ge N_u \\ Hx_i &\le h + \epsilon_i,\ \forall i \in \{0,\cdots,N_\epsilon-1\} \\ Hx_i &\le h + HA^{i-N_\epsilon}e,\ \forall i \in \{N_\epsilon,\cdots,N-1\} \\ Tx_N &\le t + TA^{N-N_\epsilon}e \end{cases} \tag{11}$$

where $U = [u_0, \cdots, u_{N_u-1}]$, $\varepsilon = [\epsilon_0, \cdots, \epsilon_{N_\epsilon-1}]$ and where

$$J(x,U,\varepsilon,e) \triangleq \|[x,U,\varepsilon,e]\|_P^2, \tag{12}$$

for some appropriate matrix P whose selection will be explained below. Here, we have introduced an additional horizon N_ϵ, which may be chosen freely so as to satisfy $N \ge N_\epsilon \ge 1$ and can be used to significantly reduce the number of slack variables ϵ_i as compared with $[P^N]$. Also, to "summarize" constraint violations beyond the prediction time $i = N_\epsilon - 1$, we have introduced a slack variable $e \in \mathbb{R}^{n_x}$. The matrices T and t describe a "terminal constraint set" whose selection will be described later. In the sequel, we use $U^*(x)$, $\varepsilon^*(x)$, $e^*(x)$ to denote the optimal values of U, ε, e, resulting from $[P^{N,N_\epsilon}]$.

To describe various conditions on the cost function matrix P, we introduce the following system:

$$\begin{bmatrix} x_{n+1} \\ U_{n+1} \\ \varepsilon_{n+1} \\ e_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A & [B\ 0 \cdots 0] & 0 & 0 \\ 0 & \Gamma(N_u, n_u) & 0 & 0 \\ 0 & 0 & \Gamma(N_\epsilon, n_h) & \bar{H} \\ 0 & 0 & 0 & A \end{bmatrix}}_{\bar{A}_0} \begin{bmatrix} x_n \\ U_n \\ \varepsilon_n \\ e_n \end{bmatrix}, \tag{13}$$

where $\bar{H} \triangleq [0, \cdots, 0, H]$, and where $\Gamma(\bar{N}, \bar{n})$ is a matrix such that using $\bar{U} = \left[\bar{u}_0, \cdots, \bar{u}_{\bar{N}-1}\right]$ we have $\Gamma(\bar{N}, \bar{n})\bar{U} = [\bar{u}_1, \cdots, \bar{u}_{N-1}, 0]$, that is,

$$\Gamma\left(\bar{N}, \bar{n}\right) = \begin{bmatrix} 0 & I_{\bar{n}} & 0 & \cdots & 0 \\ \vdots & 0 & I_{\bar{n}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & I_{\bar{n}} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar{N}\bar{n} \times \bar{N}\bar{n}}. \tag{14}$$

We also require the following matrix function:

$$\Sigma_{\{Q,R,S\}}(P) \triangleq \bar{A}_0^{\mathrm{T}} P \bar{A}_0 - P + \bar{C}_0^{\mathrm{T}} \mathrm{diag}\,[Q,\, R,\, S]\, \bar{C}_0, \tag{15}$$

where $\bar{A}_0$ is the matrix defined in (13), $Q \in \mathbb{R}^{n_x \times n_x}$, $Q \ge 0$, $R \in \mathbb{R}^{n_u \times n_u}$, $R > 0$, $S \in \mathbb{R}^{n_h \times n_h}$, $S > 0$, and where the matrix $\bar{C}_0$ is such that $\bar{C}_0[x, U, \varepsilon, e] = [x, u_0, \epsilon_0]$.

The following theorem connects the three optimization problems described above.

Theorem 2. If $T = T_\infty$, $t = t_\infty$ [as defined in (8)], $N_\epsilon = N \ge \bar{N}(x)$ [as defined in (10)] and $\Sigma_{\{Q,R,S\}}(P) = 0$ in (15), then problems $[P]$, $[P^N]$ and $[P^{N,N_\epsilon}]$ are equivalent, that is, $\Psi = \Psi^N = J^*(x)$ and $U^*(x) = \left[u_0^N, \cdots, u_{N_u-1}^N\right]$, $\varepsilon^*(x) = \left[\epsilon_0^N, \cdots, \epsilon_{N-1}^N\right]$.

Proof When $N_\epsilon = N$ and $\Sigma_{\{Q,R,S\}}(P) = 0$, the cost function (12) satisfies

$$J(x,U,\varepsilon,e) \triangleq \|x_{N_u}\|_{P_F}^2 + \sum_{i=0}^{N_u-1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2\right) + \|e\|_\Pi^2 + \sum_{i=0}^{N-1} \|\epsilon_i\|_S^2, \tag{16}$$

where $A^{\mathrm{T}} P_F A - P_F = -Q$ and $A^{\mathrm{T}} \Pi A - \Pi = -H^{\mathrm{T}} S H$, and where x_i is given by (11). Since $N \ge \bar{N}(x)$, $T = T_\infty$, $t = t_\infty$ and $T_\infty x_N^N \le t_\infty$, the solution to $[P^N]$ (i.e., $\bar{U} = \left[u_0^N, \cdots, u_{N_u-1}^N\right]$, $\bar{\varepsilon} = \left[\epsilon_0^N, \cdots, \epsilon_{N-1}^N\right]$) together with the choice $e = 0$ will yield a feasible solution to $[P^{N,N_\epsilon}]$ at a cost of $J(x, \bar{U}, \bar{\varepsilon}, 0) = \Psi^N$. Hence, we have $\Psi^N \ge J^*(x)$. Using a similar argument we can also establish the reverse inequality $\Psi^N \le J^*(x)$, since $\Pi \ge 0$, and thus we conclude that $\Psi^N = J^*(x)$. The remainder of the theorem then follows from Lemma 1 and uniqueness properties implied by strict convexity (c.f. $R > 0$, $S > 0$).

Remark 3.2. Note that, if the pair H, A is not observable, then choosing $\Sigma_{\{Q,R,S\}}(P) = 0$ leads to a semi-definite QP since the matrix Π in (16) will be singular. In this case, to ensure uniqueness of $e^*(x)$, one may modify Π or add an equality constraint of the form $T_{\text{no}}e = 0$, where T_{no} is a matrix which selects the modes of A unobservable by H.

The proposed state-feedback MPC design is based on $[P^{N,N_\epsilon}]$ as follows:

Algorithm 2. Off-line: (i) Choose any integers N, N_u and N_ϵ satisfying $N \geq N_u \geq 1$, $N \geq N_\epsilon \geq 1$. (ii) Choose any matrices $Q \geq 0$, $R > 0$, $S > 0$. (iii) Choose any matrix P satisfying $\Sigma_{\{Q,R,S\}}(P) \leq 0$. (iv) Choose any T and t such that the set $X_F \triangleq \{x \,|\, Tx \leq t\}$ satisfies

$$Ax \in X_F, \forall x \in X_F, \quad X_F \subseteq \{x \,|\, Hx \leq h\}. \tag{17}$$

On-line: At each time step $k \geq 0$, solve $[P^{N,N_\epsilon}]$, using $x = x_k$, then apply $u_k = [I \; 0 \; \cdots \; 0]U^*(x)$ to (9).

Next we establish closed-loop stability of Algorithm 2.

Theorem 3. The closed-loop system under Algorithm 2 is globally exponentially stable. Moreover, the closed-loop trajectories satisfy

$$\sum_{k=0}^{\infty} \left(\|x_k\|_Q^2 + \|u_k\|_R^2 + \|\epsilon_k^*\|_S^2\right) \leq J^*(x_0), \tag{18}$$

where ϵ_k^* denotes the first block component of $\varepsilon^*(x_k)$.

Proof Let

$$\mathbb{S} \triangleq \{[x, U, \varepsilon, e] \,|\, [x, U, \varepsilon, e] \text{ satisfy (11)}\}, \tag{19}$$

so that the constraints in $[P^{N,N_\epsilon}]$ can be written as $[x, U, \varepsilon, e] \in \mathbb{S}$. Note that problem $[P^{N,N_\epsilon}]$ is always feasible, since a particular feasible solution is given by $[U, \varepsilon, e] = K_F x$, where

$$K_F \triangleq [0, H, HA, \cdots, HA^{N_\epsilon - 1}, A^{N_\epsilon}]. \tag{20}$$

That is, we have $[x, K_F x] \in \mathbb{S}$, $\forall x$. In view of (17), it can also be verified that the set $\mathbb{S}$ is invariant for the system (13), that is,

$$\bar{A}_0[x, U, \varepsilon, e] \in \mathbb{S}, \; \forall [x, U, \varepsilon, e] \in \mathbb{S}. \tag{21}$$

Since (21) implies that

$$J^*(x_{k+1}) \leq \| [x_k, U^*(x_k), \varepsilon^*(x_k), e^*(x_k)] \|^2_{\bar{A}_0^{\mathrm{T}} P \bar{A}_0},$$

we have, from $\Sigma_{\{Q,R,S\}}(P) \leq 0$, that, in closed-loop, $\forall k \geq 0$,

$$J^*(x_{k+1}) - J^*(x_k) \leq -\left(\|x_k\|_Q^2 + \|u_k\|_R^2 + \|\epsilon_k^*\|_S^2\right). \tag{22}$$

Summation of (22) establishes (18). In cases when $Q > 0$, inequality (22) also establishes global exponential stability, since quadratic bounds on $J^*(x_k)$ always exist. In cases when $Q \geq 0$, global exponential stability can be established using a Lyapunov function of the form $\bar{J}(x_k) = \|x_k\|_{P_L} + J^*(x_k)$, for a suitably chosen matrix P_L.

Remark 3.3. The system (13) may be seen as an autonomous "prediction system" similar to that in Kouvaritakis, Rossiter, and Schuurmans (2000). The proof of Theorem 3 exploits the facts that: (i) the matrix P is a "Lyapunov matrix" for the prediction system, and (ii) the set $\mathbb{S}$ is an invariant set for the prediction system.

Remark 3.4. Algorithm 2 and Theorem 3 may be generalized to include exact penalty functions (de Oliveira and Biegler, 1994; Scokaert and Rawlings, 1999). In particular, Theorem 3 also holds whenever we add the following term to the cost function (12):

$$f(\varepsilon, e) = |\bar{\Pi} e|_p + \sum_{i=0}^{N-1} |\bar{S}\epsilon_i|_p,$$

using some norm $|\cdot|_p$ and matrices $\bar{\Pi}$, $\bar{S}$, satisfying $|\bar{\Pi}Ae|_p - |\bar{\Pi}e|_p \leq -|\bar{S}He|_p$, $\forall e$. Regarding computation and existence of such matrices $\bar{\Pi}$, $\bar{S}$, see, for example, Christophersen and Morari (2007).

Since Algorithm 2 is based on a single quadratic programme of prescribable size, we believe it to be a useful alternative to Algorithm 1, especially for fast/uncertain systems for which the maximum value of $\bar{N}(x_k)$ may become relatively large. Also note that the problem $[P^{N,N_\epsilon}]$ includes the additional design parameter N_ϵ which may be used to reduce computational complexity. A numerical example illustrating the flexibility associated with N_ϵ is presented in the extended version of this paper, Løvaas et al. (2007b).

4. UNCERTAIN CASE WITH OUTPUT-FEEDBACK

In this section, we propose a class of output-feedback MPC policies for the uncertain system described in Section 2. We also show how robust closed-loop stability can be guaranteed by choosing the cost function parameters so as to satisfy an LMI condition.

4.1 Proposed Output-Feedback MPC

The output-feedback policies we consider are based on Algorithm 2 and a state estimator. For simplicity, we consider full-order state estimators of the form

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k), \quad \hat{x}_0 = \hat{x}. \tag{23}$$

Here, u_k and y_k are the input and output of the uncertain system (4) and $L \in \mathbb{R}^{n_x \times n_y}$ is a design parameter such that the matrix $(A - LC)$ is stable (i.e., its eigenvalues are strictly inside the unit disk). The dynamics of the associated estimation error $\tilde{x}_k \triangleq x_k - \hat{x}_k$ are

$$\tilde{x}_{k+1} = (A - LC)\,\tilde{x}_k + B_w w_k, \quad \tilde{x}_0 = x - \hat{x}. \tag{24}$$

The proposed output-feedback policy for the uncertain system can then be described as follows:

Algorithm 3. Off-line: Design a state estimator (23) and an instance of Algorithm 2. *On-line*: At each time step $k \geq 0$, solve $[P^{N,N_\epsilon}]$, using $x = \hat{x}_k$, then apply $u_k = [I \; 0 \; \cdots \; 0]U^*(\hat{x}_k)$ to (4).

4.2 Robust Stability Test

Next we present an LMI condition on the cost function matrix P which is sufficient for robust closed-loop stability. To this end, we adopt the dissipativity approach proposed recently in Løvaas et al. (2007a), in which one uses SQCs of the type (5) to describe both the model uncertainty and the static nonlinearity that solves the on-line optimization.

The static nonlinearity of interest here is given by

$$K(x) \triangleq [U^*(x), \varepsilon^*(x), e^*(x)]. \tag{25}$$

In the sequel, we let K denote the memoryless system obtained by persistent use of the function $K(x)$; in closed-loop, the input to the system K is the sequence of state estimates $\{\hat{x}_k\}$, whereas the output of K is the sequence of minimizers $\{\mu_k\} \triangleq \{K(\hat{x}_k)\}$. A particular representation of the closed-loop system suited to the subsequent analysis is shown in Fig. 2, where Δ is as in Fig. 1 and the system Σ, with state $\bar{x}_k = [\tilde{x}_k, \hat{x}_k]$, is defined as follows:

$$\Sigma : \begin{cases} \bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}\mu_k + \bar{B}_w w_k, \quad \bar{x}_0 = [\tilde{x}_0, \hat{x}_0], \\ \hat{x}_k &= \bar{C}\bar{x}_k, \\ z_k &= \bar{C}_z \bar{x}_k + \bar{D}\mu_k + \bar{D}_w w_k, \end{cases}$$

and where

$$\bar{A} \triangleq \begin{bmatrix} A - LC & 0 \\ LC & A \end{bmatrix}, \; \bar{B} \triangleq \begin{bmatrix} 0 \\ BD_1 \end{bmatrix}, \; \bar{B}_w \triangleq \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \tag{26}$$

$$\bar{C} \triangleq [0 \;\, I], \; \bar{C}_z \triangleq [C_z \;\, C_z], \; \bar{D} \triangleq DD_1, \; \bar{D}_w \triangleq D_w. \tag{27}$$

Here, the matrix $D_1 \triangleq [I \;\, 0 \;\cdots\; 0]$ so that $u_k = D_1\mu_k$.

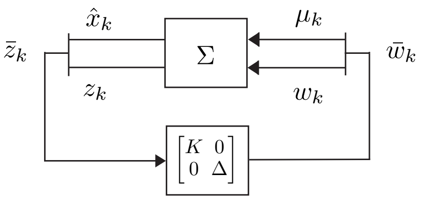

Fig. 2. Setup for stability analysis where $\bar{z}_k = [\hat{x}_k, z_k]$ and $\bar{w}_k = [\mu_k, w_k]$.

The first ingredient for our stability analysis is an SQC for the system K, provided by the following lemma.

Lemma 4. (Li (2006)). For any integer $L \geq 0$, we have

$$\sum_{k=0}^{L} \begin{bmatrix} \hat{x}_k \\ \mu_k \end{bmatrix}^{\mathrm{T}} M_2(P) \begin{bmatrix} \hat{x}_k \\ \mu_k \end{bmatrix} \geq 0, \tag{28}$$

where

$$M_2(P) \triangleq \begin{bmatrix} K_F^{\mathrm{T}} P_{21} + P_{21}^{\mathrm{T}} K_F & K_F^{\mathrm{T}} P_{22} - P_{21}^{\mathrm{T}} \\ P_{22}K_F - P_{21} & -2P_{22} \end{bmatrix}. \tag{29}$$

Here, K_F is as in (20) and P_{ij} denotes the (i,j) block entry of P.

Proof Note that it is straightforward to find a matrix G such that the set $\mathbb{S}$ in (19) can be written as $\mathbb{S} = \{[x, \mu] \,|\, G\mu - GK_F x \leq g\}$, where $g \triangleq [v, \cdots, v, h, \cdots, h, t] \geq 0$. The QP can thus be written as $\min_\mu \|[\hat{x}, \mu]\|_P^2$ s.t. $G\mu - GK_F\hat{x} \leq g$, and the result follows from the KKT (Karush-Kuhn-Tucker) optimality conditions, as shown by Li (2006).

As the next ingredient in our stability analysis we need to find matrices, F_1, F_2, F_3, such that $\mu = F_1\bar{x}_k + F_2\mu_k + F_3 w_k$ is a feasible solution to the online optimization at time $k+1$. We identify suitable choices for F_1, F_2, F_3 in the following lemma:

Lemma 5. Suppose that (17) holds and let $\bar{A}$, $\bar{B}$, $\bar{B}_w$ be as defined in (26). Then

$$\begin{bmatrix} \bar{A} & \bar{B} & \bar{B}_w \\ F_1 & F_2 & F_3 \end{bmatrix} [\tilde{x}, \hat{x}, \mu, w] \in \bar{\mathbb{S}}, \; \forall [\tilde{x}, \hat{x}, \mu] \in \bar{\mathbb{S}}, \; \forall w \in \mathbb{R}^{n_w}, \tag{30}$$

where $\bar{\mathbb{S}} \triangleq \mathbb{R}^{n_x} \times \mathbb{S}$, $F_1 \triangleq K_F\,[LC \;\, 0]$, $F_3 \triangleq 0$ and

$$F_2 \triangleq \begin{bmatrix} \Gamma(N_u, n_u) & 0 & 0 \\ 0 & \Gamma(N_\epsilon, n_h) & \bar{H} \\ 0 & 0 & A \end{bmatrix}. \tag{31}$$

Here, the set $\mathbb{S}$ is as in (19), K_F is as in (20) and $\Gamma(\cdot)$ and $\bar{H}$ are as in (13).

Proof Firstly note that the left hand side of (30) equals $[(A - LC)\tilde{x} + B_w w, \, LC\tilde{x} + A\hat{x} + BD_1\mu, \, K_F LC\tilde{x} + F_2\mu]$. We need to show that, for any $\tilde{x} \in \mathbb{R}^{n_x}$, any $[\hat{x}, \mu] \in \mathbb{S}$ and any $w \in \mathbb{R}^{n_w}$: (i) $(A - LC)\tilde{x} + B_w w \in \mathbb{R}^{n_x}$; and (ii) $[LC\tilde{x} + A\hat{x} + BD_1\mu, \, K_F LC\tilde{x} + F_2\mu] \in \mathbb{S}$. The first point is clear. To prove (ii), we let $\mathbb{S}$ be described as in the proof of Lemma 4 (i.e., $\mathbb{S} = \{[x, \mu] \,|\, G\mu - GK_F x \leq g\}$) and thus we must show that

$$\begin{aligned} &G\left(K_F LC\tilde{x} + F_2\mu\right) - GK_F\left(LC\tilde{x} + A\hat{x} + BD_1\mu\right) \\ &\quad = GF_2\mu - GK_F\left(A\hat{x} + BD_1\mu\right) \leq g, \end{aligned} \tag{32}$$

for any $\tilde{x} \in \mathbb{R}^{n_x}$ and any $[\hat{x}, \mu] \in \mathbb{S}$. To show that inequality in (32) indeed holds, we note from (13) that $[A\hat{x} + BD_1\mu, \, F_2\mu] = \bar{A}_0[\hat{x}, \mu]$. Hence, the result follows from (21).

The final step of our stability analysis it to specialize Theorem 2 of Løvaas et al. (2007a) into a robust stability test for Algorithm 3. To this end, we require the following definitions:

$$\begin{aligned} \Phi\left(\Omega_0, m_1, m_2, P\right) \triangleq \begin{bmatrix} \bar{A} & \bar{B} & \bar{B}_w \\ F_1 & F_2 & F_3 \end{bmatrix}^{\mathrm{T}} \Omega\left(\Omega_0, P\right) \begin{bmatrix} \bar{A} & \bar{B} & \bar{B}_w \\ F_1 & F_2 & F_3 \end{bmatrix} \\ - [I_q \;\, 0]^{\mathrm{T}}\, \Omega\left(\Omega_0, P\right) [I_q \;\, 0] + \bar{C}_\eta^{\mathrm{T}} M(m_1, m_2, P)\bar{C}_\eta, \end{aligned} \tag{33}$$

where $q = 2n_x + N_u n_u + N_\epsilon n_h + n_x$,

$$\bar{C}_\eta \triangleq \begin{bmatrix} \bar{C} & 0 & 0 \\ \bar{C}_z & \bar{D} & \bar{D}_w \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$\Omega\left(\Omega_0, P\right) \triangleq \begin{bmatrix} \Omega_0 & 0 \\ 0 & 0 \end{bmatrix} + D_P^{\mathrm{T}} P D_P, \quad D_P \triangleq \operatorname{diag}\left[\bar{C}, I\right],$$

$$M(m_1, m_2, P) \triangleq m_1 E_1^{\mathrm{T}} M_1 E_1 + m_2 E_2^{\mathrm{T}} M_2(P) E_2,$$

$$E_1 \triangleq \begin{bmatrix} 0 & I_{n_z} & 0 & 0 \\ 0 & 0 & 0 & I_{n_w} \end{bmatrix}, \quad E_2 \triangleq \begin{bmatrix} I_{n_x} & 0 & 0 & 0 \\ 0 & 0 & D_1 & 0 \end{bmatrix},$$

and where M_1, $M_2(P)$ are defined in (5) and (28), respectively, and m_1, m_2 are real numbers.

Remark 4.1. We note that the matrix $M(m_1, m_2, P)$, where $m_1 \geq 0$, $m_2 \geq 0$, defines an SQC for the system $\operatorname{diag}[K, \Delta]$ in the lower loop in Fig. 2. That is, for any integer $L \geq 0$:

$$\sum_{k=0}^{L} \begin{bmatrix} \bar{z}_k \\ \bar{w}_k \end{bmatrix}^{\mathrm{T}} M(m_1, m_2, P) \begin{bmatrix} \bar{z}_k \\ \bar{w}_k \end{bmatrix} + m_1\beta_1 \geq 0.$$

This can be verified using (5) and (28).

The stability test for Algorithm 3 is as follows.

Theorem 6. Suppose that (17) holds and that there exist scalars, $m_0 > 0$, $m_1 \geq 0$, $m_2 \geq 0$, and a symmetric matrix $\Omega_0 \in \mathbb{R}^{2n_x \times 2n_x}$ such that

$$\Phi\left(\Omega_0, m_1, m_2, P\right) \leq -m_0 \bar{C}_\eta^{\mathrm{T}} M_0 \bar{C}_\eta, \tag{34a}$$

$$\Omega\left(\Omega_0, P\right) \geq 0, \tag{34b}$$

where $M_0 \triangleq \operatorname{diag}[0, D_1^{\mathrm{T}} D_1, I_{n_w}]$. Then, the closed-loop system is globally ℓ_2-stable in the sense that

$$\sum_{k=0}^{\infty} \| [\tilde{x}_k, \hat{x}_k, u_k, w_k] \|^2 \leq \infty$$

for any initial condition.

Proof The result is a special case of Theorems 1 and 2 in Løvaas et al. (2007a).

4.3 Robust Design: Upper Bound on the Nominal Cost

We note that it is a standard LMI feasibility problem to search for Ω_0, $m_0 > 0$, $m_1 \geq 0$ and $m_2 \geq 0$ satisfying (34) and thereby test stability of a given instance of Algorithm 3. Alternatively, and similarly to Løvaas et al. (2008), Theorem 6 may be used for the synthesis problem of determining P subject to robust stability. For example, by solving the following semi-definite programme:

$$\inf_{P_1, P_2, \Omega_0, m_0, m_1} \Big\{ \theta(P) \text{ s.t. } \Sigma_{\{Q,R,S\}}(P) \leq 0, \ (34),$$

$$P = \text{diag}[P_1, P_2], \ m_0 > 0, \ m_1 \geq 0 \text{ and } m_2 = 0 \Big\}, \quad (35)$$

where

$$\theta(P) = \text{trace}(P_1) + \upsilon \text{trace}(P_2), \quad \upsilon > 0. \quad (36)$$

In problem (35), we have introduced the structural constraint $P = \text{diag}[P_1, P_2]$ so that the cost (12) takes the form $J(x, U, \varepsilon, e) = \|[x, U]\|^2_{P_1} + \|[\varepsilon, e]\|^2_{P_2}$. Moreover, we have introduced the constraint $m_2 = 0$ so as to ensure convexity. Nevertheless, the problem is always feasible, as shown in the following theorem.

Theorem 7. Problem (35) is feasible if and only if (3) holds.

Proof See Løvaas et al. (2007b).

In the sequel, we use P^* to denote a feasible and (near) optimal solution to (35). Setting $P = P^*$ leads to the following modified version of Algorithm 3 which is guaranteed to be both feasible and robustly stable:

Algorithm 4. Off-line: (i) Choose any integers N, N_u and N_ϵ satisfying $N \geq N_u \geq 1$, $N \geq N_\epsilon \geq 1$. (ii) Choose any T and t such that the set $X_F = \{x \,|\, Tx \leq t\}$ satisfies (17). (iii) Choose any observer gain L such that $(A - LC)$ is stable. (iv) Choose any matrices $Q \geq 0$, $R > 0$, $S > 0$ and determine $P = P^*$ by solving (35). *On-line*: At each time step $k \geq 0$, solve $[P^{N,N_\epsilon}]$, using $x = \hat{x}_k$, then apply $u_k = [I \ 0 \ \cdots \ 0]U^*(\hat{x}_k)$ to (4).

For nominal performance purposes, we would like P^* to be a "tight" upper bound on the "nominal" cost function $P_{\{Q,R,S\}}$ defined by

$$\Sigma_{\{Q,R,S\}}\left(P_{\{Q,R,S\}}\right) = 0. \quad (37)$$

Note that, since $\Sigma_{\{Q,R,S\}}(P^*) \leq 0$, we always have $P^* \geq P_{\{Q,R,S\}}$. The following theorem shows that the added weighting $P^* - P_{\{Q,R,S\}}$ will be arbitrarily small provided that the model uncertainty is sufficiently small.

Theorem 8. For any given $\epsilon > 0$, there exists a $\delta > 0$, such that, if we make the assignments $G_{21}(z) \leftarrow \delta G_{21}(z)$ and $G_{22}(z) \leftarrow \delta G_{22}(z)$, then trace$\left(P^* - P_{\{Q,R,S\}}\right) \leq \epsilon$.

Proof See Løvaas et al. (2007b).

Remark 4.2. Note that replacing $G_{21}(z)$ and $G_{22}(z)$ with $\delta G_{21}(z)$ and $\delta G_{22}(z)$, respectively, amounts to "shrinking" the uncertainty by a factor δ, see Remark 2.1.

Theorems 2 and 8 show that Algorithm 4 converges to a certainty equivalence implementation of the design of Scokaert and Rawlings (1999) as the model uncertainty decreases (i.e., $[G_{21}(z) \ G_{22}(z)] \to 0$), provided we make the choices $T = T_\infty$, $t = t_\infty$ and $N = N_\epsilon$ sufficiently large. For a numerical example illustrating Algorithm 4, see Løvaas et al. (2007b).

5. CONCLUSIONS

In this paper, we have proposed a novel robust output-feedback MPC design for open-loop stable systems. The proposed design respects hard input constraints and treats soft state constraints using quadratic penalty functions. Global robust stability in face of unstructured model uncertainty is achieved by choosing the cost function parameters so as to satisfy an LMI condition.

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