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Consensus of Dynamical Agents in Time-Varying Networks ⋆

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Abstract: Consensus problems in time-varying networks are studied in this paper. We consider two cases. In the first case, the networks are basically connected and the conditions for reaching consensus are described by means of the algebraic properties of connectivity for network graph. In the second case, the networks are possibly disconnected all time. A concept called integral connectivity of networks is used and by means of its algebraic characterization we study the consensus problems with variant time-varying network cases. Necessary and sufficient conditions of consensus over periodic time-varying networks are presented. For aperiodic time-varying network cases some sufficient conditions are given. The estimations of convergence rate are given in terms of the integral connectivity.

Keywords: Consensus, Time-Varying Network, Laplacian of Graph, Integral Connectivity.

1. INTRODUCTION

Distributed coordination of dynamical multi-agent systems in networks is an attractive topic nowadays and consensus is one of the problems in such type. The study of consensus for dynamical multi-agents under network is inspired by different research motivations such as distributed computation, wireless sensor networks, mobile robotic swarms and so on.

Recently the consensus problem over time-varying networks received great attention. [1] studied consensus stability over the network with switching topology. In their work the communication networks are assumed to be balanced and strongly connected at each time instant. [2], [3], [5], [6], [7], [8], [10], [11] studied the consensus problem of discrete-time dynamic agents over time-varying networks. Their results are established based on jointly connectivity of networks, that is, the graph $\mathcal{G}(t)$ associated to the network topology is not necessary connected at any time t, but the union graph of $\mathcal{G}(t)$, $\mathcal{G}(t+1), \cdots, \mathcal{G}(t+T)$ is connected. [9] discussed continuous-time dynamic agents over time-varying networks, where the network topology is not necessary connected, but a so-called δ-digraph associated to the integral adjacency matrix is required to be connected.

In this paper we consider the continuous-time consensus problem of swarms or dynamic multi-agent systems over time-varying networks. We first generalize the results of [1] showing that under certain conditions the agents in time-varying networks may asymptotically reach an agreement in un-exponential rate. It differs from the most results in existent literature, where only exponential stability is discussed.

Then, we allow the network topology to be possibly disconnected all time. We obtain necessary and sufficient conditions of consensus problem under periodic time-varying networks. To aperiodic time-varying networks, we propose some sufficient conditions of the consensus problem.

In our study a notion called *Integral Connectivity of time-varying graph* is used to guarantee the multi-agent system reaching consensus. This concept can be regarded as a counterpart of jointed connectivity used in discrete-time cases by [2], [8], [10].

The notion *Integral Connectivity of time-varying graph* introduced in this work has some similarity with that in [9]. For example, both discuss continuous-time systems, and the consensus conditions for both are related to connectivity of an integral of adjacency matrix over finite time interval. However, there are some essential differences between Moreau's work and ours:

(1) When the integral connectivity (or jointed connectivity) is discussed over time-interval $[t, t+T)$ in [9] as well as [2], [8], [10], the interval-length T is fixed. In our work the interval-length T is allowed to be varying, it could be arbitrarily large. This situation may occur in some

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dynamical process, for example, the system energy decays as time t tends to large. With this respect the connectivity in our framework is much weaker than that in [9].

(2) The integral graph introduced in our work is different from the δ-graph defined in [9], and the connectivity of the integral graph is characterized by some quantitative algebraic values of the integral matrix, for example, the second least eigenvalue λ_2, or some other value according to context.

(3) Our results are confined to undirected networks, which can be easily generalized to balanced networks, but Moreau's results are valid for generally directed networks. It seems that Moreau's results are more general, however, there are some implicit difficulties in generally directed networks. For example, the connectivity of δ-graph can not guarantee the convergence of consensus when the integral matrix is defined on time-intervals with varying interval-length (See Example 2 in this paper). When the time-intervals are of constant length, the convergence is proved by Moreau, however, the convergence rate may extremely slow (See Example 3 in this paper).

Our contribution can be described as follows.

(1) We not only give the necessary and sufficient condition of consensus problem under periodic time-varying networks, but also propose a general framework to investigate the continuous-time consensus problem under time-varying networks. The connectivity of network topology is allowed to be weaker than that in [2], [8], [9],[10]. The convergence analysis of consensus problem based on integral connectivity over a sequence of time-intervals with varying length plays a key issue in this work.

(2) Under our setting we show that there are variety of convergence rates for consensus according to the properties of networks. The estimations of convergence rate are given in terms of the algebraic characterization of integral connectivity.

2. THE CONSENSUS PROBLEM IN TIME-VARYING NETWORKS

The swarm under consideration consists of n dynamical members (agents) $\{v_1, v_2, \cdots, v_n\}$, the associated communication network is described by a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$. The vertex set $\mathcal{V} := \{v_1, v_2, \cdots, v_n\}$, which is the set of members in the swarm. $\mathcal{A}(t) = (a_{ij}(t))_{n\times n}$ with $a_{ij}(t) \geq 0$ is the weighted adjacency matrix of $\mathcal{G}(t)$, the edge set $\mathcal{E}(t) := \{(v_i, v_j) \in \mathcal{V} \times \mathcal{V} \mid a_{ij}(t) > 0\}$. Moreover, the neighbor set to each member v_i at time instant t is defined as

$$N_i(t) = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}(t)\}. \tag{1}$$

It is assumed that $a_{ii}(t) = 0$ for each $i \in \underline{n}$, i.e. the v_i is not a neighbor of itself, $v_i \notin N_i(t)$. In this paper, we only consider undirected graphs, i.e. $a_{ij}(t) = a_{ji}(t)$ for all i, j.

Let $x_i \in \mathbb{R}$ be the value of dynamical agent v_i, and the dynamical equation of v_i, for each $i \in \underline{n}$, is described by

$$\dot{x}_i = u_i. \tag{2}$$

The (average) consensus problem of the swarm aims for finding a (distributed) control protocol such that

$$\lim_{t\to\infty} x_i(t) = \bar{x}, \tag{3}$$

where $\bar{x}$ is the average value of $\{x_i(0) |\ i \in \underline{n}\}$.

A well-known linear consensus protocol (See [1]) is that

$$u_i = \sum_{j\in N_i} a_{ij}(t)(x_j(t) - x_i(t)).$$

Thus the behavior of the swarm is described as follows

$$\dot{x}_i = \sum_{j\in N_i} a_{ij}(t)(x_j(t) - x_i(t)) \quad \text{for all } i \in \underline{n}, \tag{4}$$

where $\{a_{ij}(t) \geq 0\}$ are the entries of the adjacency matrix $\mathcal{A}(t)$.

The time-varying degree of v_i for a time-varying graph $\mathcal{G}(t)$ is defined by

$$d_i(t) = \sum_{j=1}^{n} a_{ij}(t).$$

Let $D(t) = \mathrm{diag}(d_1(t), \cdots, d_n(t))$, then $L(t) = D(t) - \mathcal{A}(t)$ is called the *Laplacian* of $\mathcal{G}(t)$.

Let $x := (x_1, x_2, \cdots, x_n)^\tau$, then the equation (4) is written as a time-varying linear system

$$\dot{x} = -L(t)x. \tag{5}$$

We will answer the equation: For dynamic system (5) under what conditions it holds that

$$\lim_{t\to\infty} x(t) = x^*, \tag{6}$$

where $x^* = (\bar{x}, \cdots, \bar{x})^\tau \in \mathbb{R}^n$ is a constant vector in $\mathbb{R}^n$ representing the average value of x.

The following lemma ([4]) is fundamental for this work.

Lemma 1. Let $\mathcal{G}$ be an undirected weighted graph and L be its *Laplacian*, then L is a semi-positive symmetric matrix with following properties.

(1) Let $\lambda_k(L)$ be the eigenvalues of L with the order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\rho_k(e^{-Lt})$ be the eigenvalues of $\exp(-Lt)$ with the order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ then

$$\lambda_1(L) = 0, \qquad \rho_1(e^{-Lt}) = 1.$$

Moreover, the vector of ones, $\mathbf{1} = (1, \cdots, 1)^\tau \in \mathbb{R}^n$, is the eigenvector to $\lambda_1(L)$ and $\rho_1(e^{-Lt})$. In other words, $L\mathbf{1} = \mathbf{0}$, and $e^{-Lt}\mathbf{1} = \mathbf{1}$ for any t.

(2) For each $k \geq 1$,

$$\rho_k(e^{-Lt}) = e^{-\lambda_k(L)t}$$

and the eigenvector to $\rho_k(e^{-Lt})$ is same as that to $\lambda_k(L)$, i.e. for any $\xi \in \mathbb{R}^n$, $L\xi = \lambda_k(L)\xi$ if ond only if $e^{-Lt}\xi = e^{-\lambda_k(L)t}\xi$.

(3) $\lambda_2(L) > 0$ if and only if $\mathcal{G}$ is connected.

(4) By Rayleigh-Ritz Theorem,

$$\lambda_2(L) = \min\left\{ \frac{x^\tau L x}{x^\tau x} \;\middle|\; x \neq 0, x^\tau \mathbf{1} = 0 \right\},$$

$$\rho_2(e^{-Lt}) = \max\left\{ \frac{x^\tau e^{-Lt} x}{x^\tau x} \;\middle|\; x \neq 0, x^\tau \mathbf{1} = 0 \right\}.$$

The following lemma is also useful in studying the behavior of the swarm.

Lemma 2. Let L be a semi-positive symmetric real matrix, $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. If there exists $t_1 > 0$ and $\xi \in \mathbb{R}^n$ such that $\|e^{-Lt_1}\xi\| = \|\xi\|$, then $L\xi = \mathbf{0}$.

3. CONSENSUS OVER CONNECTED NETWORKS

In this section we consider the consensus problem of (5) in connected networks.

Theorem 3. Let $\mathcal{G}(t)$ be an undirected time-varying graph with *Laplacian* $L(t)$. If there exists a real number $K > 0$ such that

$$\lambda_2(L(t)) \geq K \quad \text{for all } t \geq 0,$$

where $\lambda_2(L(t))$ is the second smallest eigenvalue of $L(t)$, then the dynamics (6) is convergent exponentially.

$$\|x(t) - x^*\| \leq e^{-Kt}\|x(0) - x^*\|.$$

This result is a straightforward generalization of Theorem 9 in [1] and its proof is omitted.

The condition $\lambda_2(L(t)) \geq K > 0$ implies that not only $\mathcal{G}(t)$ for any $t \geq 0$ is connected, but also the connectivity is strong enough.

Nevertheless, under a milder assumption on the network, next result gives an asymptotical convergence theorem for consensus problem in the price that the exponential convergence may be not valid anymore.

An infinite sequence $\{t_k | k \geq 0\}$ is called a *partition* of $[0, \infty)$ if

$$0 = t_0 < t_1 < \cdots < t_k < \cdots$$

and $\lim_{k\to\infty} t_k = \infty$.

Theorem 4. Let $\mathcal{G}(t)$ be an undirected time-varying graph with *Laplacian* $L(t)$. If there exists a partition $\{t_k | k \geq 0\}$ of time interval $[0, +\infty)$, and a real number $K > 0$ such that

$$\lambda_2(L(t))\Delta t_k \geq K \quad \text{for all } t \in [t_{k-1}, t_k), \tag{7}$$

where $\Delta t_k = t_k - t_{k-1}$, then the dynamics (5) converges to the average vector, i.e. the consensus is reached.

Proof. Let $\delta(t) = x(t) - x^*$ and $V(t) = \delta(t)^\tau \delta(t)$, then $\delta(t)$ satisfies that $\dot{\delta} = -L(t)\delta$ and $\delta^\tau \mathbf{1} = 0$. By Rayleigh-Ritz Theorem, we have

$$\frac{\delta(t)^\tau L(t)\delta(t)}{\delta(t)^\tau \delta(t)} \geq \lambda_2(L(t)),$$

i.e., $\delta(t)^\tau(-L(t))\delta(t) \leq -\lambda_2(L(t))\delta(t)^\tau\delta(t)$. Then

$$\begin{aligned} \dot{V}(t) &= 2\delta(t)^\tau \dot{\delta}(t) = 2\delta(t)^\tau(-L(t))\delta(t) \\ &\leq -2\lambda_2(L(t))V(t) \leq -2\frac{K}{\Delta t_k}V(t). \end{aligned} \tag{8}$$

Hence, for any $t \in [t_{k-1}, t_k)$,

$$V(t) \leq \exp\left(-2\frac{K}{\Delta t_k}(t - t_{k-1})\right) V(t_{k-1}), \tag{9}$$

which implies $\|\delta(t_k)\| \leq e^{-K}\|\delta(t_{k-1})\|$.

Therefore

$$\|\delta(t_k)\| \leq e^{-kK}\|\delta(0)\|. \tag{10}$$

Thus, $\|\delta(t)\| \to 0$, i.e. (6) holds.

Remark 1.

(1) The coefficient e^{-kK} in (10) make the $\delta(t_k)$ looks like an exponential decaying function. In fact, the $\|\delta(t_k)\|$ does not decay exponentially to zero, as the Δt_k may tend to infinite large.

(2) With a similar proof, the assumption (7) can be replaced by a condition in integral form $\int_{t_{k-1}}^{t_k} \lambda_2(L(t))dt \geq K$, which can be simply written as

$$\int_0^\infty \lambda_2(L(t))dt = \infty. \tag{11}$$

4. CONSENSUS OVER INTEGRALLY CONNECTED NETWORKS

In this section we discuss the case that the time-varying graph $\mathcal{G}(t)$ is possibly disconnected all time. A notion called *Integrally Connected Graph* is used and defined in what follows.

Definition 1. Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{A}(t), \mathcal{E}(t))$ be a time-varying graph and assume that $\mathcal{A}(t) = (a_{ij}(t))_{n\times n}$ is integrable. Define

$$\bar{\mathcal{A}} = \frac{1}{T}\int_\tau^{\tau+T} \mathcal{A}(t)dt, \tag{12}$$

$$\bar{\mathcal{E}} = \left\{ (v_i, v_j) \in \mathcal{V} \times \mathcal{V} \,\middle|\, \int_\tau^{\tau+T} a_{ij}(t)dt > 0 \right\}.$$

Then the graph

$$\bar{\mathcal{G}}_{[\tau,\tau+T)} := (\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$$

is named the integral graph of $\mathcal{G}(t)$ over time interval $[\tau, \tau + T)$.

Definition 2. A time-varying graph $\mathcal{G}(t)$ is said to be integrally connected on $[\tau, \tau + T)$ if its integral graph $\bar{\mathcal{G}}_{[\tau,\tau+T)}$ is connected.

Lemma 5. Let $\mathcal{G}(t)$ be a time-varying graph, $L(t)$ and $\bar{L}$ be the *Laplacian* of $\mathcal{G}(t)$ and $\bar{\mathcal{G}}_{[\tau,\tau+T)}$ respectively. Then one has

$$\bar{L} = \frac{1}{T}\int_\tau^{\tau+T} L(t)dt.$$

Moreover, $\mathcal{G}(t)$ is integrally connected on $[\tau, \tau + T)$ if and only if $\lambda_2(\bar{L}) > 0$.

Proof. It is easy to be verified according to definition 2.

For simplicity, we assume that the time-varying graph $\mathcal{G}(t)$ is piecewise time-invariant, which is defined as follows.

Definition 3. A time-varying graph $\mathcal{G}(t)$ is called piecewise time-invariant if there exists a partition $\{t_k | k \geq 0\}$ of $[0, \infty)$ such that the adjacency matrix $\mathcal{A}(t)$ is constant on each interval $[t_{k-1}, t_k)$.

Note that the assumption of piecewise time-invariance is of obvious engineering significance. Besides, the methods used here can be easily generalized to some general time-varying networks such as $\mathcal{A}(t)$ is a piecewise continuous matrix or a measurable matrix.

4.1 Consensus in Periodic Cases

Let $\mathcal{G}(t)$ be a periodic graph that $\mathcal{G}(t + T) = \mathcal{G}(t)$ for all $t \geq 0$ and there exists a partition of $[0, T)$

$$0 = t_0 < t_1 < \cdots < t_m = T,$$

such that $\mathcal{A}(t)$ is time-invariant on each $[t_{k-1}, t_k)$.

Correspondingly, let $L_k = L(t)$ for $t \in [t_{k-1}, t_k)$, then

$$\dot{x}(t) = -L_k x(t) \quad \text{for all } t \in [t_{k-1}, t_k).$$

Recalling a notation in linear system theory. Given linear time-varying system (5) its transition matrix $\Phi(t,\tau)$ satisfies

$$\frac{\partial \Phi(t,\tau)}{\partial t} = -L(t)\Phi(t,\tau),\ \Phi(\tau,\tau) = I. \tag{13}$$

Then we have $x(t) = \Phi(t,\tau)x(\tau)$ for any $\tau \in [0,+\infty)$ and $t \geq \tau$.

Now we denote

$$P := \Phi(T,0) = e^{-L_m\Delta t_m} \cdots e^{-L_1\Delta t_1}. \tag{14}$$

As the matrix P is no longer symmetric, the second largest eigenvalue of $P^\tau P$, denoted by $\rho_2(P^\tau P)$, will play a key role in the analysis of consensus for the dynamical agents.

Lemma 6. $\bar{\mathcal{G}}_{[0,T)}$ is connected if and only if

$$\bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k) = \mathrm{span}\{\mathbf{1}\}. \tag{15}$$

Proof. It is obvious that $\mathrm{span}\{\mathbf{1}\} \subseteq \bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k)$ as $L_k\mathbf{1} = 0$ for all $k > 0$. Thus, we need to prove that $\bar{\mathcal{G}}_{(0,T)}$ is disconnected if and only if there exists a nonzero vector ξ such that $\xi^\tau\mathbf{1} = 0$ and $L_k\xi = 0$ for all k.

If $\bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k) \neq \mathrm{span}\{\mathbf{1}\}$, then there exists $\xi \notin \mathrm{span}\{\mathbf{1}\}$ such that $L(t)\xi = 0$ for all $t \in [0,T)$. It implies that $\bar{L}\xi = 0$. Thus there exist two independent eigenvectors of $\bar{L}$ corresponding to eigenvalue zero, which implies $\bar{\mathcal{G}}_{[0,T)}$ is unconnected.

Conversely, if $\lambda_2(\bar{L}) = 0$, then there exists $\xi \neq 0$ such that $\xi^\tau\mathbf{1} = 0$ and $\bar{L}\xi = 0$. Thus $\xi^\tau\bar{L}\xi = 0$, which implies $\int_0^T \xi^\tau L(t)\xi dt = 0$. Notice that $L(t)$ is semi-positive, then we have $\xi^\tau L(t)\xi = 0$ at almost all $t \in [0,T)$. Moreover, as $L(t)$ is piecewise constant, there is no t such that $\xi^\tau L(t)\xi \neq 0$. For a semi-positive matrix, $\xi^\tau L(t)\xi = 0$ implies that $L(t)\xi = 0$. Hence, $\xi \in \bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k)$, i.e., $\bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k) \neq \mathrm{span}\{\mathbf{1}\}$.

Lemma 7. $\bar{\mathcal{G}}_{[0,T)}$ is connected if and only if $\rho_2(P^\tau P) < 1$.

Proof. It is obvious that $\rho_2(P^\tau P) \leq 1$ whatever $\bar{\mathcal{G}}_{[0,T)}$ is connected or not. Hence, we show that $\bar{\mathcal{G}}_{[0,T)}$ is unconnected if and only if $\rho_2(P^\tau P) = 1$.

If $\bar{\mathcal{G}}_{[0,T)}$ is not connected, then there exists $\xi(\neq 0) \in \bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k)$ satisfying $\xi^\tau\mathbf{1} = 0$. It implies $L_k\xi = 0$ and $e^{-L_kt}\xi = \xi$. Since $e^{-L_kt}\xi = \xi$ for all k, we have $P\xi = \xi$. Recall the Rayleigh-Ritz Theorem,

$$\rho_2(P^\tau P) \geq \frac{\xi^\tau(P^\tau P)\xi}{\xi^\tau\xi} = 1. \tag{16}$$

As $\rho_2(P^\tau P) \leq 1$, (16) implies that $\rho_2(P^\tau P) = 1$.

Conversely, if $\rho_2(P^\tau P) = 1$, by the Rayleigh-Ritz Theorem, and notice that the set $\{x|\ x^\tau\mathbf{1} = 0, x^\tau x = 1\}$ is compact, there must exist $\xi \neq 0$ such that

$$\frac{\xi^\tau(P^\tau P)\xi}{\xi^\tau\xi} = \rho_2(P^\tau P) = 1.$$

Take $\xi_0 = \xi$ and $\xi_k = e^{-L_kt}\xi_{k-1}$. Recalling the fact $\|e^{-L_k\Delta t_k}\xi\| \leq \|\xi\|$, we have

$$\|\xi\| = \|P\xi\| = \|\xi_m\| \leq \cdots \leq \|\xi_1\| \leq \|\xi_0\| = \|\xi\|.$$

Hence, $\|e^{-L_k\Delta t_k}\xi_{k-1}\| = \|\xi_{k-1}\|$. By Lemma 2, we have $L_k\xi_{k-1} = 0$. Thus, we know $\xi_{k-1} \in \mathrm{Ker}(L_k)$ and $\xi_k = \xi_{k-1}$. This implies that $\xi = \xi_{k-1} \in \mathrm{Ker}(L_k)$ for all k, i.e., $\xi \in \bigcap_{k\in\underline{m}} \mathrm{Ker}(L_k)$, then, $\bar{\mathcal{G}}_{[0,T)}$ is not connected.

Theorem 8. Let $\mathcal{G}(t)$ be a piecewise time-invariant graph with period T, then (5) implements consensus if and only if $\bar{\mathcal{G}}_{[0,T)}$ is connected. Moreover, let $\delta(t) = x(t) - x^*$, then

$$\|\delta(t)\| \leq e^{-\gamma(t-T)}\|\delta(0)\|, \tag{17}$$

where $\gamma = -\frac{1}{2T}\ln\rho_2(P^\tau P)$.

Proof. If $\bar{\mathcal{G}}_{[0,T)}$ is unconnected, then by Lemma 6, there is $\xi \neq 0$ such that $\xi^\tau\mathbf{1} = 0$ and $L_k\xi = 0$ for all k. Let $x(0) = \xi$, then $x(t) = \xi$ for all $t > 0$, thus (6) does not hold.

When $\bar{\mathcal{G}}_{[0,T)}$ is connected, notice that $P\mathbf{1} = \mathbf{1}$, then $Px^* = x^*$. Thus $P(x(0) - x^*) = x(T) - x^*$, i.e., $P\delta(0) = \delta(T)$. By Rayleigh-Ritz Theorem,

$$\rho_2(P^\tau P) \geq \frac{\delta(0)^\tau(P^\tau P)\delta(0)}{\delta(0)^\tau\delta(0)} = \frac{\delta(T)^\tau\delta(T)}{\delta(0)^\tau\delta(0)}.$$

We have $\|\delta(T)\| \leq \rho_2(P^\tau P)^{\frac{1}{2}}\|\delta(0)\|$, Therefore,

$$\|\delta(kT)\| \leq \rho_2(P^\tau P)^{\frac{1}{2}k}\|\delta(0)\|.$$

As $\|\delta(t)\|$ is not increasing, for $t \in [kT, kT+T)$

$$\begin{aligned}\|\delta(t)\| &\leq \|\delta(kT)\| \leq \rho_2(P^\tau P)^{\frac{1}{2T}kT}\|\delta(0)\| \\ &\leq \rho_2(P^\tau P)^{\frac{1}{2T}(t-T)}\|\delta(0)\|.\end{aligned}$$

Hence (17) holds.

Example 1. Consider a graph $\mathcal{G}(t)$ which consists of three vertices v_1, v_2, v_3. Suppose $\mathcal{G}(t)$ is a periodic graph such that $\mathcal{G}(t+3h) = \mathcal{G}(t)$, and the edge set $\mathcal{E}(t)$ is defined as follows. When $t \in [0,h)$, $\mathcal{E}(t) = \{(v_1,v_2)\}$; when $t \in [h,3h)$, $\mathcal{E}(t) = \{(v_1,v_3)\}$. The network is unconnected at any time.

The *Laplacian* of $\mathcal{G}(t)$ for $t \in [kh-h, kh)$ for $k = 1,2,3$ are as follows.

$$L_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = L_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and the integral *Laplacian* $\bar{L} = \frac{1}{3}(L_1 + L_2 + L_3)$ satisfies the condition $\lambda_2(\bar{L}) > 0$.

The trajectory of the state is given by

$$x(t) = e^{-L_k(t-kh)}\ x(kh) \text{ for } t \in [kh, kh+h)$$

where $L_k = L_{(k \mod 3)+1}$.

Let $h = 0.5$ and $x(0) = (0,0,1)^\tau \in \mathcal{R}^3$. Figure 1 shows that each $x_j(t), j \in \underline{3}$, tends to the average value $\frac{1}{3}$.

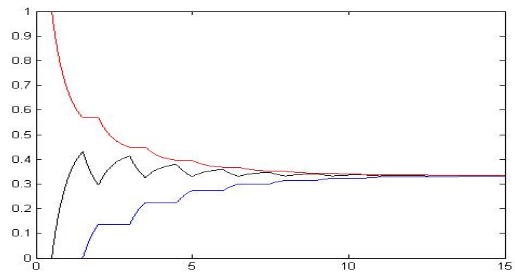

Fig 1

To verify (17), we define

$$p(t) = \frac{\|\delta(t)\|}{\|\delta(0)\|}, \; r(t) = -\frac{1}{t}\ln(p(t)),$$

and their trajectories are demonstrated in Figure 2 .

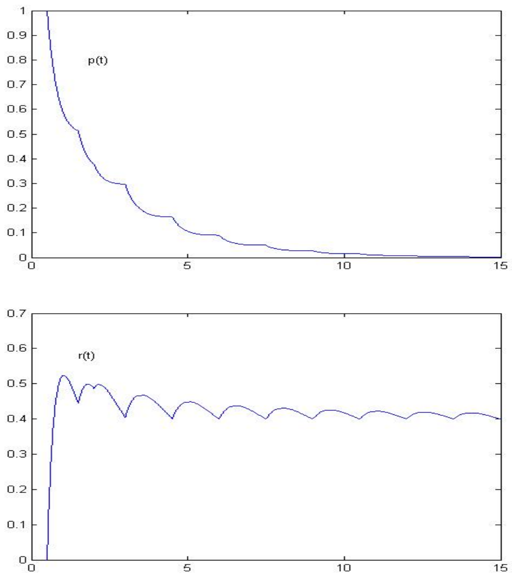

Fig 2(a) and Fig 2(b)

4.2 Consensus in Aperiodic Cases

Definition 4. A time-varying graph $\mathcal{G}(t)$ is said to be piecewise integrally connected if there is a partition $\{T_k | k \geq 0\}$ of $[0, \infty)$ such that $\mathcal{G}(t)$ is integrally connected over each interval $[T_{k-1}, T_k)$.

We emphasize that for each k, $\Delta T_k = T_k - T_{k-1} < \infty$, however it is possible that $\lim_{k\to\infty} \Delta T_k = \infty$.

According to the previous results one gets:

Lemma 9. The following statements are equivalent:

(1) $\mathcal{G}(t)$ is integrally connected over $[T_{k-1}, T_k)$.

(2) $\lambda_2(\bar{L}_k) > 0$, where $\bar{L}_k = \frac{1}{\Delta T_k}\int_{T_{k-1}}^{T_k} L(t)dt$.

(3) $\rho_2(R_k) < 1$, where $R_k = \Phi(T_k, T_{k-1})^\tau \Phi(T_k, T_{k-1})$.

It is easy to understand that the piecewise integral connectivity is a necessary condition for the consensus of system stability of (5). However, it is not a sufficient condition under general time-variant networks. Let's examine a simple example.

Given system (5) in the form that $\dot{x} = \alpha(t)Lx$ where L is a *Laplacian* matrix of a time-invariant connected graph $\mathcal{G}_0$. Let $\alpha(t) = e^{-t}$. Then the communication network is connected all time, of course, it is also piecewise integrally connected. One gets its trajectory of the system, $x(t) = \exp(Le^{-t})e^{-L}x(0)$. As

$$\lim_{t\to\infty} x(t) = e^{-L}x(0).$$

Thus, it does not reach consensus.

Therefore, we need some additional condition to solve the consensus problem under general time-variant networks. The following gives a sufficient condition for the consensus problem.

Theorem 10. Suppose $\mathcal{G}(t)$ is a piecewise time-invariant graph and integrally connected on each interval $[T_{k-1}, T_k)$. If there exists a constant number $K > 0$ such that

$$-\ln \rho_2(R_k) \geq 2K, \tag{18}$$

then (5) converges to the average vector.

The condition (18) in Theorem 10 can be replaced by a condition in more general form

$$-\sum_{k=1}^{\infty} \ln \rho_2(R_k) = \infty. \tag{19}$$

It is easy to see that (18) implies (19). On the other hand, if (19) holds, we can redefine a new partition using the union of $[T_{k-1}, T_k) \cup \cdots \cup [T_{k+j}, T_{k+j+1})$, as a new integral interval, such that (18) holds for the new partition. So essentially two conditions (19) and (18) are equivalent.

When (18) holds, it does not mean the convergence being with exponential rate, we have pointed out this before.

Because the calculation of $\lambda_2(\bar{L}_k)$ is more straightforward to check the connectivity of graph, there is an interesting problem: whether one can use the algebraic value of $\lambda_2(\bar{L}_k)$, instead of $\rho_2(R_k)$, to describe the sufficient condition of Theorem 10? The relation between $\rho_2(R_k)$ and $\lambda_2(\bar{L}_k)$ is unclear up to now. It remains a lot for our further work.

Finally, we present two examples to show the differences between our work and that of [9].

When the network is directed, consensus problem can also be described by (4). The only difference from undirected network is that $a_{ij}(t)$ may not be equal to $a_{ji}(t)$. We consider a system under directed network.

Example 2. A swarm consists of three agents v_1, v_2, v_3. Let $\{t_k | k \geq 0\}$ be a partition of $[0, \infty)$. The dynamics of the swarm is described as follows:

$$\begin{aligned}
&\dot{x}_1 = x_2 - x_1, \; \dot{x}_2 = \dot{x}_3 = 0 \quad \text{if } t \in [t_{3k}, \;\; t_{3k+1}),\\
&\dot{x}_2 = x_3 - x_2, \; \dot{x}_3 = \dot{x}_1 = 0 \quad \text{if } t \in [t_{3k+1}, t_{3k+2}),\\
&\dot{x}_3 = x_1 - x_3, \; \dot{x}_1 = \dot{x}_2 = 0 \quad \text{if } t \in [t_{3k+2}, t_{3k+3}).
\end{aligned}$$

Take a coarser partition $\{T_k | k \geq 0\}$ with $T_k = t_{6k}$, then the communication network of the swarm is integrally connected over each interval $[T_{k-1}, T_k)$. According to the definition in [9], the δ-digraph relative to $\int_{T_{k-1}}^{T_k} L(t)dt$ is connected and satisfies his consensus stability condition if $\Delta T = T_k - T_{k-1}$ for all k.

Now we discuss the case when Δt_k can be arbitrarily large and chose $\{t_k | k \geq 0\}$,

$$\begin{aligned}
t_{6k+1} &= t_{6k} + (k+1)\ln 4 + \ln(8 + 8 \cdot 4^{-k-1}),\\
t_{6k+2} &= t_{6k+1} + (k+1)\ln 4 + \ln(8 + 7 \cdot 4^{-k-1}),\\
&\vdots
\end{aligned}$$

and the states are calculated in the following values.

t	x_1	x_2	x_3
t_{6k}	$1 - 4^{-k}$	$9 + 4^{-k}$	$1 - 3 \cdot 4^{-k-1}$
t_{6k+1}	$9 + 3 \cdot 4^{-k-1}$	$9 + 4^{-k}$	$1 - 3 \cdot 4^{-k-1}$
t_{6k+2}	$9 + 3 \cdot 4^{-k-1}$	$1 - 2 \cdot 4^{-k-1}$	$1 - 3 \cdot 4^{-k-1}$
t_{6k+3}	$9 + 3 \cdot 4^{-k-1}$	$1 - 2 \cdot 4^{-k-1}$	$9 + 2 \cdot 4^{-k-1}$
t_{6k+4}	$1 - 4^{-k-1}$	$1 - 2 \cdot 4^{-k-1}$	$9 + 2 \cdot 4^{-k-1}$
t_{6k+5}	$1 - 4^{-k-1}$	$9 + 4^{-k-1}$	$9 + 2 \cdot 4^{-k-1}$

One gets

$$\lim_{k\to\infty} x_1(t_{6k}) = 1 \quad \text{but} \quad \lim_{k\to\infty} x_2(t_{6k}) = 9.$$

Thus, the consensus is not achieved.

Example 3. The system of dynamical is the same as in Example 2, but the time-interval $[t_{k-1}, t_k)$ is of constant length Δt. In such a situation, the system will achieve consensus according to the theorem in [9]. However, the convergence rate of consensus is

$$\|\delta(t)\| \leq (1 - e^{-\Delta t})\|\delta(0)\|.$$

This fact implies that the convergence speed is extremely slow for large Δt. Figure 3 demonstrates the trajectory of $x_2(t)$ when $\Delta t = 10$. Notice that it is convergent with oscillation. However, the convergence is too slow to have practice meaning.

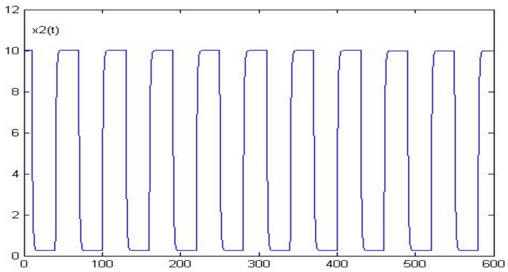

Fig 3: An example of consensus convergence using the result given by [9]

5. CONCLUSION

Consensus problem under time-variant network is a challenge research problem. There still are some fundamental problems opening to researchers. According to our results, the integral connectivity is a key concept for solving the consensus problem under time-variant network. Integral connectivity is a necessary and sufficient condition to achieve consensus over periodic networks, but not sufficient over aperiodic networks.

When the integral connectivity is valid, the convergence of consensus could be with quite different rates. In certain case we can estimate the convergence rate by means of the algebraic characteristic value of connectivity of integral graph.

Finally, some examples show that there are essential differences between undirected networks and directed networks to the consensus problem.

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