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On Stability Properties of Nonlinear Time-Varying Systems by Semi-definite Time-Varying Lyapunov Candidates

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Abstract: Stability properties (uniform stability/uniform asymptotic stability) of nonlinear time-varying systems are explored using positive semi-definite time-varying Lyapunov candidates whose derivative along trajectories is either non-positive or negative semi-definite. Once these positive semi-definite time-varying Lyapunov candidates are available, conditional stability properties on some specific sets can be used to ensure stability properties (unform stability and unform asymptotic stability) of nonlinear time-varying systems.

1. INTRODUCTION

In practice, many nonlinear systems are time-varying. For example, when a radar is tracking a missile, the dynamics of the missile is time-varying due to time-varying wind. Many mechanical systems use time-varying (periodic) excitation signals to obtain the desired performance, for example, vibration control schemes (see Bellman *et al.* (1986)) or extremum seeking control schemes (see in Ariyur and Krstić (2003)), leading to nonlinear time-varying closed-loop systems.

It is well known that uniform stability property (US) is one of the most fundamental properties of dynamic systems, whether they are linear or nonlinear. This is the reason why US plays an important role in the control analysis and design for dynamics systems. Lyapunov direct methods have been widely used to show US properties of nonlinear systems with/without controller (see Chapter 4, Khalil (2002)). *Positive definite* time-invariant/time-varying Lyapunov candidates whose derivatives are negative *semi-definite* have been employed to verify US properties of time-varying nonlinear systems (see Theorem 4.8 in Khalil (2002)). To obtain a stronger stability property: uniform asymptotic stability (UAS), which implies that the trajectories of the system converge to the equilibrium uniformly in the initial condition, it is often required that the derivatives of the time-invariant/time-varying Lyapunov candidates along the trajectories are *negative definite* (see, for instance, Theorem 4.9 in Khalil (2002)).

In general, it is hard to find a positive definite Lyapunov candidate for many nonlinear systems. Other than positive definite Lyapunov candidates, positive *semi-definite* functions can also be Lyapunov candidates. For example, positive semi-definite storage functions have been widely used in the analysis of the nonlinear *time-invariant systems* based on the concept of the passivity. Passivity comes from passive systems that the energy of systems can be increased only through the supply from an external source. Obviously, many engineered systems are passive. Using passive properties, the controller design has been explored (Khalil , 2002, Chapter 6) for nonlinear *time-invariant* systems. However, it is not straightforward to use the "passivity" concept to design a controller for nonlinear time-varying systems as LaSalle's invariance principle, which is a basic tool to obtain stability properties from passivity, is not valid for time-varying systems.

In 2004, Iggidr and Sallet showed stability properties (US/UAS) of time-varying systems in Iggidr and Sallet (2000) by using Lyapunov candidates, which are positive semi-definite, but *time-invariant.* When nonlinear time-varying systems are considered, it is very natural to use time-varying Lyapunov candidates. On the other hand, finding a time-invariant Lyapunov candidate for time-varying systems is not always easy due to the limited searching space. A question arises naturally, is it possible to ensure the stability properties (US/UAS) of nonlinear time-varying systems by using one positive semi-definite time-varying Lyapunov candidate?

This paper aims at addressing the above question. That is, stability properties of nonlinear time-varying systems are guaranteed by finding one time-varying semi-positive definite Lyapunov candidate with either non-positive or negative semi-definite derivative. Once this Lyapunov candidate is available, with appropriate conditional stability properties on some sets, Theorem 1 shows that US properties can be achieved for the time-varying system. This result can be treated as an extension of the result in (Iggidr and Sallet , 2000, Theorem 5) to a more general setting as positive semi-definite *time-varying* Lyapunov candidates are employed.

Next, we show this Lyapunov candidate along with appropriate conditional stability properties on some sets can also be used to guarantee UAS properties of the time-varying system (Theorem 2). This result extends that in (Iggidr and Sallet , 2000, Theorem 6) to the situation when the Lyapunov candidate is time-varying.

This paper is organized as follows. Problem formulation and preliminaries are provided in Section 2. The main results are stated in Section 3 followed by an illustrative example. A summary is given in Section 5. Proofs are presented in the Appendix.

2. PROBLEM FORMULATION AND PRELIMINARIES

In this paper, the set of real numbers is denoted as $\mathbb{R}$ and the sets of integers is denoted as $\mathbb{N}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. The following notations will be used in this paper: $B_\epsilon = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| < \epsilon\}$, $\bar{B}_\epsilon = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq \epsilon\}$. For any $\mathbf{x} \in \mathbb{R}^n$ and any closed set $\mathcal{A} \in \mathbb{R}^n$, $d_{\mathcal{A}}(\mathbf{x}) := \inf_{\mathbf{y} \in \mathcal{A}} \|\mathbf{x} - \mathbf{y}\|$.

Consider the following time-varying system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n$ is a domain with $\mathbf{x} = \mathbf{0} \in \mathcal{D}$ and $\mathbf{f}(t, \mathbf{0}) \equiv \mathbf{0}$ for all $t \geq t_0$. The solution of the system (1) at any time instant t is denoted as $\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)$. Sometimes, for simplicity, we also use $\mathbf{x}(t)$ when no confusion is caused.

The following assumption is needed in the sequel.

Assumption 1. $\mathbf{f}(t, \mathbf{x})$ is continuous and locally Lipschitz in $\mathbf{x}$ uniformly in t, i.e., for any compact set $\mathcal{B} \subset \mathcal{D}$, there exists a constant $L_B > 0$, independent of t, such that

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq L_B \|\mathbf{x}_1 - \mathbf{x}_2\|,$$

for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{B}$ and $t \in \mathbb{R}$.

Remark 1. Assumption 1 is exactly the same as Assumption 1 in Iggidr and Sallet (2000), which is widely used to ensure the existence and uniqueness of solutions of time-varying dynamic systems.

The following definitions will be used in the sequel.

Definition 1. (Sepulchre *et al.* , 1996, Chapter 2)

The equilibrium point $\mathbf{x} = \mathbf{0}$ of the system (1) is

- stable if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that
$$\|\mathbf{x}_0\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0; \tag{2}$$
- uniformly stable (US) if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0, such that (2) is satisfied;
- uniformly attractive (UA) if there is a positive constant c, independent of t_0, such that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ uniformly in t_0, for all $\|\mathbf{x}_0\| < c$. That is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that
$$\|\mathbf{x}(t)\| \leq \eta, \forall t \geq t_0 + T, \forall t_0 \geq 0, \forall \|\mathbf{x}_0\| < c; \tag{3}$$
- uniformly asymptotically stable (UAS) if it is US and UA.

Conditional stability properties have been employed in Iggidr *et al.* (1996); Sepulchre *et al.* (1996) to show stability properties of time-invariant systems by means of positive semi-definite time-invariant Lyapunov candidates (or storage functions). In this paper, conditional stability property also plays an important role in showing stability properties of the time-varying system (1).

Definition 2. Let $\mathcal{Z} \subset R^n$ contain $\mathbf{x} = \mathbf{0}$. The point $\mathbf{x} = \mathbf{0}$ of the system (1) is

- uniformly stable conditionally to $\mathcal{Z}$ ($\mathcal{Z}$-US) if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, independent of t_0 such that
$$\forall \mathbf{x}_0 \in \mathcal{Z} \text{ and } \|\mathbf{x}_0\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \epsilon, \tag{4}$$
for all $t \geq t_0 \geq 0$.
- uniformly attractive conditionally to $\mathcal{Z}$ ($\mathcal{Z}$-UA) if, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that
$$\|\mathbf{x}(t)\| \leq \eta, \quad \forall t \geq t_0 + T, \forall t_0 \geq 0, \tag{5}$$
for all $\|\mathbf{x}_0\| < c$ and $\mathbf{x}_0 \in \mathcal{Z}$.
- uniformly asymptotically stable conditionally to $\mathcal{Z}$ ($\mathcal{Z}$-UAS) if it is $\mathcal{Z}$-US and $\mathcal{Z}$-UA.

Remark 2. The conditional stability properties are much weaker than the stability properties. A uniformly stable equilibrium is $\mathcal{Z}$-US for any $\mathcal{Z} \subset \mathbb{R}^n$. However, if $\mathbf{0}$ is $\mathcal{Z}$-US for some $\mathcal{Z} \subset \mathbb{R}^n$, this equilibrium may not US, as can be seen in the following example:

$$\dot{x}_1 = -x_1 + x_2; \qquad \dot{x}_2 = x_2, \tag{6}$$

where $\mathcal{Z} = \left\{\mathbf{x} \in \mathbb{R}^2 \,|x_2 = 0\right\}$. It is obviously that this system is $\mathcal{Z}$-US, but it is not US. To ensure stability properties of nonlinear systems from $\mathcal{Z}$-US/$\mathcal{Z}$-UAS, more conditions are needed.

Definition 3. For system (1), a set $\mathcal{A} \subset \mathcal{D}$ is called an invariant set if any solution $\mathbf{x}(t)$ belongs to $\mathcal{A}$ at some time t_1 belongs to $\mathcal{A}$ for all future and past time:

$$\mathbf{x}(t_1) \in \mathcal{A} \Rightarrow \mathbf{x}(t) \in \mathcal{A}, \qquad \forall t \in \mathbb{R}. \tag{7}$$

It is called a positively invariant if (7) holds true for all future time $t \geq t_1$.

3. MAIN RESULTS

Our first result is an extension of (Iggidr and Sallet , 2000, Theorem 5) when the Lyapunov candidate is time-varying instead of time-invariant.

Theorem 1. Suppose Assumption 1 holds. If there exists a function $V(t, \mathbf{x}) \in \mathcal{C}^1(\mathbb{R}_{\geq t_0} \times \mathcal{D}, \mathbb{R}_{\geq 0})$ such that

$$W_1(\mathbf{x}) \leq V(t, \mathbf{x}) \leq W_2(\mathbf{x}), \tag{8}$$

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(t, \mathbf{x}) \leq 0, \tag{9}$$

for all $t \geq 0$ and $\mathbf{x} \in \mathcal{D}$, where $W_j(\mathbf{x}) \geq 0$ is a positive semi-definite function for any $j = 1, 2$. Then the equilibrium $\mathbf{x} = \mathbf{0}$ of (1) is US if it is Ω-UAS, where $\Omega \triangleq \{\mathbf{x} \in \mathcal{D} | W_1(\mathbf{x}) = 0\}$.

Proof: see Appendix. ○

Remark 3. For a general time-varying system (1), a time-varying Lyapunov candidate satisfying (8) and (9) is not sufficient to show US properties. Ω-UAS is also needed in order to guarantee uniform stability properties of the system (1). It will show in the example later that by choosing W_1 carefully and taking advantages of the knowledge of the dynamics (1), it may not hard to check Ω-UAS under some cases.

Remark 4. Ω-UAS plays an important role to show US properties of the time-varying systems. When the system is only Ω-UA, it is well-known that the convergence of trajectories of the system (1) does not imply US of the equilibrium. Therefore, Ω-US is necessary to ensure the US of the equilibrium of the system (1). On the other hand, the following example illustrates that Ω-AS is also necessary to ensure the US of the system.

$$\dot{x}_1 = 2(1+e^{-t})x_2^2, \qquad \dot{x}_2 = -(1+e^{-t})x_2^3. \tag{10}$$

By computation, the trajectories of (10) can be represented as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} ln\left(1+2x_{20}^2(t-t_0-e^{-t}+e^{-t_0}\right)+x_{10} \\ \dfrac{x_{20}}{\sqrt{1+2x_{20}^2(t-t_0-e^{-t}+e^{-t_0})}} \end{bmatrix},$$

which implies that $(0,0)$ is not US since the solution depends on initial time t_0.

Let a time-varying Lyapunov candidate be chosen as $V(t,\mathbf{x}) = (1+e^{-t})\,x_2^2$ with $W_1(\mathbf{x}) = x_2^2$ and $\Omega = \left\{\mathbf{x}\in\mathbb{R}^2 \,|x_{2,0}=0\right\}$. For any $\mathbf{x}_0 = [\,x_{10}\ 0\,]^T \in \Omega$, it is not difficult to check that the system (10) is Ω-US, however, the system (10) is not Ω-AS as $x_1(t)$ does not converge to 0. This example illustrates that Ω-US only is not sufficient to ensure US properties. In other words, even though Ω-UAS is not a necessary condition, it is a tight sufficient condition.

Remark 5. Note that the set Ω may not be positively invariant (see Definition 3) for the system (1). That is, when the initial condition is in Ω, the trajectories of the system may leave Ω at some time instants. In fact, Ω-UAS characterizes all trajectories of the system (1) starting from the set $\Omega \in \mathcal{D}$ that are uniformly bounded and uniformly attractive.

Remark 6. When the Lyapunov candidate is time-invariant, it is clear that

$$0 \le W_1(\mathbf{x}) \le V(\mathbf{x}) \Rightarrow \mathcal{A} = \Omega,$$

where $\mathcal{A} = \{\mathbf{x}\in\mathcal{D}\,|V(\mathbf{x})=0\}$. Theorem 1 becomes Theorem 5 in Iggidr and Sallet (2000).

Remark 7. Let $W_1(\mathbf{x})$ in (8) be positive definite, instead of positive semi-definite. By calculation, $\Omega \overset{\triangle}{=} \{\mathbf{0}\}$, Ω-UAS holds true for system (1). We then can conclude that the system (1) is US. This becomes Theorem 4.8 in Khalil (2002).

Remark 8. Consider a time-invariant system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{11}$$

and let $V(\mathbf{x}) \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}_{\ge 0})$ is a positive semi-definite function such that $\dot{V} \le 0$. By applying Theorem 1, the system (11) is US if it is $\mathcal{A}$-UAS where $\mathcal{A}$ is defined in Remark 6. This result is exactly the same as (Sepulchre *et al.* , 1996, Theorem 2.24).

AS is very appealing in applications due to its advantages in terms of robustness as discussed in Loria *et al.* (2005). To show the uniform attractivity for time-invariant nonlinear systems, the well-known LaSalle invariance principle (see results in LaSalle (1960); LaSalle and Lefschetz (1961) and references herein) as well as Krasovskii-LaSalle theorem (see, Vidyasagar (1993)) can be used to show the UAS properties. A lot of work has been done to show uniform attractive (UA) properties of nonlinear time-varying systems. For example, limiting equations (see work in Artstein (1976, 1978b); Lee and Jiang (2005)), which describes the limiting behavior of the original systems as initial time instants approach to infinity, was used to extend the LaSalle's invariance principle to a class of nonlinear time-varying systems. Observability or detectability have been employed in the work of Artstein (1978a); Aeyels and Peuteman (1998); Lee *et al.* (2001); Lee and Chen (2002); Khalil (2002) to show attractivity of nonlinear time-varying systems. The attractivity of the systems can be also verified by means of Matrosov' theorem (see, Matrosov (1962); Rouche and Mawhin (1980)) and its generalizations (see, Loria *et al.* (2005)) as well as persistent excitation condition in the work of Loria *et al.* (2001, 2002). Although many results are available to check AS properties, there are few results to verify the AS property on basis of positive semi-definite time-varying Lyapunov candidates.

The second result shows that conditional stability property on some set also provides a sufficient condition to ensure UAS of time-varying systems provided that positive semi-definite time-varying Lyapunov candidate has a *negative semi-definite* derivative along the trajectories (see Theorem 2). In other words, we can check UAS properties of a time-varying system by using **one** positive semi-definite time-varying Lyapunov candidate and condition stability property on some set. This result (Theorem 2) provides an alternative ways to show AS properties of time-varying systems.

In the proof procedure of Theorem 2, UA properties are first showed followed by UA properties. The proof technique is similar to results listed in Teel *et al.* (2002) which showed UA of the time-varying system once US is obtained. However, Theorem 1 and Theorem 2 provide a way to show stability properties of the time-varying system (1) by using one positive semi-definite time-varying Lyapunov candidate and the conditional stability property.

The following proposition will be needed in the proof of Theorem 2.

Proposition 1. Let $\mathcal{A} \subset \mathbb{R}^n$ be a set containing the origin. Let $W(\mathbf{x}) : \mathbb{R} \to \mathbb{R}_{\ge 0}$ be a continuous function satisfying $W(\mathbf{x})|_{\mathbf{x}\in\mathcal{A}} = 0$ and $W(\mathbf{x}) > 0$ for all $\mathbf{x} \notin \mathcal{A}$. Assume that the equilibrium $\mathbf{0}$ of the system (1) is UA. If there exists a positive constant C_W such that the following inequality holds

$$\int_{t_0}^{t} W(\boldsymbol{\phi}(s;t_0,\mathbf{x}_0))ds \le C_W, \quad \forall t \ge t_0, \forall \mathbf{x}_0 \in B_\delta, \tag{12}$$

where δ is from (2), then for any $\epsilon > 0$, there exists $T_M > 0$, independent of t_0, such that trajectories of the system (1) satisfy

$$\mathbf{x}_0 \in B_\delta \Rightarrow d_{\mathcal{A}}(\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)) < \epsilon, \forall t \geq t_0 + T_M. \tag{13}$$

Proof: see Appendix. $\circ$

Remark 9. (13) implies that $W(\boldsymbol{\phi}(t; t_0, \mathbf{x}_0))$ converges to the equilibrium $\mathbf{0}$ uniformly as $t \to \infty$. Furthermore, if W is a positive semi-definite. Then $\mathcal{A} = \{\mathbf{0}\}$. If (12) holds, using Proposition 1, it yields

$$\mathbf{x}_0 \in B_\delta \Rightarrow |\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)| < \epsilon, \forall t \geq t_0 + T_M, \tag{14}$$

which means $\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)$ is UA. It is apparent that Proposition 1 is an extension of famous Barbalat Lemma Khalil (2002), in which the *uniform* attractivity not provided. Therefore, Proposition 1 provides a very useful to show UA of the time-varying system (1).

Remark 10. Proposition 1 is quite similar to (Teel *et al.* , 2002, Theorem 1), in which UAS was showed with integral characterizations. However the result in Proposition 1 alone cannot ensure UAS of time-varying system without $\mathcal{A}$-UAS (see Theorem 2). Proposition 1 only ensures that trajectories of the time-varying system (1) uniformly converges to a set $\mathcal{A}$, instead of the equilibrium $\mathbf{0}$. If the equilibrium of system is $\mathcal{A}$-UAS, then trajectories of this system will converge to equilibrium uniformly.

With the help of Proposition 1, here comes the second main result of this paper.

Theorem 2. Suppose Assumption 1 holds. Assume that there exists a function $V(t, \mathbf{x}) \in \mathcal{C}^1(\mathbb{R}_{\geq t_0} \times \mathcal{D}, \mathbb{R}_{\geq 0})$ with positive semi-definite functions $W_j(\mathbf{x})$, $j = 1, 2$, such that

$$W_1(\mathbf{x}) \leq V(t, \mathbf{x}) \leq W_2(\mathbf{x}) \tag{15}$$

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(t, \mathbf{x}) \leq -\alpha(W_1(\mathbf{x})), \tag{16}$$

are satisfied for all $t \geq 0$ and $\mathbf{x} \in \mathcal{D}$ and $\alpha \in \mathcal{K}$. Then the equilibrium $\mathbf{x} = \mathbf{0}$ of (1) is UAS if it is Ω-UAS, where $\Omega \triangleq \{\mathbf{x} \in \mathcal{D} | W_1(\mathbf{x}) = 0\}$.

Remark 11. Although the major proof techniques are quite similar to those in (Iggidr and Sallet , 2000, Theorem 6), neither LaSalle invariant principle nor Barbalat lemma is employed to show the uniform attractivity of the equilibrium. It is worthwhile to note that LaSalle invariant principle is not applicable to time-varying systems in general. On the other hand, Barbalat lemma cannot provide enough information about uniformly attractivity with respect to the initial time instant t_0. Proposition 1 helps to ensure uniform attractivity with respect to the initial time.

After obtaining uniformly attractivity with respect to the initial time instant, using the similar proof techniques as in (Iggidr and Sallet , 2000, Theorem 6) (see the proof of Theorem 2 in Appendix), the UAS properties of the system (1) are thus obtained.

Combining Theorem 1 with Theorem 2 yields the following corollary.

Corollary 1. Suppose Assumption 1 holds. If there exists a function $V(t, \mathbf{x}) \in \mathcal{C}^1(\mathbb{R}_{\geq t_0} \times \mathcal{D}, \mathbb{R}_{\geq 0})$ such that inequality (8) holds; moreover, there exists $\alpha \in \mathcal{K}$ such that

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(t, \mathbf{x}) \leq -\alpha(V(t, \mathbf{x})). \tag{17}$$

Then the equilibrium $\mathbf{x} = \mathbf{0}$ of (1) is UAS if it is Ω-UAS, where $\Omega \triangleq \{\mathbf{x} \in \mathcal{D} | W_1(\mathbf{x}) = 0\}$.

Remark 12. Corollary 1 is an extension of (Iggidr and Sallet , 2000, Theorem 6) to a more general setting in the sense that the Lyapunov candidate becomes time-varying. When V becomes time-invariant, the result in Corollary 1 is the same as that in (Iggidr and Sallet , 2000, Theorem 6).

The following corollary is an extension of (Khalil , 2002, Theorem 4.9)

Corollary 2. Suppose Assumption 1 holds. Assume that there exists a function $V(t, \mathbf{x}) \in \mathcal{C}^1(\mathbb{R}_{\geq t_0} \times \mathcal{D}, \mathbb{R}_{\geq 0})$ such that inequality (8) holds with positive definite $W_j(\mathbf{x})$, $j = 1, 2$. Moreover, the inequality (16) holds. If the equilibrium $\mathbf{x} = \mathbf{0}$ is Ω-US, where $\Omega \triangleq \{\mathbf{x} \in \mathcal{D} | W_1(\mathbf{x}) = 0\}$, it is UAS for the time-varying system (1).

4. AN ILLUSTRATIVE EXAMPLE

The following nonholonomic system discussed in (Loria *et al.* , 2005, Example 2) is used to illustrate main results of this paper.

$$\dot{x}_1 = -x_1 + h(t, x_{2,3}) \tag{18a}$$

$$\dot{x}_2 = u(t, \mathrm{x}) x_3 \tag{18b}$$

$$\dot{x}_3 = -x_3 - u(t, \mathrm{x}) x_2, \tag{18c}$$

where $u(t, \mathrm{x}) = -x_1 + h(t, x_{2,3})$, $h(t, 0) \equiv 0$ and $x_{2,3} = col[x_2, x_3]$.

Uniform Stability Property

Let a time-invariant Lyapunov candidate is chosen to be $V_1(\mathbf{x}) = \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$ and $W_1(\mathbf{x}) = \frac{1}{2}(x_2^2 + x_3^2)$, leading to $\Omega := \{\mathbf{x} \in \mathbb{R}^3 | x_2 = 0, x_3 = 0\}$. By calculation, its derivative along trajectories of the system (18) is

$$\dot{V}(t, \mathrm{x}) \leq -x_3^2 \leq 0. \tag{19}$$

Next is to check Ω-UAS. Let $\mathbf{x}_0 = [x_{10}, 0, 0]^T \in \Omega$, as $V(\mathrm{x}(t; t_0, \mathrm{x}_0)) \leq V(\mathrm{x}_0) = 0$, it is easy to show $x_2(t) = x_3(t) \equiv 0, \forall t \geq 0$. Using the assumption as in (Loria *et al.* , 2005, Example 2) that $h(t, 0) \equiv 0$, $\lim_{t\to\infty} x_1(t) = 0$, uniformly in t_0. Thus, Ω-UAS is obtained, showing US property of the system (18) by using Theorem 1.

Uniform Asymptotic Stability Property

Let $W_1(\mathbf{x}) = \frac{1}{2}x_3^2$, the same Lyapunov candidate $V_1(\mathbf{x})$ is used. By computation, it follows that $\Omega_1 := \{\mathbf{x} \in \mathbb{R}^3 | x_3 = 0\}$. The initial condition of the system (18) is chosen to be $\bar{\mathbf{x}}_0 := [\, x_{10} \;\, x_{20} \;\, 0\,]^T \in \Omega_1 \subset \Omega$.

The remaining is to check Ω_1-UAS to conclude UAS. The following facts are obvious.

Fact 1 $x_3(t)$ converges zero uniformly in t_0.

Using the inequality (19) yields

$$\int_{t_0}^{t} x_3^2(s)ds \leq V(\bar{\mathbf{x}}_0).$$

Fact 1 holds by applying Proposition 1. Moreover, Fact 1 also indicates that $x_3(t), \forall t \in [t_0, \infty)$ is uniformly bounded in t_0.

Fact 2 $u(t, \mathbf{x}(t))x_2(t)$ converges zero uniformly in t_0.
Using (18c) leads to

$$\int_{t_0}^{t} |x_3(s) - u(s, \mathbf{x}(s))x_2(s)|\, ds \leq |x_3(t)|\,,$$

which implies that $x_3(t) - u(t, \mathrm{x}(t))x_2(t)$ converges to zero uniformly in t_0 (Proposition 1). Fact 2 holds true by using Fact 1.

Fact 3 $x_2(t)$ converges to a constant C uniformly in t_0.
This is true by using dynamics of x_2 (18b).

Assume that $C \neq 0$, using equation (18c) yields that $\lim_{t\to\infty} u(t, \mathbf{x}(t)) = 0$. Moreover $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} h(t, C, 0)$ is obtained from (18c). In other words, we cannot conclude that the equilibrium of system (18a-18c) is Ω_1-UAS for any uniformly bounded $h(t, \cdot)$ since it is hard to confirm that the constant C is zero. ◦

Remark 13. In order to shown the system (18a-18c) is UAS, $h(t, x_{2,3})$ needs to have a nice property: $U\delta$-PE property (see more detail in Loria *et al.* (2005)). We may also use a nested Matrosov theorem to check the UGAS for the example 2. The details can be found in (Loria *et al.* , 2005, Proposition 1).

Remark 14. As we can see the example that the way of showing UAS is not unique. For a general time-varying system, it is hard to compare which method is easier to use in real applications. In some situations, Ω -UAS may be easy to check while in other situations, constructing Matrosov functions to check UAS may be easy. It is an interesting research topic to clarify the link between Matrosov functions and Ω -UAS. We will explore such a link in our future research work.

5. CONCLUSION

In this paper, stability properties for nonlinear time-varying systems are investigated by using *time-varying* positive semi-definite Lyapunov candidates whose derivative along with the trajectories of the system is non-positive or negative semi-definite. Conditional stability properties on some sets, together with the available *time-varying* positive semi-definite Lyapunov candidates are used to show the US/ UAS of the time-varying systems. An illustrative example shows that main results of this paper provide useful and alternative ways in showing stability properties of time-varying systems.

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APPENDIX

Proof of Theorem 1

The proof procedure is similar to that in (Iggidr and Sallet , 2000, Theorem 5) in which contradiction was employed, though some necessary modifications are made.

Suppose that the origin is not uniformly stable. Then there exists $\epsilon > 0$ for which we can construct a sequence of initial conditions $\mathbf{x}_n^0 \in B_\epsilon$ and $\lim_{n\to\infty} \mathbf{x}_n^0 = \mathbf{0}$ such that for each $n \in \mathbb{N}$, there exists an initial time $t_n^0 \geq 0$ in such a way that the solution of (1) $\boldsymbol{\phi}(t; t_n^0, \mathbf{x}_n^0)$ does not stay with B_ϵ for all time $t \geq t_n^0$. That is, $\exists t_n > 0$ such that

$$\|\boldsymbol{\phi}(t; t_n^0, \mathbf{x}_n^0)\| \begin{cases} < \epsilon & \text{for all } 0 \leq t < t_n \\ = \epsilon & t = t_n \end{cases}.$$

The proof is completed by the following steps.

Step 1 . Since $\mathbf{x} = \mathbf{0}$ is Ω-UAS. Then for any given $\frac{\epsilon}{2} > 0$, there exists $\delta > 0$ and $T > 0$ such that the following holds.

$$\forall \mathbf{x}_0 \in \Omega \cap B_\delta \Rightarrow \|\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)\| \leq \frac{\epsilon}{2}. \tag{20}$$

for all $t \geq t_0 + T$.

Step 2 . Let $n_0 \in \mathbb{N}$ such that $\|\mathbf{x}_n^0\| \leq \eta$ for all $n \geq n_0$. Define a sequence $\{\mathbf{u}\}_{n \geq n_0}$ by $\mathbf{u}_n = \boldsymbol{\phi}(t_n^0 + t_n - T; t_n^0, \mathbf{x}_n^0)$ that converges to $\mathbf{z} \in \bar{B}_\epsilon$ as $n \to \infty$. Using the inequality (8), it follows that

$$0 \leq W_1(\mathbf{z}) = \lim_{n\to\infty} W_1(\mathbf{u}_n) \leq V(t; \mathbf{x}) \leq W_2(\mathbf{z})$$
$$W_2(\mathbf{z}) = \lim_{n\to\infty} W_2(\mathbf{u}_n) = 0,$$

which implies that $\mathbf{z} \in \bar{B}_\epsilon \cap \Omega$.

Step 3 Following the same steps in (Iggidr and Sallet , 2000, Theorem 5), we can show that there exists $p \geq n_0$ such that $\|\boldsymbol{\phi}(t_p^0 + t_p; t_p^0, \mathbf{x}_p^0)\| \leq \epsilon$.

This is a contradiction and completes the proof. □

Proof of Proposition 1.

We prove it by contradiction. Assume that there exists $\epsilon_0 > 0$ for which we can construct a sequence of initial conditions $\mathbf{x}_n^0 \in B_\delta$ and $\lim_{n\to\infty} \mathbf{x}_n^0 = \mathbf{0}$ such that for each n there exists an initial time $t_n^0 \geq t_0$ in such a way that the solution of (1) $\boldsymbol{\phi}(t; t_n^0, \mathbf{x}_n^0)$ stays with B_ϵ for all time $t \geq t_n^0$. That is, $\exists T_n \geq n > 0$ such that

$$d_{\mathcal{A}}(\boldsymbol{\phi}(t_n^0 + T_n; t_n^0, \mathbf{x}_n^0)) \geq \epsilon_0. \tag{21}$$

Noting the following facts

Fact 1 The origin is uniformly stable. For above ϵ_0, there exists $\delta_0 \in (0, \epsilon_0)$, such that

$$\|\mathbf{x}_0\| < \delta_0 \Rightarrow \|\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)\| < \epsilon_0, \forall t \geq t_0 \geq 0.$$

Fact 2 $d_{\mathcal{A}}(\mathbf{x}) \leq \|\mathbf{x}\|$ since $\mathbf{0} \in M$. It follows that

$$d_{\mathcal{A}}(\mathbf{x}_0) \leq \|\mathbf{x}_0\| < \delta_0$$
$$\Rightarrow d_{\mathcal{A}}(\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)) \leq \|\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)\| < \epsilon_0. \tag{22}$$

Fact 3 There exists $\delta_1 > 0$ such that

$$d_{\mathcal{A}}(\boldsymbol{\phi}(t_n^0 + T_n; t_n^0, \mathbf{x}_n^0)) \geq \delta_1, \tag{23}$$

for all $t \in [t_n^0, t_n^0 + T_n]$ and each $n \in \mathbb{N}$.

Proof: We prove it by contradiction. Assume that(23) does not hold, for any $\delta > 0$, we have

$$d_{\mathcal{A}}(\boldsymbol{\phi}(t_n^0 + T_n; t_n^0, \mathbf{x}_n^0)) < \delta,$$

for all $t \in [t_n^0, t_n^0 + T_n]$ and each $n \in \mathbb{N}$. By Fact 2, we have a contradiction if $\delta \leq \epsilon_0$.

Let $\Theta := \{\mathbf{x} \in B_\delta \,| d_{\mathcal{A}}(\mathbf{x}) \geq \delta_0 > 0\}$. $\Theta \subset B_\delta$ is not an empty set (see (23)) and is compact. Since $W(\mathbf{x})$ is continuous with respect to $\mathbf{x}$, $M_{min} = \min_{\mathbf{x}\in\Theta} W(\mathbf{x})$ is well-defined.

Since (12) holds true, it follows that

$$Mn \leq M_{min} T_n \leq \int_{t_n^0}^{t_n^0 + T_n} W(\boldsymbol{\phi}(s; t_n^0, \mathbf{x}_n^0)) ds \leq C_W,$$

holds for all $n \in \mathbb{N}$ and $\mathbf{x}_0 \in B_\delta \cap \Theta$. This is a contradiction and completes the proof. □

The proof of Theorem 2.

By using Theorem 1, the equilibrium of system (1) is US. Next step is to show that the equilibrium is UA. Since the equilibrium of system (1) is both US as Ω-UAS and satisfying (16), the following facts are obvious.

Fact 1 (US): For each $\epsilon > 0$, there exists $\delta > 0$ and $\delta < \epsilon$ (independent of t_0) such that

$$\|\mathbf{x}_0\| \leq \delta \Rightarrow \|\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)\| \leq \epsilon, \forall t \geq t_0 \geq 0. \tag{24}$$

Fact 2 (Ω-UA). Let $\gamma > 0$. For each δ obtained from Fact 1, there exists $T(\delta) > 0$, such that

$$\mathbf{x}_0 \in \Omega \cap B_\gamma \Rightarrow \|\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)\| \leq \delta, \tag{25}$$

for all $t \geq t_0 + T(\delta)$.

Fact 3 : Let $\gamma > 0$ is from Fact 2, generating η in Fact 1. It is apparent that

$$\mathbf{x}_0 \in B_\eta \Rightarrow \boldsymbol{\phi}(t; t_0, \mathbf{x}_0) \in B_\gamma. \tag{26}$$

Fact 4 : Since the inequality (16) holds true, let $C_W := \max_{\mathbf{x}_0 \in B_\eta} W_2(\mathbf{x})$, it follows that

$$\int_{t_0}^{t} \alpha(W_1(\boldsymbol{\phi}(s; t_0, \mathbf{x}_0)) ds \leq V(t_0, \mathbf{x}_0) \leq W_2(\mathbf{x}_0) \leq C_W.$$

Applying Proposition 1, it yields that for each $\alpha > 0$, there exists $T_\alpha > 0$ such that $d_\Omega(\boldsymbol{\phi}(t; t_0, \mathbf{x}_0)) < \alpha$, for any $\mathbf{x}_0 \in B_\gamma$ and $t \geq t_0 + T_\alpha$. This implies that

$$\exists T_\alpha > 0, \forall \mathbf{x}_0 \in B_\gamma, \exists \mathbf{z} \in B_\eta \cap \Omega$$
$$\|\boldsymbol{\phi}(t_0 + T_\alpha; t_0, \mathbf{x}_0) - \mathbf{z}\| \leq \alpha, \tag{27}$$

which is exactly the same as (Iggidr and Sallet , 2000, Equation (12)).

The proof is completed by following similar steps in the proof of (Iggidr and Sallet , 2000, Theorem 6) □