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Sliding Mode Static Output Feedback Control for Uncertain Systems: A Polytopic Approach

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Abstract: This paper presents a sliding mode output feedback control design methodology based on LMI's under a polytopic perspective for a class of uncertain dynamical systems. Both matched and mismatched uncertainties are considered. The existence and reachability problems are formulated through a polytopic description and solved using LMI's. The proposed controller is static in nature. The reaching and sliding motion are guaranteed despite the presence of both matched and mismatched uncertainties. A simulation example is presented to demonstrate the effectiveness of the proposed approach.

Keywords: Sliding Mode Control, Output Feedback, Uncertain Dynamical Systems, Mismatched Uncertainty, Linear Matrix Inequalities (LMI's), Polytopic Models.

1. INTRODUCTION

A *Variable Structure System* (VSS) is a type of nonlinear system that varies its structure according to a switching function. *Sliding Mode Control* (SMC) is a particular class of VSS which uses a switched or discontinuous control action across a sliding surface. The sliding surface defines the desired system performance and the control methodology is robust. SMC theory has been extensively studied, see for example [Utkin et al., 1999] [Edwards and Spurgeon, 1998].

Linear Matrix Inequalities (LMI's) [Boyd et al., 1994] represent a powerful mathematical tool for formulating and solving problems in control and systems engineering which consist of a set of matrix variables that may have specific structures defined by the designer. In [Gahinet et al., 1995] it is asserted that LMI's have the following attractive features: several problems can be recast as LMI's, such problems can be solved numerically in an efficient way by means of convex optimization algorithms, moreover, some mathematical programming problems with multiple constraints or objective functions which cannot be solved analytically can be tractable using LMI techniques. The LMI framework has been applied successfully to state-feedback SMC [Choi, 1997] [Edwards, 2004].

In many practical engineering applications the state vector cannot be measured because some states do not have physical meaning and/or software and hardware overhead costs may be high. There are two ways of overcoming this problem: the *observer-based approach* and the *output feedback approach*. The former requires extra dynamics for estimating the state vector given the input and output signals of the real plant. This increases the complexity of the control system. Furthermore, observers frequently undermine the robustness properties of state feedback control. The latter uses only measured output signals and can be either *static* or *dynamic* in nature. The *Static Output Feedback* (SOF) approach is the simplest and cheapest approach since the software and hardware costs are less than in the observer-based case and the dynamic output feedback approach. Static output feedback is an important and still open problem in control theory although several approaches have been applied [Syrmos et al., 1997]. While state feedback control corresponds to a convex problem, the static output feedback problem represents a more complex problem due to its non-convexity.

Many contributions explore the development of *Sliding Mode Output Feedback Control* (SMOFC) approaches. These approaches can be classified into *Sliding Mode Dynamic Output Feedback* (SMDOF) and *Sliding Mode Static Output Feedback* (SMSOF). The SMDOF, in turn, can be classified into *observer-based control* and *compensator-based control*. The last one is employed when the so-called Kimura-Davison condition is not satisfied. Additional details on the limitations of some existing SMOFC designs are discussed in [Edwards and Spurgeon, 2000]. Most of the cited papers consider a class of *Linear Time Invariant* (LTI) systems either without uncertainties or with *matched uncertainties*, disturbance signals or nonlinearities. Only a few papers have been devoted to the class of systems with mismatched uncertainties. SMDOF controllers have been proposed in [Shyu et al., 2000] and [Shyu et al., 2001]. Sliding mode static output feedback control (SMSOFC) systems have been developed in [Hui

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and Żak, 1993], [Choi, 2002] and [Xiang et al., 2006]. In [Shyu et al., 2000] a SMDOF control system where two sets of switching surfaces are considered and a new invariance property is established. The sliding sector is defined by the set of switching surfaces. In [Shyu et al., 2001] a dynamic output feedback controller is developed considering the usual style sliding surface instead of a set of two as in [Shyu et al., 2000]. With respect to SMSOFC, [Hui and Żak, 1993] proposed a parameterised bounded output feedback controller for the class of uncertain systems where uncertainty is decomposed in to matched and mismatched terms. Such uncertain components are bounded. In addition, physical and technical constraints, which are generally imposed upon the design of the controller, *e.g.*, constrained gains, are related to the controller bounds. The switching gain matrix design is based on eigenstructure assignment. The VSC proposed by Hui and Żak corresponds to a discontinuous control law which guarantees *practical stability* or the so-called *uniform ultimate boundedness* of the closed-loop system [Hui and Żak, 1993]. Such a VSC corresponds to a static output feedback variable structure controller (SOFVSC). A SOFVSC based upon LMI's is proposed in [Choi, 2002]. The class of system considered has matched and mismatched uncertainties. The latter are norm bounded but they are not required to be bounded by a known function of output signals. The control law proposed yields high control effort which is not desirable in many real engineering applications due to physical constraints on actuators and the plant to be controlled. As remarked by Choi, a trade-off between complexity and control effort can be made using a dynamic variable structure output feedback control law [Choi, 2002]. However, as argued above, such a dynamic output feedback approach incurs additional software and hardware overhead costs. An iterative LMI approach is presented in [Xiang et al., 2006] which neither requires coordinate transformations nor solves a SOF problem. Since the proposed control law corresponds to the class of high gain control laws, an optimization problem is formulated for designing the switching function and to avoid high control effort. Nevertheless, the LMI's involved are not easy to solve. In addition, due to the iterative nature of the algorithm, convergence depends upon the initial condition.

In this paper a new framework based on LMI's is developed for designing a *Sliding Mode Static Output Feedback Controller*. This approach extends the work in [Edwards et al., 2000] [Edwards et al., 2001] for a class of uncertain systems, which includes mismatched uncertainties. The existence and reaching problems are formulated from a polytopic perspective. The switching surface design problem is recast as an output feedback problem in terms of LMI's using a polytopic formulation. Several available numerical algorithms can be applied to such a problem. However, in this paper the Benton and Smith non-iterative algorithm [Benton Jr. and Smith, 1999] is considered because of its simplicity. The proposed control law consists of both linear and nonlinear components. It is important to point out that the control law is different to the high gain control law proposed in other cited references for SMSOFC with mismatched uncertain systems, where the approach proposed in this work is less complex. Moreover, the results obtained demonstrate the efficacy of this new approach. Therefore, the major contribution of this paper is to present a polytopic formulation for the static output feedback problem as well as to synthesise numerically the controller gains based on LMI's.

The structure of this paper is as follows. Section 2 defines the class of systems to be considered and provides the problem statement. Section 3 is devoted to Sliding Mode Static Output Feedback Control where the sliding surface design and control law synthesis are dealt with. Next, the efficacy of the new approach is demonstrated in Section 4 using a numerical example. Section 5 presents concluding remarks.

Throughout this paper $||\cdot||$ stands for the *Euclidean* norm of a vector and the induced spectral norm of a matrix.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider an uncertain dynamical system described in state-space form $\forall\, t \geq 0$ by

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \Big(\mathbf{A} + \Delta\mathbf{A}(t)\Big)\mathbf{x}(t) + \mathbf{B}\Big(\mathbf{u}(t) + \xi(t, \mathbf{x}, \mathbf{u})\Big) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned}\right\} \quad (1)$$

where $\mathbf{x} \in \mathscr{X} \subseteq \Re^n$ is the state vector with $\mathscr{X}$ an open set, $\mathbf{u} \in \mathscr{U} \subseteq \Re^m$ is the control vector with $\mathscr{U}$ the set of all admissible control signals and $\mathbf{y} \in \mathscr{Y} \subseteq \Re^p$ is the output vector of measurable signals. The uncertain vector function $\xi(t, \mathbf{x}, \mathbf{u})$ represents the lumped sum of matched nonlinearities and/or uncertainties.

Throughout the paper the following is assumed:

A.1 The order of the system and the number of input and output signals are such that

$$n > p > m \quad (2)$$

In the square case, *i.e.*, $p = m$, there exists no design freedom with respect to the synthesis of a switching gain matrix to determine the sliding mode.

A.2 The input and output distribution matrices are both full rank, *i.e.*, $rank(\mathbf{B}) = m$ and $rank(\mathbf{C}) = p$.

A.3 In the nominal triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, $rank(\mathbf{CB}) = m$.

As $rank(\mathbf{CB}) = m$ there exists a change of coordinates in which the triple $\big(\mathbf{A}, \mathbf{B}, \mathbf{C}\big)$ can be written in the canonical form proposed in [Edwards and Spurgeon, 1995]

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}, \ \mathbf{C} = [\,\mathbf{0} \quad \mathbf{T}\,] \quad (3)$$

where $\mathbf{A}_{11} \in \Re^{(n-m)\times(n-m)}$, $\mathbf{A}_{12} \in \Re^{(n-m)\times m}$, $\mathbf{A}_{21} \in \Re^{m\times(n-m)}$, $\mathbf{A}_{22} \in \Re^{m\times m}$, $\mathbf{B}_2 \in \Re^{m\times m}$ and $\mathbf{T} \in \Re^{p\times p}$ are assumed to be known constant matrices. Furthermore, $\mathbf{B}_2$ is non-singular and $\mathbf{T}$ is orthogonal.

Consequently, the system matrix $\mathbf{A}_\Delta(t) = \mathbf{A} + \Delta\mathbf{A}(t)$ has the structure

$$\mathbf{A}_\Delta(t) = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{A}_{11}(t) & \Delta\mathbf{A}_{12}(t) \\ \Delta\mathbf{A}_{21}(t) & \Delta\mathbf{A}_{22}(t) \end{bmatrix} \quad (4)$$

The matrix sub-blocks $\Delta\mathbf{A}_{11}(t)$, $\Delta\mathbf{A}_{12}(t)$, $\Delta\mathbf{A}_{21}(t)$ and $\Delta\mathbf{A}_{22}(t)$ depend upon uncertain and/or time-varying real parameters θ_i with $i = 1, 2, \cdots, r$.

Let $\Theta \subseteq \Re^r$ be the parameter space and let $\theta = [\theta_1 \;\; \theta_2 \;\; \cdots \;\; \theta_r]^T$ be the vector of real uncertain parameters where the uncertain parameter bounds

$$\underline{\theta}_i \leqslant \theta_i \leqslant \overline{\theta}_i \quad \text{for} \quad i = 1, 2, \cdots, r \tag{5}$$

define a *hyper-rectangle* in Θ. Such a hyper-rectangle is said to be a *parameter box.* The uncertain parameter vector θ can describe two kinds of uncertainties. On the one hand, θ could correspond to physical parameters which are constant but unknown and for which only extreme values are known up to some accuracy. On the other hand, θ could represent a continuous time real vector-valued function, *i.e.*, $\theta = \theta(t) : \Re_+ \rightarrow \Theta$. In this case, the uncertain parameters vary continuously or piece-wise continuously within the parameter box. Such an uncertain model with this sort of uncertainty is called a *Linear Time Varying* (LTV) *System.* For the sake of generality, uncertain time-varying parameters $\theta_i(t)$ with $i = 1, 2, \cdots, r$ are considered since time invariant uncertain parameters can be seen as the particular case when $\theta_i(t) = \theta_i \;\; \forall\, t$ with $i = 1, 2, \cdots, r$.

Since $\Delta\mathbf{A}_{21}(t) = \mathbf{B}_2\Delta\tilde{\mathbf{A}}_{21}(t)$ and $\Delta\mathbf{A}_{22} = \mathbf{B}_2\Delta\tilde{\mathbf{A}}_{22}(t)$, the system matrix (4) is taken to have the form

$$\tilde{\mathbf{A}}_\Delta(t) = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{A}_{11}(t) & \Delta\mathbf{A}_{12}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{6}$$

and the matched uncertainty term $\tilde{\xi} : \Re_+ \times \Re^n \times \Re^m \rightarrow \Re^m$ is given by

$$\tilde{\xi}(t, \mathbf{x}, \mathbf{u}) = \xi(t, \mathbf{x}, \mathbf{u}) + \Delta\tilde{\mathbf{A}}_{21}(t)\mathbf{x}_1(t) + \Delta\tilde{\mathbf{A}}_{22}(t)\mathbf{x}_2(t) \tag{7}$$

For the remainder of the paper, it is assumed that:

A.4 The matched uncertainty term is bounded by

$$||\tilde{\xi}(t, \mathbf{x}, \mathbf{u})|| \leq k_1||\mathbf{u}(t)|| + \phi(t, \mathbf{y}(t)) + k_2 \tag{8}$$

where $\phi(t, \mathbf{y}(t))$ is a known function such that $\phi : \Re_+ \times \Re^p \rightarrow \Re_+$. In addition, $0 < k_1 < 1$ and $k_2 \in \Re_+$.

The *sliding surface* $\mathscr{S}$ is defined as follows

$$\mathscr{S} = \{\mathbf{x} \in \Re^n : \sigma(t) = \mathbf{\Gamma}\mathbf{y}(t) = \mathbf{\Gamma}\mathbf{C}\mathbf{x}(t) = 0\} \tag{9}$$

where $\sigma(t) \in \Re^m$ is the *switching function* and $\mathbf{\Gamma} \in \Re^{m\times p}$ is the *switching gain matrix* to be designed.

Let

$$\mathbf{\Gamma}\mathbf{T} = [\,\mathbf{\Gamma}_1 \quad \mathbf{\Gamma}_2\,] \tag{10}$$

where $\mathbf{\Gamma}_1 \in \Re^{m\times(p-m)}$ and $\mathbf{\Gamma}_2 \in \Re^{m\times m}$ is so that $\det(\mathbf{\Gamma}_2) \neq 0$.

Define $\mathbf{C}_1 \in \Re^{(p-m)\times(n-m)}$ as

$$\mathbf{C}_1 \triangleq \left[\,\mathbf{0}_{((p-m)\times(n-p))} \qquad \mathbf{I}_{(p-m)}\,\right] \tag{11}$$

and gain $\mathbf{K} \in \Re^{m\times(p-m)}$ as

$$\mathbf{K} \triangleq \mathbf{\Gamma}_2^{-1}\mathbf{\Gamma}_1 \tag{12}$$

Since $\mathbf{x}_2 = -\mathbf{K}\mathbf{C}_1\mathbf{x}_1(t)$ in the sliding mode then

$$\dot{\mathbf{x}}_1(t) = \big(\tilde{\mathbf{A}}_{11}(t) - \tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1\big)\mathbf{x}_1(t) \tag{13}$$

where $\tilde{\mathbf{A}}_{11}(t) = \big(\mathbf{A}_{11} + \Delta\mathbf{A}_{11}(t)\big)$ and $\tilde{\mathbf{A}}_{12}(t) = \big(\mathbf{A}_{12} + \Delta\mathbf{A}_{12}(t)\big)$.

This reduced-order sliding mode dynamics corresponds to an output feedback problem involving mismatched uncertainties.

From (10) and (12) the switching gain matrix $\mathbf{\Gamma}$ is parameterized as

$$\mathbf{\Gamma} = \mathbf{\Gamma}_2\,[\,\mathbf{K} \quad \mathbf{I}_m\,]\,\mathbf{T}^T \tag{14}$$

where $\mathbf{\Gamma}_2 \in \Re^{m\times m}$.

The problem of synthesizing the gain matrix $\mathbf{K}$ even when $\Delta\mathbf{A}_{11}(t) = \Delta\mathbf{A}_{12}(t) = \mathbf{0}$ is still an open problem. Note that the matrix $\mathbf{\Gamma}_2$ represents a scaling of the switching matrix $\mathbf{\Gamma}$. In this paper, it is assumed that $\mathbf{\Gamma}_2$ is chosen so that $\mathbf{\Gamma}\mathbf{C}\mathbf{B} = \mathbf{I}_m$. An appropriate choice is $\mathbf{\Gamma}_2 = \mathbf{B}_2^{-1}$; this is helpful in the solution of the reachability problem. A control law for the uncertain dynamical system (1) is required in order to guarantee that the sliding surface $\mathscr{S}$ is reached and the sliding motion takes place.

3. SLIDING MODE OUTPUT FEEDBACK CONTROL

In this section the existence and reachability problems are formulated from a polytopic perspective via LMI's.

Consider an uncertain matrix $\mathbf{\Pi}(t) \in \Re^{q\times q}$. A *polytope* $\mathscr{P}$, such that $\Pi(t) \in \mathscr{P}$, is the convex hull

$$\left.\begin{aligned} \mathscr{P} &= \mathrm{Co}\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \cdots, \mathbf{\Pi}_N\} \\ &= \left\{ \mathbf{\Pi} = \sum_{i=1}^{N} \mu_i\mathbf{\Pi}_i : \sum_{i=1}^{N} \mu_i = 1, \mu_i \geq 0 \right. \\ &\qquad \left. \text{for } \; i = 1, 2, \cdots, N \right\} \end{aligned}\right\} \tag{15}$$

where N is the number of vertices of the polytope $\mathscr{P}$ and μ_j with $j = 1, 2, \cdots, N$ are said to be the polytopic coordinates of $\mathbf{\Pi}$.

At this point it is pertinent to highlight that the contribution of this paper lies in formulating the sliding mode static output feedback problem for systems with mismatched uncertainties in a polytopic fashion as well as to present an extension of the approach in [Edwards et al., 2000] [Edwards et al., 2001] for synthesizing the control law and hence to solve the reachability problem.

3.1 Sliding Surface Design

Let $\mathbf{\Pi}(t)$ be the triple $(\tilde{\mathbf{A}}_{11}(t), \tilde{\mathbf{A}}_{12}(t), \mathbf{C}_1)$ and let $\mathbf{\Pi}_j$ be the triple $(\tilde{\mathbf{A}}_{11j}, \tilde{\mathbf{A}}_{12j}, \mathbf{C}_1)$ then a polytope $\mathscr{P}$ can be defined according to (15) where the vertices are given by

$$\tilde{\mathbf{A}}_{11j} = \mathbf{A}_{11} + \sum_{i=1}^{r} \theta_i\Delta\mathbf{A}_{11i}\Big|_{\theta_i=\{\underline{\theta}_i, \overline{\theta}_i\}} \tag{16}$$

$$\tilde{\mathbf{A}}_{12j} = \mathbf{A}_{12} + \sum_{i=1}^{r} \theta_i\Delta\mathbf{A}_{12i}\Big|_{\theta_i=\{\underline{\theta}_i, \overline{\theta}_i\}} \tag{17}$$

for $j = 1, 2, \cdots, N = 2^r$ and $\mathbf{C}_1$ is defined in (11).

The following is assumed:

A.5 The triple $(\tilde{\mathbf{A}}_{11j}, \tilde{\mathbf{A}}_{12j}, \mathbf{C}_1)$ for $j = 1, 2, \cdots, N$ is stabilisable and detectable for all admissible uncertainties in the hyper-rectangle Θ.

The reduced-order system (13) is static-output feedback stabilisable if and only if there exists a positive definite *Lyapunov* matrix $\mathbf{P}_1 = \mathbf{P}_1^T \in \Re^{(n-m)\times(n-m)}$ and a gain matrix $\mathbf{K}$ such that

$$\left(\tilde{\mathbf{A}}_{11j}-\tilde{\mathbf{A}}_{12j}\mathbf{K}\mathbf{C}_1\right)^T\mathbf{P}_1+\mathbf{P}_1\left(\tilde{\mathbf{A}}_{11j}-\tilde{\mathbf{A}}_{12j}\mathbf{K}\mathbf{C}_1\right)<0 \quad (18)$$

for $j = 1, 2, \cdots, N$.

The vertices of the polytope $\mathscr{P}$ are said to be simultaneously stabilized by the gain matrix $\mathbf{K}$ if (18) holds for $\mathbf{P}_1 = \mathbf{P}_1^T > 0$ and $\mathbf{K}$.

Here the non-iterative LMI-based algorithm proposed in [Benton Jr. and Smith, 1999] is applied. An important feature of Benton and Smith's approach is its simplicity and reduced computing time. Benton and Smith assert that their algorithm produces satisfactory results and is applicable in many situations. The main drawback of this approach is the difficulty in finding a suitable state feedback gain $\mathbf{K}_{sf}$ such that the system is *Simultaneously K-Stabilisable and Detectable*.

Considering the existence problem as an output feedback problem under the polytopic formulation, the aim of Benton and Smith's algorithm is to synthesize a gain matrix $\mathbf{K}$ and a *Lyapunov* matrix $\mathbf{P}_1$ as follows

Step 1: Define N vertices of the polytopic model.
Step 2: Define a degree of stability such that

$$\tilde{\mathbf{A}}_{11\alpha j} = \tilde{\mathbf{A}}_{11j} + \alpha\mathbf{I} \quad \text{for} \quad j = 1, 2, \cdots, N$$

Step 3 : Solve the following optimization problem
min *trace*$(\mathbf{Q}_{sf})$
s.t.

$$\mathbf{Q}_{sf} - \mathbf{I} > \mathbf{0}$$
$$\mathbf{Q}_{sf}\tilde{\mathbf{A}}_{11\alpha j}^T + \tilde{\mathbf{A}}_{11\alpha j}\mathbf{Q}_{sf} + \mathbf{Y}_{sf}^T\tilde{\mathbf{A}}_{12j}^T + \tilde{\mathbf{A}}_{12j}\mathbf{Y}_{sf} < 0$$
$$\text{for} \quad j = 1, 2, \ldots, N$$

Step 4 : Set $\mathbf{K}_{sf} = \mathbf{Y}_{sf}\mathbf{Q}_{sf}^{-1}$.
Step 5 : Solve the LMI feasibility problem
find σ and $\mathbf{P}_1$
s.t.

$$\mathbf{P}_1 > \mathbf{I} \quad , \quad \sigma > 0$$
$$\left(\tilde{\mathbf{A}}_{11\alpha j} + \tilde{\mathbf{A}}_{12j}\mathbf{K}_{sf}\right)^T\mathbf{P}_1 + \mathbf{P}_1\left(\tilde{\mathbf{A}}_{11\alpha j} + \tilde{\mathbf{A}}_{12j}\mathbf{K}_{sf}\right) < 0$$
$$\text{for} \quad j = 1, 2, \ldots, N$$

Step 6 : Solve the following LMI problem
find $\mathbf{K}$
s.t.

$$\left(\tilde{\mathbf{A}}_{11\alpha j} - \tilde{\mathbf{A}}_{12j}\mathbf{K}\mathbf{C}_1^T\right)^T\mathbf{P}_1+\mathbf{P}_1\left(\tilde{\mathbf{A}}_{11\alpha j} - \tilde{\mathbf{A}}_{12j}\mathbf{K}\mathbf{C}_1\right) < 0$$
$$\text{for} \quad j = 1, 2, \ldots, N.$$

When synthesizing the gain matrix $\mathbf{K}$, an optimization problem can be considered instead of the previous feasibility problem. Such an optimization problem involves minimising a norm defined by the designer.

3.2 Control Law Synthesis

If a switching gain matrix $\mathbf{\Gamma}$ exists such that the sliding dynamics (13) is stable, then a nonsingular change of coordinates $\mathbf{x} \mapsto \hat{\mathbf{T}}\mathbf{x}$ where

$$\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{I}_{(n-m)} & \mathbf{0} \\ \mathbf{B}_2^{-1}\mathbf{K}\mathbf{C}_1 & \mathbf{B}_2^{-1} \end{bmatrix} \quad (19)$$

is such that the triple $\left(\tilde{\mathbf{A}}_\Delta(t), \mathbf{B}, \mathbf{C}\right)$ from (3) and (6) can be transformed into

$$\hat{\mathbf{A}}_\Delta(t) = \hat{\mathbf{A}} + \Delta\hat{\mathbf{A}}(t) = \begin{bmatrix} \hat{\mathbf{A}}_{\Delta 11}(t) & \hat{\mathbf{A}}_{\Delta 12}(t) \\ \hat{\mathbf{A}}_{\Delta 21}(t) & \hat{\mathbf{A}}_{\Delta 22}(t) \end{bmatrix} \quad (20)$$

$$\hat{\mathbf{B}} = \begin{bmatrix} 0 & \mathbf{I}_m \end{bmatrix}^T \quad (21)$$

$$\mathbf{\Gamma}\hat{\mathbf{C}} = \begin{bmatrix} 0 & \mathbf{I}_m \end{bmatrix} \quad \text{where} \quad \hat{\mathbf{C}} = \begin{bmatrix} 0 & \overline{\mathbf{T}} \end{bmatrix} \quad (22)$$

with $\overline{\mathbf{T}} \in \Re^{(p\times p)}$ such that $\det\{\overline{\mathbf{T}}\} \neq 0$. Noting the form of (19) it is obvious that (21) holds. From $\mathbf{B}_2^{-1} = \mathbf{\Gamma}_2$, (22) follows straightforwardly.

Let $\mathbf{\Pi}(t)$ be the triple $(\hat{\mathbf{A}}_\Delta(t), \hat{\mathbf{B}}, \hat{\mathbf{C}})$ and let $\mathbf{\Pi}_j$ be the triple $(\hat{\mathbf{A}}_{\Delta j}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ thus a polytope $\mathscr{P}$ is defined according to (15) where the vertices are given by

$$\hat{\mathbf{A}}_{\Delta j} = \hat{\mathbf{A}} + \sum_{i=1}^{r}\theta_i\Delta\hat{\mathbf{A}}_i\Big|_{\theta_i=\{\underline{\theta}_i,\overline{\theta}_i\}} = \begin{bmatrix} \hat{\mathbf{A}}_{\Delta 11j} & \hat{\mathbf{A}}_{\Delta 12j} \\ \hat{\mathbf{A}}_{\Delta 21j} & \hat{\mathbf{A}}_{\Delta 22j} \end{bmatrix} \quad (23)$$

for $j = 1, 2, \cdots, N = 2^r$ and $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ are defined in (21) and (22) respectively.

The sliding mode dynamics is implicit in $\hat{\mathbf{A}}_{\Delta 11j} = \tilde{\mathbf{A}}_{11j} - \tilde{\mathbf{A}}_{12j}\mathbf{K}\mathbf{C}_1$ for $j = 1, 2\cdots, N$ which is stable by design and in turn $\hat{\mathbf{A}}_{\Delta 11}(t) = \tilde{\mathbf{A}}_{11}(t) - \tilde{\mathbf{A}}_{12}(t)\mathbf{K}\mathbf{C}_1$ by the convexity property of the polytope $\mathscr{P}$.

Proposition: If there exists a Lyapunov matrix $\mathbf{P}$ partitioned as follows

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{bmatrix} \quad (24)$$

where $\mathbf{P}_1 \in \Re^{(n-m)\times(n-m)}$ and $\mathbf{P}_2 \in \Re^{m\times m}$, and a parametrised gain matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix}\overline{\mathbf{T}}^{-1} \quad (25)$$

where $\mathbf{G}_1 \in \Re^{m\times(p-m)}$ and $\mathbf{G}_2 \in \Re^{m\times m}$ such that the following matrix inequality holds for $j = 1, 2, \cdots, N$

$$\mathcal{A}_j^T\mathbf{P} + \mathbf{P}\mathcal{A}_j < 0 \quad (26)$$

where

$$\mathcal{A}_j = \begin{bmatrix} \hat{\mathbf{A}}_{\Delta 11j} & \hat{\mathbf{A}}_{\Delta 12j} \\ \hat{\mathbf{A}}_{\Delta 21j} - \hat{\mathbf{B}}_2\mathbf{G}_1\mathbf{C}_1 & \hat{\mathbf{A}}_{\Delta 22j} - \hat{\mathbf{B}}_2\mathbf{G}_2 \end{bmatrix} \quad (27)$$

Then, the control law

$$\mathbf{u}(t) = \mathbf{u}_L(t) + \mathbf{u}_{NL}(t) \quad (28)$$

with the linear component $\mathbf{u}_L(t)$ of the form

$$\mathbf{u}_L(t) = -\mathbf{G}\mathbf{y}(t) \quad (29)$$

and the nonlinear component given by

$$\mathbf{u}_{NL}(t) = \begin{cases} -\rho(\cdot)\mathbf{P}_2^{-1}\dfrac{\mathbf{\Gamma}\mathbf{y}(t)}{||\mathbf{\Gamma}\mathbf{y}(t)||} & \text{if } \mathbf{\Gamma}\mathbf{y}(t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (30)$$

where

$$\rho\big(t, \mathbf{y}(t), \mathbf{u}(t)\big) = \frac{k_1||\mathbf{u}_L(t)|| + \phi(t, \mathbf{y}(t)) + k_2 + \eta}{(1 - k_1)} \quad (31)$$

guarantees a sliding motion on the surface $\mathscr{S}$ inside *the sliding patch*

$$\Omega = \left\{\left(\hat{\mathbf{x}}_1 \in \Re^{n-m}, \hat{\mathbf{x}}_2 \in \Re^m\right) \; : \; ||\hat{\mathbf{x}}_1|| < \eta\gamma^{-1}\right\} \quad (32)$$

where η is a design scalar and

$$\gamma = max\left\{||\mathbf{P}_2\big(\hat{\mathbf{A}}_{\Delta 12j} - \mathbf{G}_1\mathbf{C}_1\big)||\right\} \quad (33)$$

for $j = 1, 2, \cdots, N$. ♦

Proof:
Consider *the Lyapunov function* $V(t) = \hat{\mathbf{x}}^T(t)\mathbf{P}\hat{\mathbf{x}}(t)$. Since $\mathbf{P}$ has the partition shown in (24) then $\mathbf{P}\hat{\mathbf{B}} = \big(\mathbf{\Gamma}\hat{\mathbf{C}}\big)^T\mathbf{P}_2$.

After a few manipulations considering (30) and (31) as well as (26) and $\eta > 0$, then $\dot{V} < 0 \quad \forall\, \hat{\mathbf{x}}(t) \neq 0$. Therefore, the system is *quadratically stable.*

Partition the state vector $\hat{\mathbf{x}}(t)$ as $\left[\,\hat{\mathbf{x}}_1^T(t) \quad \hat{\mathbf{x}}_2^T(t)\,\right]^T$. Using matrix inequality (26) with the previous partition, the following quadratic form can be written

$$\hat{\mathbf{x}}_2^T(t)\Big(\big(\hat{\mathbf{A}}_{\Delta 22j} - \mathbf{G}_2\big)^T\mathbf{P}_2 + \mathbf{P}_2\big(\hat{\mathbf{A}}_{\Delta 22j} - \mathbf{G}_2\big)\Big)\hat{\mathbf{x}}_2(t) < 0$$

for $j = 1, 2, \cdots, N = 2^r$.

Consider the Lyapunov function $\tilde{V}(t) = \hat{x}_2^T(t)\mathbf{P}_2\hat{x}_2(t)$. Its derivative along the trajectories is given by

$$\begin{aligned}\dot{\tilde{V}}(t) &= 2\hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\big(\hat{\mathbf{A}}_{\Delta 21j} - \mathbf{G}_1\mathbf{C}_1\big)\hat{\mathbf{x}}_1(t) + \\ &+ \hat{\mathbf{x}}_2^T(t)\Big(\big(\hat{\mathbf{A}}_{\Delta 22j} - \mathbf{G}_2\big)^T\mathbf{P}_2 + \mathbf{P}_2\big(\hat{\mathbf{A}}_{\Delta 22j} - \mathbf{G}_2\big)\Big)\hat{\mathbf{x}}_2(t) + \\ &+ 2\hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\mathbf{u}_{NL}(t) \qquad \text{for} \quad j = 1, 2, \cdots, N\end{aligned}$$

Since (31) implies $||\xi(t, \mathbf{y}(t), \mathbf{u}(t))|| \leq \rho(t, \mathbf{y}(t), \mathbf{u}(t)) - \eta$ then

$$\dot{\tilde{V}}(t) < 2\hat{\mathbf{x}}_2^T(t)\mathbf{P}_2\big(\hat{\mathbf{A}}_{\Delta 21j} - \mathbf{G}_1\mathbf{C}_1\big)\hat{\mathbf{x}}_1(t) - 2\eta||\hat{\mathbf{x}}_2(t)||$$

which means that the sliding motion occurs inside the sliding patch Ω defined in (32) with (33).

Therefore, the sliding motion occurs in finite time since the system is quadratically stable. *Q.E.D.*

Note that (27) can be written as $\mathcal{A}_j \triangleq \hat{\mathbf{A}}_{\Delta j} - \hat{\mathbf{B}}\mathbf{G}\hat{\mathbf{C}}$, then

$$\mathcal{A}_j^T\mathbf{P} + \mathbf{P}\mathcal{A}_j = \begin{bmatrix} \Lambda_{11j} & \Lambda_{12j} \\ \Lambda_{21j} & \Lambda_{22j} \end{bmatrix} \tag{34}$$

where

$$\left.\begin{aligned}\Lambda_{11j} &= \hat{\mathbf{A}}_{\Delta 11j}^T\mathbf{P}_1 + \mathbf{P}_1\hat{\mathbf{A}}_{\Delta 11j} \\ \Lambda_{12j} &= \mathbf{P}_1\hat{\mathbf{A}}_{\Delta 12j} + \hat{\mathbf{A}}_{\Delta 21j}^T\mathbf{P}_2 - \mathbf{C}_1^T\mathbf{L}_1^T \\ \Lambda_{21j} &= \mathbf{P}_2\hat{\mathbf{A}}_{\Delta 21j} + \hat{\mathbf{A}}_{\Delta 12j}^T\mathbf{P}_1 - \mathbf{L}_1\mathbf{C}_1 \\ \Lambda_{22j} &= \mathbf{P}_2\hat{\mathbf{A}}_{\Delta 22j} - \mathbf{L}_2 + \hat{\mathbf{A}}_{\Delta 22j}^T\mathbf{P}_2 - \mathbf{L}_2^T\end{aligned}\right\} \tag{35}$$

for $j = 1, 2, \cdots, N$.

Consider a partition of

$$\hat{\mathbf{A}}_{\Delta 21j} = \left[\,\hat{\mathbf{A}}_{\Delta 211j} \;\; \hat{\mathbf{A}}_{\Delta 212j}\,\right] \quad \text{for } j = 1, 2, \cdots, N \tag{36}$$

where $\hat{\mathbf{A}}_{\Delta 211j} \in \Re^{m\times(n-p)}$ and $\hat{\mathbf{A}}_{\Delta 212j} \in \Re^{m\times(p-m)}$.

Choose any $\gamma > max\Big\{||\hat{\mathbf{A}}_{\Delta 211j}||\Big\}$ and solve the following optimization problem in order to find $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{P}_1$ and $\mathbf{P}_2$

$$\left.\begin{aligned}&min\;\; \lambda \\ &s.t. \\ &\begin{bmatrix} -\lambda\mathbf{I} & [\,\mathbf{L}_1\;\mathbf{L}_2\,]\,\hat{\mathbf{T}}^{-1} \\ \big([\,\mathbf{L}_1\;\mathbf{L}_2\,]\,\hat{\mathbf{T}}^{-1}\big)^T & -\lambda\mathbf{I}\end{bmatrix} < 0 \\ &\begin{bmatrix} -\gamma\mathbf{I} & \mathbf{P}_2\hat{\mathbf{A}}_{\Delta 21j} - \mathbf{L}_1\mathbf{C}_1 \\ \hat{\mathbf{A}}_{\Delta 21j}^T\mathbf{P}_2 - \mathbf{C}_1^T\mathbf{L}_1^T & -\gamma\mathbf{I}\end{bmatrix} < 0 \\ &\begin{bmatrix} -r_d\mathbf{P} & \mathbf{P}\mathcal{A}_j + c_n\mathbf{P} \\ c_n\mathbf{P} + \mathcal{A}_j^T\mathbf{P} & -r_d\mathbf{P}\end{bmatrix} < 0 \\ &\mathbf{P}\mathcal{A}_j + \mathcal{A}_j^T\mathbf{P} + 2h\mathbf{P} < 0 \\ &\mathbf{P} > \mathbf{I} \\ &\text{for } \; j = 1, 2, \cdots, N\end{aligned}\right\} \tag{37}$$

where $(-c_n, 0)$ and r_d represent the center and the radius of a disc whilst $(-h, 0)$ corresponds to a vertical line in the complex plane defining an LMI region for the minimization problem (37).

If this optimization problem has a solution then

$$\mathbf{G}_1 = \mathbf{P}_2^{-1}\mathbf{L}_1 \quad , \quad \mathbf{G}_2 = \mathbf{P}_2^{-1}\mathbf{L}_2 \tag{38}$$

and the proposed control law (28) with (29) and (30) guarantees that the sliding mode takes place inside the sliding patch. In addition, the state trajectories will reach the sliding patch in finite time and will remain on it.

4. NUMERICAL EXAMPLE

The system is taken from [Xiang et al., 2006]. Such a system belongs to the class of uncertain systems with mismatched uncertainties. Another feature of this plant is that only a subset of the state variables is available for measurement. The mathematical representation of the plant to be considered is

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} -3 + \sin(t) & 0 & 1 + \sin(2t) \\ 1 & 2 & \sin(4t) \\ 0 & 1 + \sin(3t) & -2 \end{bmatrix}\mathbf{x}(t) + \\ &+ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\big(u(t) + \sin(5t)\big)\end{aligned} \tag{39}$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}(t)$$

The initial condition considered is the same as in [Xiang et al., 2006]: $[\,1 \;\; 0 \;\; -1\,]^T$. Since $rank\big(\mathbf{CB}\big) = m$, then (39) can be written considering (3) and consequently (4), (6) and (7). Then the proposed polytopic approach is applied. The switching gain matrix is given by

$$\mathbf{\Gamma} = [\,0.7327 \quad 0.2673\,] \tag{40}$$

and the gain of (29) corresponds to

$$\mathbf{G} = [\,1.8829 \quad 1.1597\,] \tag{41}$$

The eigenvalues of $\mathcal{A}_j$ for $j = 1, \cdots, 8$ considering the plant (39) are shown in Fig. 1.

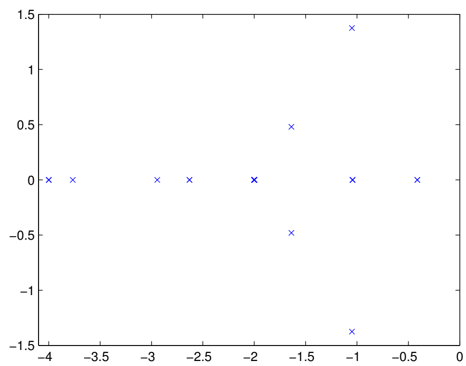

Fig. 1. Closed-loop poles

The nonlinear component (30) with (31) is designed such that

$$\begin{aligned}\rho(t, \mathbf{y}(t), \mathbf{u}(t)) = 3.1213|\mathbf{y_1(t)}| + \\ + 2.1213|\mathbf{y}_2(t)| + 1.4242\end{aligned} \tag{42}$$

Furthermore, $\mathbf{P}_2 = 1.0030$. In order to avoid high frequency oscillations in the control signal, *i.e*, chattering, the nonlinear component (30) has been replaced by

$$\mathbf{u}_{NL}(t) = \begin{cases} -\rho(\cdot)\mathbf{P}_2^{-1}\dfrac{\mathbf{\Gamma y}(t)}{||\mathbf{\Gamma y}(t)|| + \epsilon} & \text{if } \mathbf{\Gamma y}(t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (43)$$

and ϵ has been chosen as $\epsilon = 0.00001$.

The closed-loop time response is shown in Fig. 2. The control effort and the time evolution of the switching function $\sigma(t)$ is shown in Fig. 3. As expected there is no chattering in the control signal because of the smoothed unite vector considered in (43). The results obtained demonstrate the efficacy of the proposed sliding mode static output feedback control system.

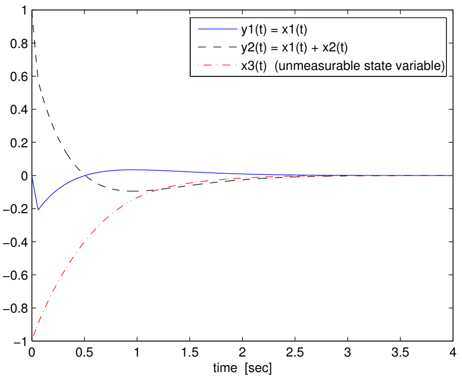

Fig. 2. Closed-loop response

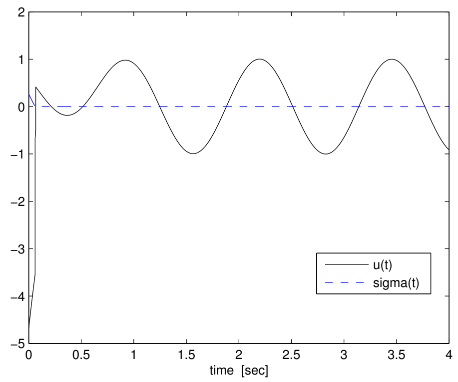

Fig. 3. Control signal $u(t)$ and switching function $\sigma(t)$

5. CONCLUSIONS

A new sliding mode static output feedback controller based on LMI's for systems with matched and mismatched uncertainties has been proposed in this paper. The existence problem and the reaching problem have been formulated using a polytopic description. Once the existence problem has been formulated from a polytopic perspective, the switching gain matrix can be designed using one of several numerical algorithms. The linear gains of the control law are numerically synthesised after formulating the reaching problem from a polytopic perspective as in the existence problem. The control law does not incur high control effort. The design methodology can be implemented in a straightforward way. Computer simulations have shown the efficacy of the new proposed sliding mode static output feedback controller.

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