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Control of mechanical systems with constraints: two pendulums case study ⋆

M. S. Ananyevskiy, * **A. L. Fradkov,** ** **H. Nijmeijer** ***

* *Saint Petersburg State University, Saint Petersburg, Russia (e-mail: msa@rusycon.ru).*
** *Institute for Problems of Mechanical Engineering of RAS, Saint Petersburg, Russia.*
*** *Eindhoven University of Technology, Department of Mechanical Engineering, Eindhoven, The Netherlands.*

Abstract: A method for control of mechanical systems under phase constraints, applicable to energy control of Hamiltonian systems is proposed. The constrained energy control problem for two pendulums by a single control action is studied both analytically and numerically. It is shown that for a proper choice of penalty parameter of the algorithm any energy level for the one pendulum under any specified constraint on the energy of the other pendulum can be achieved. Simulation results confirm fast convergence rate of the algorithm.

1. INTRODUCTION

Practical control problems for physical and mechanical systems often require taking into account phase constraints. However solving problems with constraints may be very hard, especially for optimal control problems. Constraints which should hold in every time instant may significantly change dynamical nature of the system. Difficulties become still stronger for nonlinear and uncertain systems. Complex mechanical systems usually consists of several subsystems. An important practical problem is the selective excitation, when it is needed to increase the energy of one subsystem and to constraint the energy of the other: a passing through resonance Tomchin et al. [2005], selective molecule excitation Anan'evskii [2007], etc.

A unified and powerful method for solving estimation and control problems for nonlinear system is the so called *speed-gradient method* Fradkov [1979, 1990], Fradkov and Pogromsky [1998]. Speed-gradient method applies to the case when the control goal is specified as an asymptotic minimization of a scalar *goal function* of the system state. It allows to design a state feedback allowing to achieve the control goal under certain natural conditions. The method was applied to a variety of nonlinear and adaptive control problems for physical and mechanical systems Fradkov and Pogromsky [1998], Fradkov et al. [1999], Fradkov [2007]. However, taking into account inequality phase constraints were not taken into account before (the case of equality constraints is examined in Fradkov [2007]).

In the present paper we propose an approach to control under inequality phase constraints based on an extended version of the speed-gradient method. The constraints are taken into account via a version of the penalty (barrier) functions, well known in the mathematical programming Fiacco and McCormick [1990]. A general result providing conditions for achievement of the control goal for constraints specified by a scalar *constraint function* is given. Application of the method is illustrated by example of a pair of pendulums affected by a single controlling force. The goal is to achieve the prespecified value of the one pendulum, while constraint function is the energy of the other pendulum. Both analytical examination and numerical results confirm good performance of the proposed controller.

2. CONTROL ALGORITHM: FORMULATION OF THE PROBLEM AND APPROACH

Consider a control plant described by a system of states equations

$$\frac{dx}{dt} = F(x, u), \quad x(0) \in \mathbb{X}_0,\, x \in \mathbb{R}^n,\, u \in \mathbb{R}^m,\, t \in \mathbb{R}, \quad (1)$$

here x is the state vector, u is the control input, t is time, $\mathbb{X}_0$ is the set of possible initial states, $F(x, u)$ is smooth function in both arguments. Control goal is specified by means of a non-negative smooth function $Q(x)$ (further it is called *goal function*)

$$\lim_{t\to+\infty} Q(x(t)) = 0, \quad (2)$$

where $x(t)$ is the solution of (1) with some admissible $u(t)$ and $x(0) \in \mathbb{X}_0$. Phase constraints are specified by an inequality for a smooth function $B(x)$ (further it is called *function of constraints*)

$$B(x(t)) > 0, \quad \text{for all} \quad t \geq 0, \quad \text{and} \quad x(0) \in \mathbb{X}_0. \quad (3)$$

It is assumed that $B(x) > 0$ for $x \in \mathbb{X}_0$. The control goal is to minimize goal function $Q(x)$ without crossing set $B^{-1}(0)$ [1]. In order to design the control algorithm the idea of speed-gradient algorithm for unconstraint case is used. To this end it is suggested to entroduce the penalty function for minimization without constraints

$$V(x, \alpha) = Q(x) + \frac{\alpha}{B(x)}, \quad (4)$$

where $\alpha > 0$ is the penalty parameter. The idea of $V(x, \alpha)$ is similar to penalty (barrier) functions in mathematical

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[1] $B^{-1}(0) = \{x \in \mathbb{X}_0 \colon B(x) = 0\}$

programming. If $V(x,\alpha)$ decreases along trajectories of the system (1) with some admissible $u(t)$, then according to the property

$$B(x_*) = 0 \Rightarrow \lim_{x\to x_*, B(x)>0} V(x,\alpha) = +\infty, \quad (5)$$

one can conclude that set $B^{-1}(0)$ will be never crossed. Under some additional assumptions (as for interio point method) the minimization of $V(x,\alpha)$ would ensure minimization of $Q(x)$.

According to the speed-gradient method the scalar function $w(x,u,\alpha)$ is introduced

$$w(x,u,\alpha) = \mathcal{L}_F V(x,\alpha) = \\ = \left(\frac{\partial Q}{\partial x} - \frac{\alpha}{B(x)^2}\frac{\partial B}{\partial x}\right) F(x,u), \quad (6)$$

where $\mathcal{L}_F$ means the derivative along the trajectories of the system (1) and $\frac{\partial Q}{\partial x}$, $\frac{\partial B}{\partial x}$ are the row-vectors of partial derivatives. Then the gradient of $w(x,u,\alpha)$ with respect to input variables is evaluated

$$\nabla_u w(x,u,\alpha) = \left(\frac{\partial w}{\partial u}\right)^T = \\ = \left(\frac{\partial F}{\partial u}\right)^T \left(\nabla_x Q(x) - \frac{\alpha}{B(x)^2}\nabla_x B(x)\right). \quad (7)$$

Finally, the algoritm of changing $u(t)$ is determined according to the equation

$$u(t) = u_0 - \Gamma \nabla_u w(x(t),u(t),\alpha), \quad (8)$$

where u_0 is some initial value of control variable (e.g. $u_0 = 0$), and $\Gamma = \Gamma^T > 0$ is a positive definite gain matrix. If $\alpha = 0$, then (8) transformes to the well-known speed-gradient algorithm in finite form without phase constrains. A more general form of (8) is

$$u(t) = u_0 - \gamma\psi(x(t),u(t),\alpha), \quad (9)$$

where $\gamma > 0$ is a scalar gain parameter and vector-function $\psi(x,u,\alpha)$ satisfies the so-called *pseudogradient* condition

$$\psi(x,u,\alpha)^T \nabla_u w(x,u,\alpha) \geq 0. \quad (10)$$

Special case of (9) is called *sign-like* or *relay-like* algorithm

$$u(t) = u_0 - \gamma sign \nabla_u w(x(t),u(t),\alpha), \quad (11)$$

where sign of a vector is understood component-wise. The solution of differential equation with discontinuous right hand sides is understood in Filippov sense Filippov [1988]. Of course, control algorithms (8), (9), (11) are defined only for area where $B(x) > 0$ (because all trajectories starting from $\mathbb{X}_0$ should belong to this area according to phase constraints).

2.1 Special case: energy control for Hamiltonian systems

Consider the Hamiltonian system

$$\frac{dp^k}{dt} = -\frac{\partial H(p,q,u)}{\partial q^k}, \quad \frac{dq^k}{dt} = \frac{\partial H(p,q,u)}{\partial p^k}, \\ k = 1,\ldots,n, \quad (q(0),p(0)) \in \mathbb{X}_0. \quad (12)$$

where $q = (q^1,\ldots,q^n)^T$, $p = (p^1,\ldots,p^n)^T$ are the vectors of generalized coordinates and momenta constituting the state vector (p,q) of the system, $H(p,q,u)$ is the controlled Hamiltonian function, $u \in \mathbb{R}^m$ is the input (generalized force), t is time ($t \in \mathbb{R}$), $\mathbb{X}_0$ is the set of possible initial states. Assume that Hamiltonian is affine in control

$$H(p,q) = H_0(p,q) + H_1(p,q)^T u, \quad (13)$$

where $H_0(p,q)$ is the internal Hamiltonian and $H_1(p,q)$ is an $m-$dimensional vector (column) of interaction potentials Nijmeijer and van der Schaft [1990]. Complex mechanical systems usually consists of several subsystems. An important practical problem is the selective excitation, when it is needed to increase the energy of one subsystem and to constraint the energy of the other: a passing through resonance Tomchin et al. [2005], selective molecule excitation Anan'evskii [2007], etc. Having in mind these applications split q, p: let q_1, p_1 be vectors of generalized coordinates and momenta of the first subsystem and q_2, p_2 be corresponded vectors of the second one. The following assumption is introduced

$$H_0(p,q) = H_0^1(p_1,q_1) + H_0^2(p_2,q_2) + H_0^{1,2}(p,q), \quad (14)$$

where $H_0^1(p_1,q_1)$ is the Hamiltonian of the first subsystem, $H_0^2(p_2,q_2)$ is the Hamiltonian of the second subsystem, and $H_0^{1,2}(p,q)$ is the Hamiltonian of interaction. Subsystems are called independent, if $H_0^{1,2}(p,q) \equiv 0$.

The selective excitation problem can be formalized as follows. The control goal is to stabilize energy of the first subsystem on the given goal value E_1

$$\lim_{t\to+\infty} H_0^1(p_1(t),q_1(t)) = E_1, \quad (15)$$

the phase constraint is to bound energy of the second subsystem with the given value E_2

$$H_0^2(p_2(t),q_2(t)) < E_2, \quad t \geq 0, \quad (16)$$

where $q_1(t)$, $p_1(t)$, $q_2(t)$, $p_2(t)$ are solutions of (12) with some admissible $u(t)$ and $(q(0),p(0)) \in \mathbb{X}_0$.

It is suggested to introduce the goal function and the function of constraints

$$Q(p_1,q_1) = \frac{1}{2}\left(H_0^1(p_1,q_1) - E_1\right)^2, \quad (17)$$

$$B(p_2,q_2) = E_2 - H_0^2(p_2,q_2). \quad (18)$$

Then the control goal (15) takes the form (2), and the phase constraint (16) takes the form (3).

According to (4) the penalty function is

$$V(p,q,\alpha) = Q(p_1,q_1) + \frac{\alpha}{B(p_2,q_2)} = \\ = \frac{\left(H_0^1(p_1,q_1) - E_1\right)^2}{2} + \frac{\alpha}{E_2 - H_0^2(p_2,q_2)}. \quad (19)$$

For selective excitation problem (15), (16) of Hamiltonian system (12) the control law (8) takes form

$$u(t) = u_0 - \Gamma\nabla_u \mathcal{L}_F V(p(t),q(t),\alpha), \quad (20)$$

where $\mathcal{L}_F$ is the derivative along trajectories of the system (12), u_0 is some initial value of control variable, and $\Gamma = \Gamma^T > 0$ is a positive definite gain matrix, $\alpha > 0$ is the parameter.

3. TWO PENDULUMS UNDER A SINGLE FORCE

To demonstrate the proposed algorithms an energy selective control problem for two pendulum systems is studed. Consider two independent nonlinear pendulums affected by a single force, schematically depicted in Fig. 1. The system equations are as follows

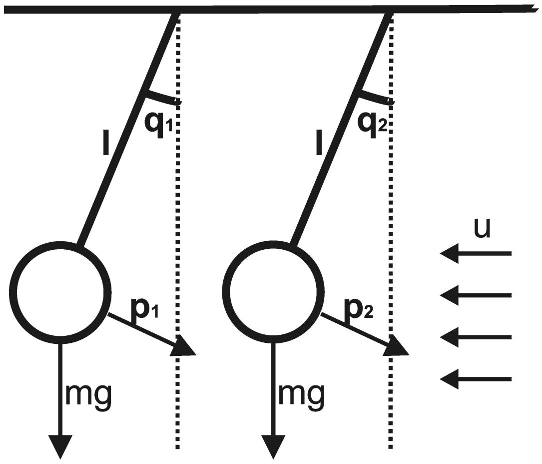

Fig. 1. Two independent nonlinear pendulums under a single control force u.

$$\begin{cases} \dot{q}_1 = (ml^2)^{-1} p_1, \\ \dot{p}_1 = -mgl \sin q_1 + ul \cos q_1, \end{cases}$$
$$\begin{cases} \dot{q}_2 = (ml^2)^{-1} p_2, \\ \dot{p}_2 = -mgl \sin q_2 + ul \cos q_2, \end{cases} \quad (21)$$

where dot means the derivative by t; q_1, p_1 are angular coordinate and momentum of the first pendulum, q_2, p_2 are angular coordinate and momentum of the second; g is the gravity acceleration. Pendulums have the same mass m, length l and controlling torque u acts horizontally.

Rewrite system (21) in the Hamiltonian form (12), (13), (14). The Hamiltonian function of each pendulum is

$$H_0^k(p_k, q_k) = \frac{1}{2ml^2} p_k^2 + mgl(1 - \cos q_k), \quad k = 1, 2. \quad (22)$$

Pendulums are independent, so

$$H_0^{1,2}(p, q) \equiv 0. \quad (23)$$

The Hamiltonian of interaction potential is

$$H_1(p, q) = -l(\sin q_1 + \sin q_2). \quad (24)$$

The system (21) is underactuated, nonlinear and uncontrollable, because if once $q_1(t)$ is equal to $q_2(t)$ and $p_1(t)$ is equal to $p_2(t)$ then they are equal for all t with any control function $u(t)$.

3.1 Control problem formulation

The problem is to design a controller to swing the first pendulum to the desired energy level E_1 and to constrain the energy of the second one by E_2 during always ($E_2 > 0$). The control objective is formalized by the relation (15). The phase constraint is formalized by the relation (16). The problem is to find a feedback control law $u(p, q)$ ensuring the control goal (15) and phase constraint (16).

Suppose that initial conditions of pendulums are different

$$(p_1(0), q_1(0)) \neq (p_2(0), q_2(0)), \quad (25)$$

and satisfy to the phase constraint (16)

$$H_0^2(p_2(0), q_2(0)) < E_2. \quad (26)$$

3.2 Control algorithm design

According to the approach presented in the previous section the algorithm is designed using the equation (20) with $u_0 = 0$

$$\begin{aligned} u(p, q) &= -\Gamma \nabla_u \mathcal{L}_F V(p, q, \alpha) = \\ &= -\Gamma \nabla_u \mathcal{L}_F \left[\frac{\left(H_0^1(p_1, q_1) - E_1\right)^2}{2} + \frac{\alpha}{E_2 - H_0^2(p_2, q_2)} \right] = \\ &= -\Gamma \left(\frac{p_1 \cos q_1}{ml} (H_0^1(p_1, q_1) - E_1) + \right. \\ &\quad \left. + \alpha \frac{p_2 \cos q_2}{ml} (H_0^2(p_2, q_2) - E_2)^{-2} \right), \quad (27) \end{aligned}$$

where $\Gamma > 0$, $\alpha > 0$ are design parameters.

3.3 Control algorithm analysis

Control algorithm analysis is based on Lyapunov function approach and LaSalle principle. Consider $V(p(t), q(t), \alpha)$ as Lyapunov function. The derivative of $V(p, q, \alpha)$ along the system trajectories (21) is non-positive

$$\frac{d}{dt} V(p(t), q(t), \alpha) = -\Gamma^{-1} u(p(t), q(t))^2 \leq 0. \quad (28)$$

Consequently, for any initial conditions from the area $\{(p, q) \colon H_0^2(q_2, p_2) < E_2\}$ the solution of the closed-loop system (21), (27) is unique and exists [2] for all $t \in [0, +\infty)$. From (5) and inequality (28) follows, that phase constraint (16) always holds.

According to La Salle principle (it is assumed that phase space of each pendulum is cylindrical) any solution tends to the largest invariant set of the closed-loop system (21), (27). For $(p(t), q(t))$ to be in the invariant set, $\dot{V} = -\Gamma^{-1} u^2 = 0$ which means that

$$\begin{aligned} &\frac{p_1(t) \cos q_1(t)}{ml} (H_0^1(p_1(t), q_1(t)) - E_1) + \\ &\quad + \alpha \frac{p_2(t) \cos q_2(t)}{ml} (H_0^2(p_2(t), q_2(t)) - E_2)^{-2} \equiv 0 \quad (29) \end{aligned}$$

must be satisfied with (21), (27). Since Hamiltonian is constant while $u = 0$, therefore the equation (29) can be rewritten

$$\frac{A}{ml^2} p_1(t) \cos q_1(t) + \frac{B}{ml^2} p_2(t) \cos q_2(t) \equiv 0, \quad (30)$$

where the value of A, B

$$A = -l(H_0^1(p_1(t), q_1(t)) - E_1), \quad (31)$$

$$B = -\alpha l(H_0^2(p_2(t), q_2(t)) - E_2)^{-2} \quad (32)$$

are constants. Indeed,

$$\frac{d}{dt} \sin q_k(t) = \frac{1}{ml^2} p_k(t) \cos q_k(t), \quad k = 1, 2, \quad (33)$$

consequenty

$$A \sin q_1(t) + B \sin q_2(t) \equiv const. \quad (34)$$

According to (21) for $u = 0$ it follows from (34) that

$$\begin{aligned} &A \sin q_1(t) + B \sin q_2(t) \equiv \\ &\quad \equiv -\frac{A}{mgl} \dot{p}_1(t) - \frac{B}{mgl} \dot{p}_2(t) \equiv const. \quad (35) \end{aligned}$$

[2] This note is important, because control function (27) is not defined when $H_0^2(p_2, q_2) = E_2$. But according to (19), (26), (28) the trajectories will never cross this set. The right hand of the equations (21) is smooth and bounded in the area $\{(p, q) \colon V(p, q, \alpha) \leq V(p(0), q(0), \alpha)\}$, so the solution exists and unique for any initial conditions from the area $\{(p, q) \colon H_0^2(p_2, q_2) < E_2\}$ and for all $t \in [0, +\infty)$ (but may be not for $t \in \mathbb{R}$).

It follows from (28) that $p(t)$ is a bounded function for $t \geq 0$. Consequently

$$A \sin q_1(t) + B \sin q_2(t) \equiv \\ \equiv -\frac{A}{mgl}\dot{p}_1(t) - \frac{B}{mgl}\dot{p}_2(t) \equiv 0. \quad (36)$$

From the properties of solutions of pendulum equations (21) without control ($u = 0$) it follows, that if $A \neq 0$, $B \neq 0$ then equation (36) can be true if $H_0^1(p_1(t), q_1(t)) \equiv H_0^2(p_2(t), q_2(t))$ or pendulums are in equilibrium states [3].

Assumed that control goal (15) is not fulfilled and $|A| \neq |B|$. Then the equation (36) is true only when pendulums are in equilibrium states. The inequality $|A| \neq |B|$ is always true for the area $H_0^2(p_2, q_2) < E_2$, when

$$\alpha > E_2^2 max\{E_1, E_2 - E_1\}. \quad (37)$$

Consequently, if inequality (37) is true, then the solution of the closed-loop system achieves the control goal (15) or converges to an equilibrium state of uncontrolled system (21). There are only four equiliblium states

$$(q_1, p_1, q_2, p_2) = \\ = \{(0,0,0,0), (0,0,\pi,0), (\pi,0,0,0), (\pi,0,\pi,0)\}. \quad (38)$$

Analysis of equilibrium states. Matrix of linear approximation of the system (21), (27) near the state $(\pi, 0, 0, 0)$ is

$$\begin{pmatrix} 0 & (ml^2)^{-1} & 0 & 0 \\ mgl & \Gamma A & 0 & -\Gamma B \\ 0 & 0 & 0 & (ml^2)^{-1} \\ 0 & -\Gamma A & -mgl & \Gamma B \end{pmatrix}. \quad (39)$$

Its characteristic polynomial is

$$l^2x^4 - l^2\Gamma(B + A)x^3 + gl\Gamma(B - A)x - g^2 = 0. \quad (40)$$

Obviously, this equation has a positive root for any real A, B and an unstable manifold of the point $(\pi, 0, 0, 0)$ is not trivial, according to the center manifold theorem, see Khalil [2002]). Therefore, initial conditions for the trajectories converging to this state should have zero projection onto the unstable manifold and the set of such initial conditions has zero Lebesgue measure.

Matrix of linear approximation of the system (21), (27) near the state $(0, 0, \pi, 0)$ is

$$\begin{pmatrix} 0 & (ml^2)^{-1} & 0 & 0 \\ -mgl & \Gamma A & 0 & -\Gamma B \\ 0 & 0 & 0 & (ml^2)^{-1} \\ 0 & -\Gamma A & mgl & \Gamma B \end{pmatrix}. \quad (41)$$

Its characteristic polynomial is

$$l^2x^4 - l^2\Gamma(B + A)x^3 - gl\Gamma(B - A)x - g^2 = 0. \quad (42)$$

Obviously, this equation has a positive root for any real A, B and an unstable manifold of the point $(\pi, 0, 0, 0)$ is not trivial, according to the center manifold theorem, see Khalil [2002]). Therefore, initial conditions for the trajectories converging to this state should have zero projection onto the unstable manifold and the set of such initial conditions has zero Lebesgue measure.

[3] The linear combination of periodic functions can be zero only if periods are the same. The linear combination of $\sin q_1(t)$ and $\sin q_2(t)$ can not be zero if one pendulum makes circle oscillations and the other pendulum doesn't.

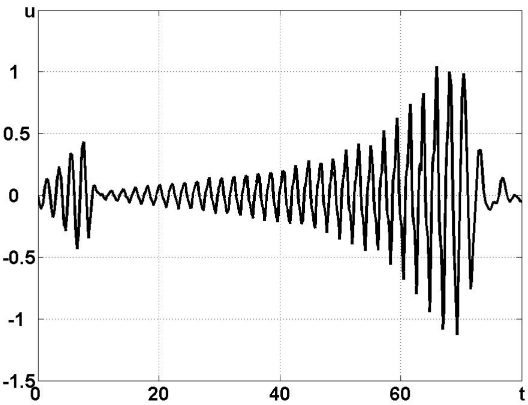

Fig. 2. Control function $u(q_1(t), p_1(t), q_2(t), p_2(t))$.

Matrix of linear approximation of the system (21), (27) near the state $(\pi, 0, \pi, 0)$ is

$$\begin{pmatrix} 0 & (ml^2)^{-1} & 0 & 0 \\ mgl & \Gamma A & 0 & \Gamma B \\ 0 & 0 & 0 & (ml^2)^{-1} \\ 0 & \Gamma A & mgl & \Gamma B \end{pmatrix}. \quad (43)$$

Its characteristic polynomial is

$$l^2x^4 - l^2\Gamma(B+A)x^3 - 2lgx^2 - gl\Gamma(B+A)x + g^2 = 0. \quad (44)$$

Obviously, this equation has a positive root for any real A, B and an unstable manifold of the point $(\pi, 0, 0, 0)$ is not trivial, according to the center manifold theorem, see Khalil [2002]). Therefore, initial conditions for the trajectories converging to this state should have zero projection onto the unstable manifold and the set of such initial conditions has zero Lebesgue measure.

Near the state $(0, 0, 0, 0)$ it is suggested to switch to another algorithm, which move the system from the area where $V(q_1, q_2, p_1, p_2) > V(0, 0, 0, 0)$ to the area where $V(q_1, q_2, p_1, p_2) < V(0, 0, 0, 0)$. For example a resonance control designed on one pendulum could be used. During simulations we never need such a switching.

Simulation. To demonstrate the ability of the controller to achieve the control goal and to fulfill the phase constraints we carried out computer simulation. The following value of system parameters and initial conditions were chosen: $m = 1$, $l = 1$, $g = 10$, $q_1 = 0$, $q_2 = 0.05$, $p_1 = 0$, $p_2 = 0$. Energy goal value for the first pendulum was taken $E_1 = 20$, energy constraint for the second one was taken $E_2 = 5$. Algorithm parameters were: $\Gamma = 0.015$, $\alpha = 10$. Time for simulating was 80 seconds. As predicted by control algorithm analysis the energy of the first pendulum converged to the goal value E_1, and energy of the second was constrainted by E_2. The simulating results are presented in Fig. 2, 3.

4. CONCLUSION

In the paper a method for control of mechanical systems under phase constraints, applicable to energy control of Hamiltonian systems is proposed. The constrained energy control problem for two pendulums by a single control action is studied both analytically and numerically. It is shown that for a proper choice of penalty parameter

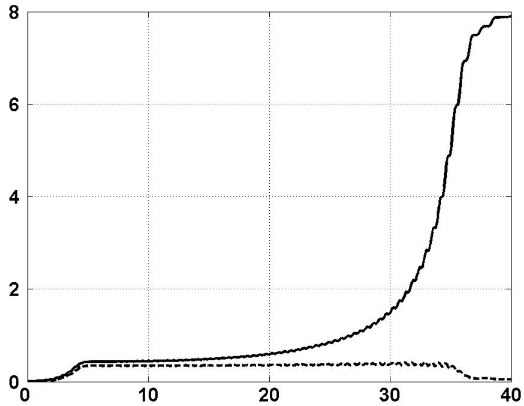

Fig. 3. Energy of pendulums. Solid line corresponds to the energy of the first pendulum $H_0^1(q_1(t), p_1(t))$, dash line — energy of the second one $H_0^2(q_2(t), p_2(t))$.

of the algorithm any energy level for the one pendulum under specified constraint on the energy of the other pendulum can be achieved. Simulation results confirm fast convergence rate of the algorithm.

Future research is aimed on application of the method to constrained energy control of Huijgens' pendula system Pogromsky et al. [2006].

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