Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Necessary and Sufficient Conditions for Delay-Dependent Asymptotic Stability of Linear Discrete Time Delay Autonomous Systems

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Abstract: This paper offers new, necessary and sufficient conditions for delay-dependent asymptotic stability of systems of the form $x(k+1) = A_0 x(k) + A_1 x(k-h)$. The time-dependent criteria are derived by Lyapunov's direct method. Two matrix equations have been derived: matrix polynomial equation and discrete Lyapunov matrix equation. Also, modifications of the existing sufficient conditions of convergence of Traub and Bernoulli algorithms for computing the dominant solvent of the matrix polynomial equation are derived. Numerical computations are performed to illustrate the results obtained.

1. INTRODUCTION

Stability problem of linear systems with time delays has been investigated by many researches. It is obvious that there are much more published papers in the area of continuous than discrete time delay systems. Certainly, one of the basic reasons for that lies in the fact that discrete time delay systems are of finite dimensions so, the equivalent systems of considerably high order can be easily built (Mori *et al.,* 1982, Trinh and Alden, 1997, Boutayeb and Darouach, 2001and Gorecki *et al*., 1989). The majority of stability conditions in the literature available, of both continual and discrete time delay systems, are sufficient conditions independent of time delay. Only a small number of works provide both necessary and sufficient conditions (Lee and Diant, 1981, Xu *et al.* 2001 and Boutayeb and Darouach, 2001), which are in their nature mainly dependent of time delay.

Basic inspiration for our investigation is based on paper (Lee and Diant, 1981), however, the stability of discrete time delay systems is considered herein. In this paper, we first propose modification of the existing sufficient condition for nonsingularity of block Vandermonde matrix $V(S_1,\ldots,S_{h+1})$. This condition has weaker hypothesis than similar condition from (Dennis et al., 1976) and represents the generalization of results presented in (Kim, 2000).

It has been then demonstrated that condition of nonsingularity of block Vandermonde matrix $V(S_2,\ldots,S_{h+1})$ is the direct outcome of nonsingularity of block matrix $V(S_1,\ldots,S_{h+1})$. Likewise, we have arrived at a new sufficient condition for the convergence of Traub and Bernoulli algorithms. This condition has weaker hypothesis than similar condition in (Dennis et al., 1978).

At the end, we propose new necessary and sufficient conditions for delay dependent stability of discrete linear time delay system, which as distinguished from the criterion based on eigenvalues of the equivalent system matrix (Gantmacher, 1960) use matrices of considerably lower dimension.

2. NOTATION AND PRELIMINARIES

$\mathbb{R}$	Real vector space
$\mathbb{T}^+$	All the non-negative integers
$\mathbb{C}$	Complex vector space
λ^*	Conjugate of $\lambda \in \mathbb{C}$
F^*	Conjugate transpose of matrix $F \in \mathbb{C}^{n\times n}$
$F > 0$	Positive definite matrix
$\det(F)$	Determinant of matrix F
$\lambda_i(F)$	Eigenvalue of matrix F
$\lambda(F)$	$\{\lambda \mid \det(F-\lambda I)=0\}$
$\sigma(F)$	Spectrum of matrix F
$\rho(F)$	Spectral radius of matrix F

A linear, discrete time-delay system can be represented by the difference equation

$$x(k+1) = A_0 x(k) + A_1 x(k-h) \tag{1}$$

with an associated function of initial state

$$x(\theta) = \psi(\theta),\ \theta \in \{-h, -h+1, \ldots, 0\} \tag{2}$$

The equation (1) is referred to as homogenous or the unforced state equation. Vector $x(k) \in \mathbb{R}^n$ is a state vector and $A_0, A_1 \in \mathbb{R}^{n\times n}$ are constant matrices of appropriate dimensions, and pure system time delay is expressed by integers $h \in \mathbb{T}^+$. System (1) can be expressed with the following representation without delay, (Mori *et al.,* 1982, Malek-Zavarei and Jamshidi, 1978, and Gorecki *et al*., 1989).

$$x_{eq}(k)=\left[x^T(k-h)\,x^T(k-h+1)\,\cdots\,x^T(k)\right]\in\mathbb{R}^N$$
$$x_{eq}(k+1)=A_{eq}x_{eq}(k),\ N\hat{=}n(h+1) \tag{3}$$
$$A_{eq}=\begin{bmatrix}0 & I_n & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & I_n\\ A_1 & 0 & \cdots & A_0\end{bmatrix}\in\mathbb{R}^{N\times N}$$

The system defined by (3) is called the equivalent system, while matrix A_{eq}, the matrix of equivalent system. Characteristic polynomial of system (1) is given with:

$$f(\lambda)\hat{=}\det M(\lambda)=\sum_{j=0}^{n(h+1)}a_j\lambda^j,\ a_j\in\mathbb{R},\quad M(\lambda)=I_n\lambda^{h+1}-A_0\lambda^h-A_1 \tag{4}$$

Denote with

$$\Omega\hat{=}\{\lambda\,|\,f(\lambda)=0\}=\lambda(A_{eq}) \tag{5}$$

the set of all characteristic roots of system (1). The number of these roots amounts to $n(h+1)$. A root λ_m of Ω with maximal module:

$$\lambda_m\in\Omega:\ |\lambda_m|=\max\left|\lambda(A_{eq})\right| \tag{6}$$

let us call maximal root (eigenvalue). If scalar variable λ in the characteristic polynomial is replaced by matrix $X\in\mathbb{C}^{n\times n}$ the two following monic matrix polynomials are obtained

$$M(X)=X^{h+1}-A_0X^h-A_1 \tag{7}$$

$$F(X)=X^{h+1}-X^hA_0-A_1 \tag{8}$$

It is obvious that $F(\lambda)=M(\lambda)$. For matrix polynomial $M(X)$, the matrix of equivalent system A_{eq} represents *block companion matrix* (Dennis *et al*., 1976).

A matrix $S\in\mathbb{C}^{n\times n}$ is a *right solvent* of $M(X)$ (Dennis *et al*., 1976) if

$$M(S)=0 \tag{9}$$

If

$$F(R)=0 \tag{10}$$

then $R\in\mathbb{C}^{n\times n}$ is a *left solvent* of $M(X)$ (Dennis *et al*., 1976).

We will further use S to denote right solvent and R to denote left solvent of $M(X)$.

In the present paper the majority of presented results start from left solvents of $M(X)$. In contrast, in the existing literature right solvents of $M(X)$ were mainly studied. The mentioned discrepancy can be overcome by the following lemma.

***Lemma* 1**. Conjugate transpose value of left solvent of $M(X)$ is also, at the same time, right solvent of the following matrix polynomial

$$M_T(X)=X^{h+1}-A_0^TX^h-A_1^T \tag{11}$$

Proof. Let R be right solvent of $M(X)$. Then it holds

$$\begin{aligned}M_T(R^*)&=(R^*)^{h+1}-A_0^T(R^*)^h-A_1^T\\&=(R^{h+1}-R^hA_0-A_1)^*=F^*(R)=0\end{aligned} \tag{12}$$

so R^* is right solvent of $M_T(X)$. Q.E.D

***Conclusion* 1**. Based on *Lemma* 1, all characteristics of left solvents of $M(X)$ can be obtained by the analysis of conjugate transpose value of right solvents of $M_T(X)$.

The following proposed factorization of the matrix $M(\lambda)$ will help us to better understand the relationship between eigenvalues of left and right solvents and roots of the system.

***Lemma* 2**. The matrix $M(\lambda)$ can be factorized in the following way

$$\begin{aligned}M(\lambda)&=\left(\lambda^hI_n+(S-A_0)\sum_{i=1}^{h}\lambda^{h-i}S^{i-1}\right)(\lambda I_n-S)\\&=(\lambda I_n-R)\left(\lambda^hI_n+\sum_{i=1}^{h}\lambda^{h-i}R^{i-1}(R-A_0)\right)\end{aligned} \tag{13}$$

Proof.

$$\begin{aligned}M(\lambda)-M(X)&=\lambda^{h+1}I_n-X^{h+1}-A_0(\lambda^hI_n-X^h)\\&=\left(\sum_{i=0}^{h}\lambda^{h-i}X^i-A_0\sum_{i=0}^{h-1}\lambda^{h-1-i}X^i\right)(\lambda I_n-X)\end{aligned} \tag{14}$$

If S is a right solvent of $M(X)$, from (14) follows (13). Similarly, if R is a left solvent of $M(X)$, from

$$\begin{aligned}&M(\lambda)-F(X)\\&=(\lambda I_n-X)\left(\lambda^hI_n+\sum_{i=1}^{h}\lambda^{h-i}X^{i-1}(X-A_0)\right)\end{aligned} \tag{15}$$

follows (13). Q.E.D

***Conclusion* 2**. From (4) and (13) follows $f(S)=f(R)=0$, e.g. the characteristic polynomial $f(\lambda)$ is *annihilating polynomial* for right and left solvents of $M(X)$. Therefore, $\lambda(S)\subset\Omega$ and $\lambda(R)\subset\Omega$ hold.

Eigenvalues and eigenvectors of the matrix have a crucial influence on the existence, enumeration and characterization of solvents of the matrix equation (9) (Dennis *et al*., 1976 and Pereira, 2003).

***Definition* 1**. (Dennis *et al*., 1976 and Pereira, 2003) Let $M(\lambda)$ be a matrix polynomial in λ. If $\lambda_i \in \mathbb{C}$ is such that $\det(M(\lambda_i)) = 0$, then we say that λ_i is a *latent root* or an *eigenvalue* of $M(\lambda)$. If a nonzero $v_i \in \mathbb{C}^n$ is such that

$$M(\lambda_i) v_i = 0 \tag{16}$$

then we say that v_i is a (right) *latent vector* or a (right) *eigenvector* of $M(\lambda)$, corresponding to the eigenvalue λ_i.

Eigenvalues of matrix $M(\lambda)$ correspond to the characteristic roots of the system, i.e. eigenvalues of its block companion matrix A_{eq} (Dennis *et al*., 1976). Their number is $n(h+1)$. Since $F^*(\lambda) = M_T(\lambda^*)$ holds, it is not difficult to show that matrices $M(\lambda)$ and $M_T(\lambda)$ have the same spectrum.

In papers (Dennis *et al*., 1976, 1978, Kim, 2000, Pereira, 2003 and Lancaster and Tismenetsky, 1985) some sufficient conditions for the existence, enumeration and characterization of right solvents of $M(X)$ were derived. They show that the number of solvents can be *zero*, *finite* or *infinite*.

For the needs of system stability (1) only the so called maximal solvents are usable, whose spectrums contain maximal eigenvalue λ_m. A special case of maximal solvent is the so called dominant solvent (Dennis *et al*., 1978 and Kim, 2000), which, unlike maximal solvents, can be computed in a simple way.

***Definition* 2**. Every solvent S_m of $M(X)$, whose spectrum $\sigma(S_m)$ contains maximal eigenvalue λ_m of Ω is a *maximal solvent*.

***Definition* 3**. (Dennis *et al*., 1978 and Kim, 2000) Matrix A dominates matrix B if all the eigenvalues of A are greater, in modulus, then those of B. In particular, if the solvent S_1 of $M(X)$ dominates the solvents $S_2,\ldots,S_l$ we say it is a *dominant solvent*. (Note that a dominant solvent cannot be singular.)

***Conclusion* 3**. The number of maximal solvents can be greater than one. Dominant solvent is at the same time maximal solvent too.

The dominant solvent S_1 of $M(X)$, under certain conditions, can be determined by the *Traub* (Dennis *et al*., 1978) and *Bernoulli iteration* (Dennis *et al*., 1978 and Kim, 2000).

3. MAIN RESULTS

We will further provide improvements for some existing sufficient conditions related to nonsingularity of block Vandermonde matrix and existence of dominant solvent.

The following lemma gives sufficient condition for the regularity of block Vandermonde matrix and has weaker hypothesis than *Theorem* 6.1 in (Dennis *et al*., 1976). This lemma represents the generalization of the corresponding result presented in (Kim, 2000).

***Lemma* 3**. If $S_1,\ldots,S_{h+1}$ are solvents of $M(X)$ with $\sigma(S_1)\cap\ldots\cap\sigma(S_{h+1}) = \varnothing$ then $V(S_1,\ldots,S_{h+1})$ is nonsingular.

Proof. It is derived by the generalization of proof given in paper (Kim, 2000), for the case $h=1$. Q.E.D.

It is demonstrated by the following lemma that condition of nonsingularity of matrix $V(S_2,\ldots,S_{h+1})$ in (Dennis *et al*., 1978) is superfluous, since it results directly from nonsingularity of matrix $V(S_1,\ldots,S_{h+1})$.

***Lemma* 4**. If the block Vandermonde matrix $V(S_1,\ldots,S_{h+1})$ is nonsingular, then $V(S_2,\ldots,S_{h+1})$ is also nonsingular.

Proof. If the block Vandermonde matrix $V(S_2,\ldots,S_{h+1})$ is nonsingular, then

$$\det\begin{bmatrix} I & I & \cdots & I \\ S_1 & S_2 & \cdots & S_{h+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_1^h & S_2^h & \cdots & S_{h+1}^h \end{bmatrix} = (-1)^{nh}\det V(S_2,\ldots,S_{h+1})\cdot$$
$$\cdot\det\left\{ S_1^h - \begin{bmatrix} S_2^h & \cdots & S_{h+1}^h \end{bmatrix} V^{-1}(S_2,\ldots,S_{h+1}) \begin{bmatrix} I \\ \vdots \\ S_1^h \end{bmatrix} \right\} \tag{17}$$

From $\det V(S_1,\ldots,S_{h+1}) \neq 0$, follows $V(S_2,\ldots,S_{h+1}) \neq 0$, so $V(S_2,\ldots,S_{h+1})$ is nonsingular, when $V(S_1,\ldots,S_{h+1})$ is regular. Q.E.D.

By combining Lemmas 3-4 one can modify some existing conditions for convergence of Traub and Bernoulli algorithms presented in (Dennis *et al*., 1978). These conditions have weaker hypothesis than conditions given in (Dennis *et al*., 1978).

***Lemma* 5**. If $M(X)$ is a matrix polynomial of degree $(h+1)$ such that

(i) it has solvents $S_1,\ldots,S_{h+1}$

(ii) S_1 is a dominant solvent

(iii) $\sigma(S_1)\cap\ldots\cap\sigma(S_{h+1}) = \varnothing$

then Traub and Bernoulli algorithms (Dennis *et al*., 1978) converge.

Proof. The first two conditions of this lemma are identical with conditions (*i*)-(*ii*) of *Theorems* 2.1 and 3.2 in (Dennis *et al*., 1978). From *Lemmas* 3-4 follows that $V(S_1,\ldots,S_{h+1})$ and $V(S_2,\ldots,S_{h+1})$ are nonsingular, whereby the third condition has been fulfilled too of *Theorems* 2.1 and 3.2 in (Dennis *et al*., 1978). So, Traub and Bernoulli algorithms converge to a dominant solvent. Q.E.D.

Similar to the definition of right solvents S_m and S_1 of $M(X)$, the definitions of both maximal left solvent, R_m, and dominant left solvent, R_1, of $M(X)$ can be provided. These left solvents of $M(X)$ are used in a number of theorems to follow. Owing to *Lemma* 1, they can be determined by proper right solvents of $M_T(X)$. Generally, all aforementioned about the existence, enumeration and characterization of right solvents of $M(X)$, holds also for right solvents of $M_T(X)$, therefore for left solvents of $M(X)$ too.

Necessary and sufficient conditions for asymptotic stability of linear discrete time-delay systems (1) are to follow.

***Theorem* 1**. Suppose that there exists at least one left solvent of $M(X)$ and let R_m denote one of them. Then, linear discrete time delay system (1) is *asymptotically stable* if and only if for any matrix $Q = Q^* > 0$ there exists *Hermitian* matrix $P = P^* > 0$ such that

$$R_m^* P R_m - P = -Q \tag{18}$$

Proof. Define the following vector discrete functions

$$x_k = x(k+\theta), \theta \in \{-h, -h+1, \ldots, 0\} \tag{19}$$

$$z(x_k) = x(k) + \sum_{j=1}^{h} T(j)x(k-j) \tag{20}$$

where, $T(k) \in \mathbb{C}^{n \times n}$ is, in general, some time varying discrete matrix function. The conclusion of the theorem follows immediately by defining Lyapunov functional for the system (1) as

$$V(x_k) = z^*(x_k) P z(x_k), \; P = P^* > 0 \tag{21}$$

It is obvious that $z(x_k) = 0$ if and only if $x_k = 0$, so it follows that $V(x_k) > 0$ for $\forall x_k \neq 0$. The forward difference of (21), along the solutions of system (1) is

$$\begin{aligned} \Delta V(x_k) &= \Delta z^*(x_k) P z(k) \\ &\quad + z^*(x_k) P \Delta z(x_k) + \Delta z^*(x_k) P \Delta z(x_k) \end{aligned} \tag{22}$$

A difference of $\Delta z(x_k)$ can be determined in the following manner

$$\Delta z(x_k) = \Delta x(k) + \sum_{j=1}^{h} T(j) \Delta x(k-j) \tag{23}$$

with

$$\Delta x(k) = (A_0 - I_n)x(k) + A_1 x(k-h) \tag{24}$$

and

$$\begin{aligned} \sum_{j=1}^{h} T(j)\Delta x(k-j) &= T(1)\left[x(k) - x(k-1)\right] + \cdots \\ &\quad + T(h)\left[x(k-h+1) - x(k-h)\right] \end{aligned} \tag{25}$$

Then simple manipulations lead to

$$\begin{aligned} \sum_{j=1}^{h} T(j)\Delta x(k-j) &= T(1)x(k) - T(h)x(k-h) \\ &\quad + \big(T(2) - T(1)\big)x(k-1) + \cdots \\ &\quad + \big(T(h) - T(h-1)\big)x(k-h+1) \end{aligned} \tag{26}$$

Define a new matrix R by

$$R = A_0 + T(1) \tag{27}$$

If

$$\Delta T(h) = A_1 - T(h) \tag{28}$$

then $\Delta z(x_k)$ has a form

$$\Delta z(x_k) = (R - I_n)x(k) + \sum_{j=1}^{h} \{\Delta T(j) \cdot x(k-j)\} \tag{29}$$

If one adopts

$$\Delta T(j) = (R - I_n)T(j), \quad j = 1, 2, \ldots, h \tag{30}$$

then $\Delta z(x_k)$ becomes

$$\Delta z(x_k) = (R - I_n) z(x_k) \tag{31}$$

Therefore, (22) becomes

$$\Delta V(x_k) = z^*(x_k)\big(R^* P R - P\big) z(x_k) \tag{32}$$

It is obvious that if the following equation is satisfied

$$R^* P R - P = -Q, \quad Q = Q^* > 0 \tag{33}$$

then $\Delta V(x_k) < 0, x_k \neq 0$. In the Lyapunov matrix equation (33), of all possible solvents R of $M(X)$, only one of maximal solvents is of importance, for it is the only one that contains maximal eigenvalue $\lambda_m \in \Omega$ (*Conclusion* 2), which has dominant influence on the stability of the system. So, (18) represent stability *sufficient condition* for system given by (1). Matrix $T(1)$ can be determined in the following way. From (30) follows

$$T(h+1) = R^h T(1) \tag{34}$$

and using (27)-(28) one can get (10), and for the sake of brevity, instead of matrix $T(1)$, one introduces simple notation T.

If solvent which is not maximal is integrated into Lyapunov equation, it may happen that there will exist positive definite solution of Lyapunov matrix equation (18), although the system is not stable (see *Example* 4). Conversely, if the system (1) is asymptotically stable then all roots $\lambda_i \in \Omega$ are located within unit circle. Since $\sigma(R_m) \subset \Omega$, follows $\rho(R_m) < 1$, so the positive definite solution of Lyapunov matrix equation (18) exists (*necessary condition*). Q.E.D.

***Corollary* 2**. Suppose that there exists at least one maximal left solvent of $M(X)$ and let R_m denote one of them. Then, system (1) is asymptotically stable if and only if $\rho(R_m)<1$.

Proof. Follows directly from *Theorem* 1. Q.E.D.

***Conclusion* 4**. Corollary 2 may be proved in the following way. From Conclusion 2 follows $\sigma(R)\subset\Omega=\lambda(A_{eq})$ and based on properties of maximal solvent R_m follows $\rho(R_m)=\rho(A_{eq})$. So, if the maximal solvent is discrete stable then A_{eq} will be also discrete stable matrix and vice versa.

***Corollary* 3**. Suppose that there exists dominant left solvent R_1 of $M(X)$. Then, system (1) is asymptotically stable if and only if $\rho(R_1)<1$.

Proof. Follows directly from *Corollary* 2, since *dominant solution* is, at the same time, *maximal solvent*. Q.E.D.

***Conclusion* 5**. In the case when dominant solvent R_1 may be deduced by Traub or Bernoulli algorithm, *Corollary* 3 represents a quite simple method. If aforementioned algorithms are not convergent but still there exists at least one of maximal solvents R_m, then one should use *Corollary* 2. The maximal solvents may be found, for example, using the concept of eigenpair (Pereira, 2003). If there exists no maximal solvent R_m, then proposed necessary and sufficient conditions *can not be used* for system stability investigation.

***Conclusion* 6**. In great time delay of the system it holds

$$\dim(R_1)=\dim(R_m)=\dim(A_i)=n\ll\dim(A_{eq})=n(h+1)$$

For example, if time delay amounts to $h=100$, and the row of matrices of the system is $n=2$, then: $R_1, R_m\in\mathbb{C}^{2\times 2}$ and $A_{eq}\in\mathbb{C}^{202\times 202}$. To check the stability by eigenvalues of matrix A_{eq}, it is necessary to determine 202 eigenvalues, which is not numerically simple. On the other hand, if dominant solvent can be computed by Traub or Bernoulli algorithm, *Corollary* 3 requires a relatively small number of additions, subtractions, multiplications and inversions of the matrix format of only 2×2.

So, in the case of great time delay in the system, by applying *Corollary* 3, a smaller number of computations are to be expected compared with a traditional procedure of examining the stability by eigenvalues of companion matrix A_{eq}.

An accurate number of computations for each of the mentioned method requires additional analysis, which is not the subject-matter of our considerations herein.

4. NUMERICAL EXAMPLES

***Example* 1**. Let us consider linear discrete system with delayed state (1) with

$$A_0=\begin{bmatrix}0.1 & 0.3\\ 0.1 & -0.15\end{bmatrix},\ A_1=\begin{bmatrix}0.3 & 0.4\\ 0.2 & 0.25\end{bmatrix},\ h=1$$

and let us check stability properties of the system under consideration, based on the application of *Theorem* 1, *Corollaries* 2 and 3.

Application of Theorem 1. By the left solvents S_i of $M_T(X)$, applying the concept of eigenpair (Pereira, 2003), left solvents R_i of $M(X)$ are calculated:

$$R_1=S_1^*=\begin{bmatrix}3.548 & 4.759\\ -2.408 & -3.391\end{bmatrix},\ R_2=S_2^*=\begin{bmatrix}-1.812 & 2.490\\ -1.171 & 1.604\end{bmatrix},$$
$$R_3=S_3^*=\begin{bmatrix}0.453 & 0.576\\ 0.342 & 0.326\end{bmatrix},\ R_4=S_4^*=\begin{bmatrix}0.402 & 0.620\\ 0.388 & 0.287\end{bmatrix},$$
$$R_5=S_5^*=\begin{bmatrix}-0.345 & -0.502\\ -0.191 & -0.394\end{bmatrix},\ R_6=S_6^*=\begin{bmatrix}-0.386 & -0.417\\ -0.167 & -0.443\end{bmatrix}$$

The solvents R_1, R_3 and R_4 are maximal solvents, since they contain eigenvalue $\lambda_m=0.838\in\Omega$. From solved Lyapunov equation (18), for example, $R_m=R_1$ and $Q=I_2$, we can conclude that system under consideration is asymptotically stable.

Application of Corollary 2. By adopting, for example, $R_m=R_3$ as a maximal solvent, we conclude that inequality $\rho(R_m)=0.838<1$ is satisfied, therefore the observed system is asymptotically stable.

Application of Corollary 3. If for a set of $h+1=2$ solvents, we choose R_1 and R_2, the conclusion is that R_1 is a dominant solvent, whereby the condition has been fulfilled $\det(V(R_1,R_2))\neq 0$. Therefore, the *Traub* or *Bernoulli* algorithm can be used for the determination of dominant solvent. By *Traub* algorithm (Dennis *et al*., 1978), after only three iterations upon G_i and three iterations upon X_i (3+3), identical value, as above calculated, was obtained for dominant solvent R_1. Similarly, by applying *Bernoulli* algorithm (Dennis *et al*., 1978), after 12 iterations upon X_i, identical value, as above calculated, was obtained for dominant solvent R_1. Since $\rho(R_1)=0.838<1$, based on *Corollary* 3, it follows that the system under consideration is *asymptotically stable*.

***Example* 2**. Let us consider linear discrete systems with delayed state (1) with

$$A_0=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix},\ A_1=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},\ h=1.$$

and let us check stability properties of the system under consideration.

Application of *Corollary* 2. The left solvents R_i of $M(X)$ are

$$R_1=\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix},\ R_2=\begin{bmatrix}-1 & -2\\ 0 & 0\end{bmatrix},\ R_3=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Since $\lambda(R_1)=\{-1,1\}$, $\lambda(R_2)=\{-1,0\}$ and $\lambda(R_3)=\{1,0\}$ *there exists no dominant solvent*, but all the three solvents are maximal solvents. Because $\rho(R_i)=1$, $1 \le i \le 3$, based on *Corollary* 2, the system is not asymptotically stable.

***Example* 3**. Let us consider linear discrete systems with delayed state (1) with

$$A_0 = \begin{bmatrix} 7/10 & 1/2 \\ 1/2 & 17/10 \end{bmatrix},\ A_1 = \begin{bmatrix} -1/75 & -1/3 \\ 1/3 & 49/75 \end{bmatrix}$$

There are two left solvents of matrix polynomial equation (10)

$$R_1 = \begin{bmatrix} 19/30 & 1/6 \\ -1/6 & 29/30 \end{bmatrix},\ R_2 = \begin{bmatrix} 1/15 & 1/3 \\ -1/3 & 11/15 \end{bmatrix}$$

Since $\lambda(R_1)=\left\{\frac{4}{5},\frac{4}{5}\right\}$, $\lambda(R_2)=\left\{\frac{2}{5},\frac{2}{5}\right\}$, dominant solvent is R_1. As we have $V(R_1,R_2)$ nonsingular, *Traub* or *Bernoulli* algorithm may be used.

Application of Corollary 3. Only after $(4+3)$ iterations for *Traub* and 17 iterations for *Bernoulli* algorithm, dominant solvent can be found with accuracy of 10^{-4}. Since $\rho(R_1)=\frac{4}{5}<1$, based on *Corollary* 3, it follows that the system under consideration is *asymptotically stable*.

***Example* 4**. Let us consider linear discrete systems with delayed state (1) with

$$A_0 = \begin{bmatrix} 17/6 & -11/6 \\ 1/3 & 2/3 \end{bmatrix},\ A_1 = \begin{bmatrix} -5/3 & 17/12 \\ -2/3 & 5/12 \end{bmatrix},\ h=1\,.$$

The eigenvalues of matrices $M(X)$ are given with $\{0.5, 0.5, 0.5, 2\}=\Omega$. There is *only one* solvent of matrix polynomial equation (10):

$$R = \begin{bmatrix} 12/7 & 1/7 \\ -4/7 & 16/7 \end{bmatrix}$$

with $\lambda(R)=\{0.5, 0.5\}$. It can be seen that there exist no dominant and maximal solvents of (10), so the proposed stability conditions *can not be applied*. If we, disregarding the assumption on the existence of maximal solvent R_m, apply Corollary 2, based on $\rho(R)=0.5<1$, we would arrive at the wrong conclusion that the system is asymptotically stable. But, the system is unstable since it possesses characteristic root $\lambda_m = 2 > 1$.

5. CONCLUSION

In this paper, we have established new, necessary and sufficient, conditions for the asymptotic stability of a particular class of linear discrete time delay systems. The time-dependent criteria are derived by Lyapunov's direct method and are exclusively based on the maximal and dominant solvents of particular matrix polynomial equation. It has been demonstrated that with great time delay of the system, if dominant solvent can be computed by Traub or Bernoulli algorithm, a decrease in the number of computations is to be expected in favour of derived stability criteria compared with the existing ones.

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