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Constructive Invariant Manifolds to Stabilize Pendulum–like systems Via Immersion and Invariance

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Abstract: The Immersion and Invariance control technique (I&I), is a method to design asymptotically stabilizing control laws for nonlinear systems, proposed in Astolfi and Ortega [2003]. The three design steps of I&I are: the definition of a target dynamics; the construction of an invariant manifold; and the design of a control law. The second step requires the solution of a partial differential equation (PDE) that may be difficult to obtain. Here we show a constructive procedure to obviate the solution of the PDE, through the well–known cart and pendulum system. The procedure follows interlacing the first and second steps and invoking physical considerations.

Keywords: Nonlinear systems; stabilization; pendular systems; immersion and invariance.

1. INTRODUCTION TO IMMERSION AND INVARIANCE

The method of I&I for stabilization of nonlinear systems originated in Astolfi and Ortega [2003] and was further developed in a series of publications that have been recently summarized in Astolfi et al. [2007], see also Karagiannis et al. [2004]. The major result of Astolfi and Ortega [2003], that constitutes the basis of the present note, is the following theorem.

Theorem 1. Consider the system [1]

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$, with an equilibrium point $x_* \in \mathbb{R}^n$ to be stabilized. Let $p < n$ and assume we can find mappings

$$\alpha(\cdot) : \mathbb{R}^p \to \mathbb{R}^p, \quad \pi(\cdot) : \mathbb{R}^p \to \mathbb{R}^n, \quad c(\cdot) : \mathbb{R}^p \to \mathbb{R}^m,$$
$$\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-p}, \quad \psi(\cdot,\cdot) : \mathbb{R}^{n\times(n-p)} \to \mathbb{R}^m,$$

such that the following hold.

(H1) *(Target system)* The system

$$\dot{\xi} = \alpha(\xi), \tag{2}$$

with state $\xi \in \mathbb{R}^p$, has an asymptotically stable equilibrium at $\xi_* \in \mathbb{R}^p$ and $x_* = \pi(\xi_*)$.

(H2) *(Immersion condition)* For all $\xi \in \mathbb{R}^p$

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi). \tag{3}$$

(H3) *(Implicit manifold)* The set identity

$$\{x \in \mathbb{R}^n \mid \phi(x) = 0\} = \{x \in \mathbb{R}^n \mid x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^p\}. \tag{4}$$

holds.

(H4) *(Manifold attractivity and trajectory boundedness)* All trajectories of the system

$$\dot{z} = \frac{\partial \phi}{\partial x}\left[f(x) + g(x)\psi(x,z)\right] \tag{5}$$
$$\dot{x} = f(x) + g(x)\psi(x,z) \tag{6}$$

are bounded and satisfy

$$\lim_{t\to\infty} z(t) = 0. \tag{7}$$

Then x_* is an asymptotically stable equilibrium of the closed loop system

$$\dot{x} = f(x) + g(x)\psi(x,\phi(x))$$

Theorem 1 lends itself to the following interpretation. Given the system (1) and the target dynamical system (2) find, if possible, a manifold $\mathcal{M}$, described implicitly by $\{x \in \mathbb{R}^n \mid \phi(x) = 0\}$, and in parameterized form by $\{x \in \mathbb{R}^n \mid x = \pi(\xi), \xi \in \mathbb{R}^p\}$, which can be rendered invariant and attractive, and such that the restriction of the closed loop system to $\mathcal{M}$ is described by $\dot{\xi} = \alpha(\xi)$.

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[1] Throughout the paper it is assumed that all functions and mappings are C^∞.

Notice, however, that we do not propose to apply the control $u = c(\pi(\xi))$ that renders the manifold invariant, instead we design a control law $u = \psi(x, z)$ that drives to zero the coordinate z and keeps the system trajectories bounded. Notice from (5) that z, called off–the–manifold coordinate, is a measure of the distance of the system trajectories to the manifold $\mathcal{M}$.

In standard applications of I&I the target system is a priori defined, hence condition (H1) is automatically satisfied. Given the target system, the equation (3) of condition (H2) defines a PDE in the unknown π, where c is a free parameter. Note that, if the linearization of (1) (at $x = x_*$) is controllable (and all functions are locally analytic), it has been shown in Kravaris and Kravaris [2000], using Lyapunov Auxiliary theorem and under some *non-resonance* conditions, that we can always find c such that the solution exists locally. Nevertheless, finding the explicit analytic solution of this equation is—in general—a difficult task.

The main objective of this note is to propose a procedure to obviate the solution of the PDE. Towards this end, we propose to interlace the steps of definition of the target dynamics (H1) and generation of the manifold (H2). More specifically, we propose to leave α as a free parameter and to view the PDE (3) as an algebraic equation relating α with π (and its partial derivatives). We then select suitable expressions for π that ensures the desired stability of the target dynamics. We illustrate this idea with the classical cart and pendulum system for which we propose to select the target dynamics as a simple pendulum whose potential energy and dissipation functions are viewed as functions of π. We then select suitable expressions for π that ensure that the potential energy has a minimum at the upward position of the pendulum and the damping function is non–negative around this point. The design is completed selecting a control law that ensures condition (H4) of Theorem 1, which in this example turns out to be a trivial task.

2. UPWARD STABILIZATION OF THE CART AND PENDULUM SYSTEM

2.1 Model

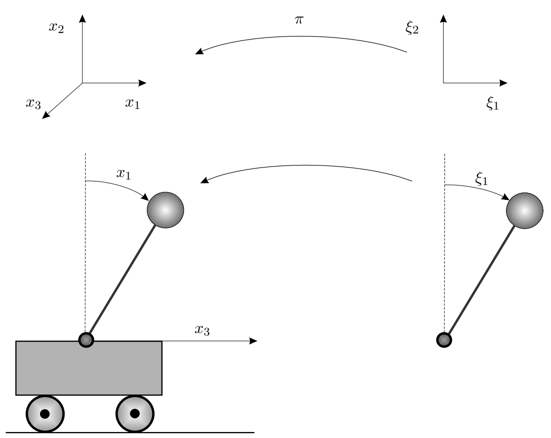

Fig. 1. Pendulum on a cart and target dynamics.

We consider the classical cart–pendulum system depicted in Fig. 1, and assume that a partial feedback linearization stage has been applied [2]. After normalization this yields

$$\Sigma : \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = a \sin x_1 - u \, b \cos x_1, \\ \dot{x}_3 = u, \end{cases} \tag{8}$$

where $(x_1, x_2) \in \mathcal{S}^1 \times \mathbb{R}$ are the pendulum angle with respect to the upright vertical and its velocity, respectively, and $x_3 \in \mathbb{R}$ is the *velocity* of the cart, $u \in \mathbb{R}$ is the input, and $a > 0$ and $b > 0$ are physical parameters. The equilibrium to be stabilized is the upward position of the pendulum with the cart stopped, which corresponds to $x_\star = 0$.

2.2 Controller design

We proceed to verify the hypothesis H1–H4 of Theorem 1.

(H1) *(Target system)* The key idea is to immerse a two dimensional system—which describes a pendulum dynamics whose potential energy and damping functions are left to be designed (see Fig. 1 (right))—into a three dimensional one. Thus, we define the target dynamics as

$$\Sigma_T : \begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = -V'(\xi_1) - R(\xi_1, \xi_2)\xi_2, \end{cases} \tag{9}$$

which are the dynamical equations of a single pendulum with energy function $H(\xi_1, \xi_2) = \frac{1}{2}\xi_2^2 + V(\xi_1)$ and, possibly nonlinear, damping function R—that, for generality, we have defined as a function of ξ_1 and ξ_2.

To ensure that the target dynamics have an asymptotically stable equilibrium at the origin we introduce the following assumption.

Assumption A.1

(i) The potential energy function $V(\xi_1)$, satisfies $V'(0) = 0$ and $V''(0) > 0$.
(ii) The damping function is such that $R(0, 0) > 0$.

(H2) *(Immersion condition)* Given the control objectives and our choice of target dynamics a natural selection of the mapping π is

$$\pi(\xi) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \pi_3(\xi_1, \xi_2) \end{bmatrix}, \tag{10}$$

where π_3 is a function to be defined. With this choice of π and the target dynamics above (3) reduces to

$$a \sin \xi_1 - b \cos \xi_1 c(\pi(\xi)) = -V'(\xi_1) - R(\xi_1, \xi_2)\xi_2 \tag{11}$$

$$c(\pi(\xi)) = \frac{\partial \pi_3}{\partial \xi_1}\xi_2 - \frac{\partial \pi_3}{\partial \xi_2}[V'(\xi_1) + R(\xi_1, \xi_2)\xi_2], \tag{12}$$

where we recall that c is the controller that renders the manifold invariant. Replacing c from (12) in (11), and after doing some rearrangements, yields the PDE to be solved, namely

$$b \cos \xi_1 \left(\xi_2 \frac{\partial \pi_3}{\partial \xi_1} - [V'(\xi_1) + R(\xi_1, \xi_2)\xi_2] \frac{\partial \pi_3}{\partial \xi_2} \right) = a \sin \xi_1 - V'(\xi_1) - R(\xi_1, \xi_2)\xi_2. \tag{13}$$

In the standard application of I&I, V' and R would be fixed and then we would need to solve the PDE (13) for

[2] See Acosta et al. [2005], Teel [1996] for further details.

the unknown π_3. Here, we let V' and R free—viewed as functions of π_3 and its derivatives—and introduce two conditions on π_3 so that (13) can be trivially solved. Towards this end, we find convenient to rewrite (13) in the form

$$\left(b\cos\xi_1 \frac{\partial \pi_3}{\partial \xi_1} - R(\xi_1,\xi_2)\Delta(\xi)\right)\xi_2 = a\sin\xi_1 + \Delta(\xi)V'(\xi_1), \tag{14}$$

where we have defined the "key" function Δ as

$$\Delta(\xi) \triangleq 1 + \frac{\partial \pi_3}{\partial \xi_2} b\cos\xi_1. \tag{15}$$

We will see below that this function also plays a fundamental role on the stabilization step, (H4), of the I&I procedure. Consider now the following assumptions

Assumption A.2 There exists an $\epsilon > 0$ such that

$$|\Delta(0)| = \left|1 + b\frac{\partial \pi_3}{\partial \xi_2}(0)\right| \geq \epsilon > 0.$$

Assumption A.3 $\frac{\partial \pi_3}{\partial \xi_2}$ is a function of ξ_1 only, and consequently Δ does not depend on ξ_2.

If Assumptions A.2 and A.3 hold, the PDE (14) is solved selecting [3]

$$V'(\xi_1) = -\frac{a\sin\xi_1}{\Delta(\xi)}, \qquad R(\xi_1,\xi_2) = \frac{b\cos\xi_1}{\Delta(\xi)}\frac{\partial \pi_3}{\partial \xi_1}. \tag{16}$$

The equations above provide a parametrization of V and R in term of the (free) manifold function π_3. We can proceed at this stage with the selection of functions π_3 such that Assumptions A.1–A.3 hold, but let us first investigate the remaining conditions of Theorem 1.

(H3) *(Implicit manifold)* It is easy to verify that The manifold $\mathcal{M}$ can be implicitly described by $\mathcal{M} = \{x \in \mathbb{R}^3 \mid \phi(x) = 0\}$, with

$$\phi(x) = x_3 - \pi_3(x_1,x_2).$$

(H4) *(Manifold attractivity and trajectory boundedness)* The off–the–manifold coordinates are $z = \phi(x)$ and straightforward calculations show that

$$\begin{aligned} \dot z &= \dot x_3 - \dot\pi_3(x_1,x_2) \\ &= \psi(x,z) - \frac{\partial \pi_3}{\partial x_1}x_2 - \frac{\partial \pi_3}{\partial x_2}\left(a\sin x_1 - b\cos x_1 \psi(x,z)\right) \\ &= -\frac{\partial \pi_3}{\partial x_1}x_2 - \frac{\partial \pi_3}{\partial x_2}a\sin x_1 + \Delta(x_1,x_2)\,\psi(x,z), \end{aligned} \tag{17}$$

where we recall that $\psi(x,\phi(x))$ is the actual controller that we apply. From the last equation we note that, under Assumption A.2, the task of driving z to zero is trivialized. Indeed, dividing by Δ we can assign arbitrarily the off–the–manifold dynamics, for instance, fix it to $\dot z = -\gamma z$, with γ a positive constant. This yields

$$\psi(x,z) = \frac{1}{\Delta(x_1,x_2)}\left(-\gamma z + \frac{\partial \pi_3}{\partial x_1}x_2 + \frac{\partial \pi_3}{\partial x_2}a\sin x_1\right),$$

that, upon evaluation on the manifold, defines the controller

$$\psi(x,\phi(x)) = \frac{1}{\Delta(x_1,x_2)}\Big(-\gamma(x_3 - \pi_3(x_1,x_2)) + \frac{\partial \pi_3}{\partial x_1}x_2 + \frac{\partial \pi_3}{\partial x_2}a\sin x_1\Big). \tag{18}$$

To complete our design it remains to propose functions π_3 that verify Assumptions A.2 and A.3 and such that Assumption A.1 with V and R as in (16) holds. In Table 1 we provide three possibilities, which are naturally suggested by Assumptions A.1–A.3 [4]. For the sake of comparison we also give the expressions of Δ and the potential energy V.

To enforce a particular behavior to the target dynamics, we can also proceed dually, that is, fix the desired potential energy V and then work backwards to compute π_3, Δ and R. A particularly interesting choice is $V(x_1) = \frac{k_1}{2}\tan^2 x_1$, with $k_1 > 0$, which has a unique minimum at zero and is radially unbounded on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Replacing in (16) yields

$$\Delta(x_1) = -\frac{a}{k_1}\cos^3 x_1,$$

which clearly satisfies Assumption A.2. From (15), and after some simple calculations we obtain

$$\pi_3(x_1,x_2) = -\frac{1}{b}\left(\frac{1}{\cos x_1} + \frac{a}{k_1}\cos^2 x_1\right)x_2 + \Phi(x_1), \tag{19}$$

where Φ is a free function. As it can be easily shown $R(0,0) = -\frac{\beta b}{a}\Phi'(0)$, hence $\Phi'(0) < 0$ to ensure the damping is positive — e.g., $\Phi(x_1) = -k_2 x_1$, with $k_2 > 0$.

2.3 Stability result

Following Theorem 1 the stability analysis is completed proving that there exists a set of initial conditions $(x(0), z(0))$ such that the corresponding trajectories $x(t)$ of (6) are bounded.

Proposition 1. For any function π_3 verifying Assumptions A.2 and A.3, and such that Assumption A.1 holds for the functions V and R given in (15) and (16), the zero equilibrium of the cart–pendulum system (8) in closed loop with the I&I controller (18) with $\gamma > 0$, is locally asymptotically stable.

Proof. We prove that, for some suitable set of initial conditions, the trajectories of the system (6), which in our example has the form,

$$\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= a\sin x_1 - b\cos x_1\ \psi(x,z) \\ \dot x_3 &= \psi(x,z), \end{aligned} \tag{20}$$

with ψ given by (18), are bounded. Towards this end, define a set

$$D := \left\{x \in \mathcal{S}^1 \times \mathbb{R}^2 \mid |1 + b\cos(x_1)\frac{\partial \pi_3}{\partial x_2}(x_1)| > 0\right\}.$$

From (15) and (16) we see that there exist $\epsilon_1, \epsilon_2 > 0$ such that, for all $x \in D$, we have

$$|\Delta(x_1)| \geq \epsilon_1, \qquad R(x_1,x_2) \geq \epsilon_2. \tag{21}$$

Now, we use the functions V' and R, defined in (16), to rewrite the first two equations of (20) in the form

[3] Assumption A.3 is needed to ensure that V' is a function of ξ_1 only. Remark that if $\frac{\partial \pi_3}{\partial \xi_1}$ is a independent of ξ_2 then also R depends only on ξ_1—but this is not necessary for stability of the target dynamics.

[4] To guide on the selection we note that Assumptions A.1–A.3 impose that $\frac{\partial \pi_3}{\partial x_1}(0) < 0$ and $\frac{\partial \pi_3}{\partial x_2}(0) < 0$.

$\pi_3(x_1,x_2)$	$\Delta(x_1)$	$V(\xi_1)$
$-k_1x_1 - k_2x_2$	$1-k_2b\cos x_1$	$-a(\ln\|1-k_2b\cos\xi_1\| - \ln\|1-k_2b\|)/(k_2b)$
$-k_1x_1 - k_2x_2\cos x_1$	$1-k_2b\cos^2 x_1$	$a(-\tanh^{-1}(k_2b/\sqrt{k_2b}) + \tanh^{-1}(k_2b\cos\xi_1/\sqrt{k_2b}))/\sqrt{k_2b}$
$-k_1x_1 - k_2\frac{x_2}{\cos x_1}$	$1-k_2b$	$-a(-1+\cos\xi_1)/(k_2b-1)$

Table 1. Proposed candidates for π_3 and corresponding Δ and V, where k_1 and k_2 are positive gains.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -V'(x_1) - R(x_1,x_2)x_2 + \frac{\gamma b\cos x_1}{\Delta(x_1)}z(t),\end{aligned}\qquad (22)$$

which, in view of Assumption A.1, is a system with an asymptotically stable equilibrium perturbed by an additive term containing an exponentially decaying function $z(t) = z(0)\exp^{-\gamma t}$ (recall that $\dot{z} = -\gamma z$). Consider the function $H(x_1,x_2) = \frac{1}{2}x_2^2 + V(x_1)$, which is positive definite in $D - \{0\}$ and whose derivative along the trajectories of (22), for all $x \in D$, satisfies

$$\begin{aligned}\dot{H} &= -R(x_1,x_2)x_2^2 + \frac{\gamma b}{\Delta(x_1)}x_2\cos x_1 z(0)\exp^{-\gamma t}\\ &\le -\epsilon_2 x_2^2 + \frac{\gamma b}{\epsilon_1}|x_2||z(0)|\exp^{-\gamma t}\\ &\le -\frac{\epsilon_2}{2}x_2^2 + \frac{\gamma^2b^2}{2\epsilon_1^2\epsilon_2}z^2(0)\exp^{-2\gamma t}\end{aligned}$$

where the first inequality is obtained using (21) and the second follows from Young's inequality, i.e. $2de \le kd^2 + \frac{1}{k}e^2$, selecting $k = \epsilon_2$ and $d = x_2$. From the last inequality above we conclude that, there exists a time t_f such that $\dot{H} \le -\frac{\epsilon_2}{4}x_2^2$, for all $t \ge t_f$. Hence, there exists a ball around zero, strictly contained in D, such that all trajectories starting in this set satisfy $H(x_1(t),x_2(t)) \le H(x_1(0),x_2(0))$—ensuring boundedness of (x_1,x_2). Finally, boundedness of x_3 follows from the fact that $x_3 = z + \pi_3(x_1,x_2)$ and both terms are bounded. $\triangleleft$

3. SIMULATIONS

Extensive simulations have been carried out for the four selections of π_3 (the three on Table 1 and the one on equation (19)) described above with different values of k_1, k_2 and γ. The largest domain of attraction among these controllers was achieved by the one calculated from the third line of Table 1. This gives the control law

$$u = -\frac{1}{1-k_2b}\Big(\gamma k_1 + k_1x_2 + \gamma x_3 + \frac{\gamma k_2}{\cos x_1}x_2 + k_2\tan x_1\big(\frac{x_2^2}{\cos x_1} + a\sin x_1\big)\Big),$$

with $k_1 > 0, k_2 > \frac{1}{b}$ and $\gamma > 0$. Notice that V has an isolated global minimum at zero and Δ is a constant. The controller is not globally defined because π_3 has a singularity at $\frac{\pi}{2}$. Simulations for this controller have bee carried out with the normalized values $a = b = 1$, the controller gains $k_1 = 3, k_2 = 4$ and $\gamma = 1, 10$ and the initial conditions $x(0) = (\frac{\pi}{2} - 0.1, 0, 0)$—that is, we start with zero velocities and with the pendulum *practically horizontal*. The results are shown in Fig. 2, which clearly exhibit the desired closed–loop behavior: first, convergence towards the manifold, i.e., $z(t) \to 0$ at a speed determined by γ, and then, once close to the manifold where the cart–pendulum system behaves like a simple pendulum, convergence towards the equilibrium. We note from Fig. 2 (left) that, increasing the speed of convergence to the manifold does not necessarily leads to a faster overall transient response. This is due to the fact that, even though the closed–loop system (22) is the cascade connection of an exponentially stable and an asymptotically stable system, the peaking phenomenon appears when we increase the rate of convergence of the former—in this simple example this is revealed by the presence of the multiplying coefficient γ in (22) (or in the bound of $\dot{H}$).

We should underscore the simplicity of the control laws that should be contrasted with other schemes proposed in the literature, e.g., Bloch et al. [2000], Acosta et al. [2005]. Also, we would like to bring to the readers attention the excellent transient performance depicted in Fig. 2 (right), and in particular the nice shape of the control action, which is a smooth low amplitude signal *that moves the cart at the right time instants in the right direction.* Again, this should be compared with other controllers, e.g., those stabilizing the homoclinic orbit, where the control action is essentially bang–bang—even when the initial conditions of the pendulum are in the upper half plane.

4. CONCLUDING REMARKS

The main stumbling block for application of the I&I methodology of Astolfi and Ortega [2003] is the need to solve the PDE of the immersion condition—i.e., the computation of the function π that defines the manifold $\mathcal{M}$.[5] To overcome this problem we have proposed in this paper to transform this PDE into an algebraic equation where the target dynamics is viewed as a function of π (and its partial derivatives), and then propose functions π that ensure the target dynamics has the desired stability property. The procedure has shown to be easily applicable for the upward stabilization of the cart and pendulum system, where physical considerations can be invoked to select the target dynamics. We are currently investigating the application of this construction to general underactuated mechanical systems and, in particular, to pendular systems.

There is a nice interpretation, in terms of passivity–based control, of the construction proposed in this paper. In I&I a stabilising control law is derived starting from the selection of a target (asymptotically stable) dynamical system. As explained in Astolfi and Ortega [2003] a different perspective can be taken: given the mapping $x = \pi(\xi)$, hence the mapping $z = \phi(x)$, find a control

[5] Solving PDEs is the stumbling block of *all* constructive procedures to stabilize nonlinear systems including forwarding, backstepping, feedback linearization, output regulation, energy–shaping, etc.

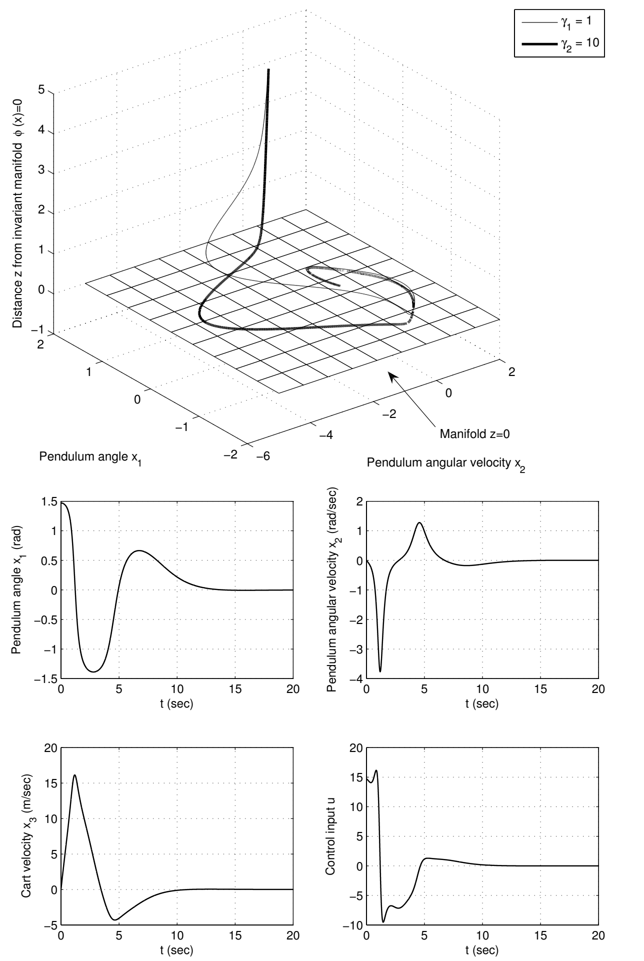

Fig. 2. Top: Attractivity of the invariant manifold. Botom: Transient performance for $\gamma = 1, k_1 = 3,\ k_2 = 4$.

law which renders the manifold $z = 0$ invariant and asymptotically stable, and an asymptotically stable vector field $\dot{\xi} = \alpha(\xi)$ such that equation (3) holds. If this goal is achieved then the system (1) with output $z = \phi(x)$ is minimum-phase and its zero dynamics are given by (2). In this respect, the result in Theorem 1 can be regarded as a *dual* of the classical stabilisation methods based on the construction of passive or minimum-phase outputs. In the cart and pendulum example we consider the output $x_3 - \pi_3(x_1, x_2)$. Assumption A.2 ensures that its relative degree is 1. Then, the selection (16) imposes as zero dynamics precisely the target system (2). Finally, Assumption A.1 guarantees that the zero dynamics are stable. We have in this way verified the conditions for feedback equivalence to a passive system of Byrnes et al. [1991] and passivity–based techniques can be applied for stabilization.

Many controller designs that have been reported for the cart and pendulum system proceed in two steps, first swing the cart to lift up the pendulum, and then regulate the cart position. An imprecisely formulated "two–time scale behavior" is often invoked to justify this widespread practice. Although we do not address in this note the problem of swinging up the pendulum (see below) and restrict ourselves to the stabilization on the upper half plane. Indeed, I&I ensures that all trajectories of the cart and pendulum system are asymptotically mapped into the trajectories of a single pendulum via $x = \pi(\xi)$—or in other words, it generates the manifold, $\mathcal{M}$, such that the restriction of the cart–pendulum system to $\mathcal{M}$ is a simple pendulum with suitable potential energy and dissipation functions.

We wrap–up the paper indicating a fundamental obstacle that hampers the application of the proposed construction to swing–up the pendulum. Indeed, from (16) we have that

$$V''(\xi_1) = -\frac{a}{\Delta^2(\xi_1)}[\Delta(\xi_1)\cos\xi_1 - \Delta'(\xi_1)\sin\xi_1],$$

which yields $V''(0) = -\frac{a}{\Delta(0)}$. To satisfy Assumption A.1, we must then have $\Delta(0) < 0$. On the other hand, if $\frac{\partial \pi_3}{\partial \xi_2}$ is finite, from (15) we have that $\Delta(\frac{\pi}{2}) = 1$, which means that Δ has to cross to zero in the interval $[0, \frac{\pi}{2}]$, inducing a singularity in the control law (18). This situation can be avoided, as done in the third option of Table 1, making $\frac{\partial \pi_3}{\partial \xi_2}(\frac{\pi}{2}) = \infty$, but this transfers the singularity to π_3.

In Acosta et al. [2008] we extend the result to a broader class of systems. Current research is under way to extend, further, this "constructive" result to different classes of well–known systems.

REFERENCES

J.Á. Acosta, R. Ortega, A. Astolfi and A.D. Mahindrakar. Interconnection and damping assignment passivity-based control of mechanical systems with underactuation degree one. *IEEE Trans. Automat. Contr.*, Vol. 50, pp. 1936-1955, December 2005.

J.Á. Acosta, R. Ortega, A. Astolfi and I. Sarras. A Constructive Solution for Stabilization Via Immersion and Invariance: The Cart and Pendulum System. *Automatica*, Vol. 44:9, 2008. (To appear)

A. Astolfi, D. Karagiannis and R. Ortega, **Nonlinear and adaptive control design and applications**, Springer-Verlag.

A. Astolfi and R. Ortega. Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems. *IEEE Trans. Automat. Contr.*, Vol. 48, pp. 590-606, April 2003.

A. Bloch, N. Leonard and J. Marsden. Controlled lagrangians and the stabilization of mechanical systems I: The first matching theorem. *IEEE Trans. Automat. Contr.*, Vol. 45, pp. 2253-2270, No. 12, December 2000.

C. I. Byrnes, A. Isidori, and J. C. Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Automat. Contr.*, 36:1228–1240, November 1991.

D. Karagiannis, R. Ortega and A. Astolfi. Nonlinear adaptive stabilization via system immersion: Control design and applications, Chapter in *Advanced topics in control systems theory: Lectures notes from FAP 2004*. LNCIS Vol. 311, eds. F. Lamnabhi–Lagarrigue, A. Loria and E. Panteley, Springer–Verlag, London, 2005, pp. 1–22.

N. Kravaris and C. Kravaris. Singular PDEs and the single-step formulation of feedback linearization with pole placement. *Systems and Control Letters*, Vol. 39, pp. 115-122, 2000.

A.R. Teel. A nonlinear small gain theorem for analysis of systems with saturation. *IEEE Trans. Automat. Contr.*, Vol. 41, No. 9, pp. 1256-1270, Sept. 1996.