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A Unified Analysis of Switching Multiple Model Adaptive Control - Virtual Equivalent System Approach

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Abstract: Based on virtual equivalent system concept and methodology, a unified analysis of the stability and convergence of switching multiple model adaptive control (MMAC) of linear time-invariant discrete plant is presented. The main results are expressed by three criteria. Two of them are applicable to switching MMAC systems with arbitrary control strategy, arbitrary parameter estimation algorithm and arbitrary switching index function. The third one is applicable to switching MMAC systems with one-step-ahead control strategy, arbitrary parameter estimation algorithm and arbitrary switching index function. We wanted to show that virtual equivalent system concept and methodology could be a general theory of switching multiple model adaptive control.

1. INTRODUCTION

Up to now, many switching multiple model adaptive control (MMAC) algorithms have been put forward. Generally speaking, there are mainly two types of switching MMAC: indirect switching (B. Martensson, 1986; D. E. Miller, 1994; D. E. Miller and E. J. Davison, 1989; K. Poolla and S. J. Cusumano, 1988; M. Fu and B. R. Barmish, 1986) and direct switching (A.S. Morse, 1993; 1992; K.S. Narendra and S. Mukhopadhyay, 1994; R. H. Middleton, et *al*., 19886; S. R. Weller and G. C. Goodwin, 1994). Indirect switching control can also be viewed as supervisory control because a supervisory function is used to decide when and which controller should be switched. Narendra, K. S. and Autenrieth, T. have used this method to improve the transient response of adaptive control systems (K.S. Narendra and J. Balakrishnan, 1997; 1994; T. Autenrieth and E. Rogers, 1999). As for direct switching control, the choice of when to switch to the next controller in a predetermined sequence is based directly on the output of the system. Since mid 1980's, papers about switching MMAC have covered continuous time system, discrete time system (K. S. Narendra and C. Xiang, 2000; Xiaoli Li, *et al.*, 2001), nonlinear system (Lingji Chen, K.S. Narendra, 2001), stochastic system (K.S. Narendra, and Osvaldo Driollet, 2001), etc, and there are also successful practical applications in this field.

Despite the fundamental progress achieved so far, there is still no a unified theory on adaptive control (conventional adaptive control and multiple model adaptive control); Here we list some remarks to support the viewpoint.

"In spite of 40 years of research, several books and hundreds of articles we still lack, in our view, a universally accepted design methodology for adaptive control which is based on sound theoretical issues and suitable for engineering implementations in real-life control systems." (Sajjad Fekri, Michael Athans and Antonio Pascoal, 2006)

"A good theory should give also good clues to the construction of new algorithms. …Unfortunately, there is no collection of results that can be called a theory of adaptive control in the sense specified." (K. J. Astrom and B. Wittenmark, 1995)

"Despite a significant number of practical applications and significant supporting theory, we are still a long way from having a full understanding of this important class of control strategies." (G. C. Goodwin, 2000)

"Despite the vast literature on the subject, there is still a general feeling that adaptive control is a collection of unrelated technical tools and tricks." (Ioannou P, Sun J., 1996)

With the help of virtual equivalent system concept (Weicun Zhang, Tianguang Chu, Long Wang, 2005; Weicun Zhang and Jin Young Choi, 2007a, b), we have developed three criteria to judge the stability and convergence of different switching MMAC algorithms. To a certain extent, these criteria are independent of specific control law and parameter estimation algorithm, and can thus provide a unified theoretical framework for understanding and evaluating different kinds of switching MMAC schemes.

The rest of the paper is organized as follows. Section 2 gives the description of switching MMAC. Section 3 introduces the virtual equivalent system of a general switching MMAC system. The main results are developed in Section 4. Some conclusions are drawn in Section 5.

2. DESCRIPTION OF SWITCHING MMAC

The basic architectures of switching MMAC system is shown in Fig. 1 (Sajjad Fekri, Michael Athans and Antonio Pascoal, 2006), which is concerned with continuous time plant in state space. Generally speaking, there are three components of a switching MMAC system: model set, controller set and switching logic or mechanism. With model set $\mathbb{M} = \{M_i, i = 1, ..., N\}$, we want to cover the uncertainty of plant P to be controlled. There are one or two estimated models in $\mathbb{M}$. According to each M_i, C_i is designed to satisfy some performance index. $\mathbb{C} = \{C_i, i = 1, ..., N\}$ is the controller set. There is an adaptive controller in $\mathbb{C}$ according to the estimated model. Switching mechanism is used to decide when and which controller should be switched.

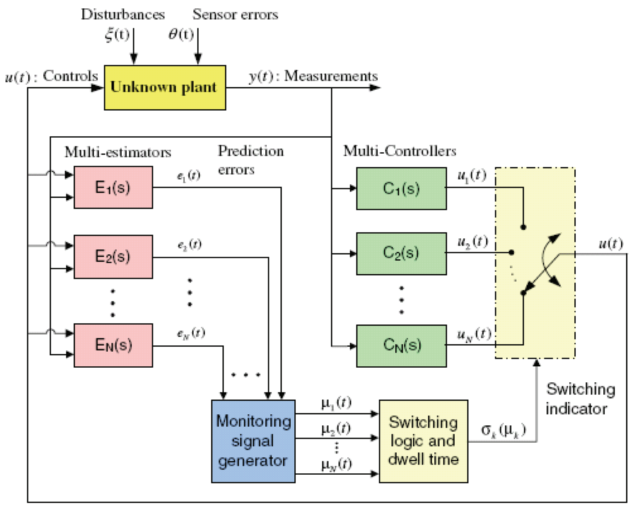

Fig. 1 Switching MMAC cited from Ref. (Sajjad Fekri, etc)

From analysis point of view, we use a very simple block diagram to represent a switching MMAC system, see Fig. 2. The essential characteristic of switching MMAC is that the controller is time-varying. And the details of controller switching is minor for the analysis of stability and convergence.

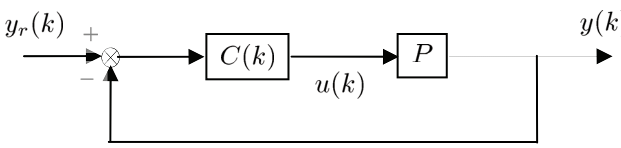

Fig. 2 Simplified Block Diagram of Switching MMAC

In Fig. 2, $y_r(k) < \infty$ is the reference input of the closed-loop control system. $C(k)$ denotes the time-varying controller of switching MMAC. P is the plant to be controlled, which takes the form

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) \quad (1a)$$

$$A(q^{-1}) = 1 + a_1q^{-1} + ... + a_nq^{-n} \quad (1b)$$

$$B(q^{-1}) = b_0 + b_1q^{-1} + ... + b_mq^{-m} \quad (1c)$$

3. VIRTUAL EQUIVALENT SYSTEM

In this section we give three kinds of virtual equivalent systems of switching MMAC according to the situations of parameter estimates.

3.1 Parameter estimates converge to the true values of plant

If parameter estimates converge to its true values, i.e. the parameters of P, and the switching mechanism switches to and stop at the adaptive controller finally, the time-varying controller $C(k)$ in Fig. 2 will converge to a certain time-invariant controller $C = f(P)$, if only the mapping

$$f: M \to \mathbb{C} \quad (2)$$

is continuous. Then we can construct a virtual equivalent system of switching MMAC in the input-output equivalence sense; see Fig. 3, where $\Delta u(k)$ is a complementary signal and it will play a very important role in the analysis.

$$\begin{aligned} \Delta u(k) &= u(k) - u_0(k) \\ &= \phi_c^T(k)\theta_c(k) - \phi_c^T(k)\theta_c \end{aligned} \quad (3)$$

$\phi_c^T(k)$ is the regression vector of control signal, generally speaking we have

$$\phi_c^T(k) = [y(k), y(k-1), ..., u(k-1), ..., y_r(k), ...] \quad (4)$$

The number of elements of $\phi_c^T(k)$ is limited.

$\theta_c(k)$ and θ_c are the parameter vectors of time-varying controller $C(k)$ and time-invariant controller C respectively. Then we have

$$\Delta u(k) = o(\|\phi_c(k)\|) \quad (5)$$

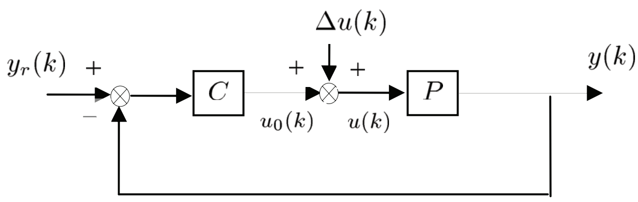

Fig. 3 Equivalent System I

3.2 Parameter estimates converge to untrue values

If parameter estimates converge to untrue values, denoted by P_0 (vector form θ_0), and the switching mechanism switch to adaptive controller finally, the time-varying controller $C(k)$ (vector form $\theta_c(k)$) in Fig. 2 will converge to a certain time-invariant controller $C_0 = f(P_0)$ (vector form θ_{c0}), if only the mapping $f(.)$ is continuous. Then we can construct a virtual equivalent system of switching MMAC in the input-output sense, see Fig. 4, where

$$\begin{aligned} e(k) &= y(k) - \phi^T(k-d)\theta_0 \\ &= y(k) - \phi^T(k-d)\hat{\theta}(k) \\ &+ \phi^T(k-d)\hat{\theta}(k) - \phi^T(k-d)\theta_0 \end{aligned} \quad (6)$$

$$\Delta u(k) = u(k) - u_0(k) = \phi_c^T(k)\theta_c(k) - \phi_c^T(k)\theta_{c0} \quad (7)$$

$\phi^T(k-d)$ is the regression vector of parameter estimation, $\hat{\theta}(k)$ is the estimated parameter vector, and the corresponding transfer function of $\hat{\theta}(k)$ is $P_m(k)$. As parameter estimates converge, i.e. $\hat{\theta}(k) \to \theta_0$, we have $\theta_c(k) \to \theta_{c0}$, that means

$$\Delta u(k) = o(\|\phi_c(k)\|) \tag{8}$$

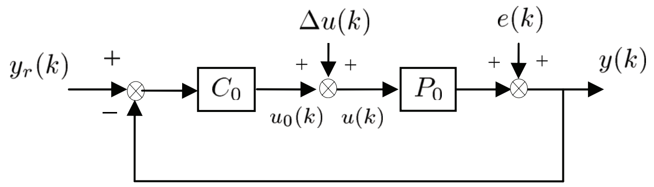

Fig. 4 Equivalent System II

3.3 Parameter estimates may not converge

In this situation, we have to limit the adaptive controller in switching MMAC as designed by one-step-ahead strategy. Otherwise we cannot have the property of $\Delta u(k)$ as in (5) and (8), which is critical to virtual equivalent system method. We may still use Fig. 3. as the virtual equivalent system of this situation.

4. MAIN RESULTS

With the help of the virtual equivalent systems of switching MMAC, we have the following theorems for different situations of parameter estimation.

4.1 Parameter estimates converge to its true values

Theorem 1

If a switching MMAC system has the following properties:

1) The parameter estimates of adaptive model converge to

the true values of plant;

2) After limited switching, the MMAC stops at adaptive controller;

3) The mapping from estimated parameters into controller parameters is continuous;

4) The controller is well defined such that $< C, P >$ constitutes a stable closed-loop system.

Then the switching MMAC system is stable and convergent.

Proof.

Conditions 1), 2), 3) guarantee that the virtual equivalent system (as shown in Fig. 3) exists. And condition 3) also guarantees the property of $\Delta u(k)$ expressed by equation (5).

Decompose the virtual equivalent system (Fig. 3) into two subsystems; see Fig. 5 and Fig. 6.

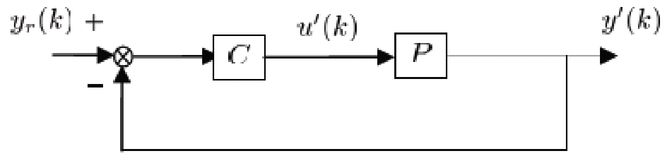

Fig. 5 Subsystem 1

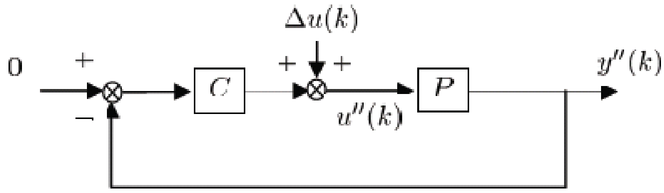

Fig. 6 Subsystem 2

By superposition principle of linear system, we have

$$y(k) = y'(k) + y''(k) \tag{9}$$

$$u(k) = u'(k) + u''(k) \tag{10}$$

By condition 4), subsystem 1 is a stable system. Then we get

$$|y'(k)| < \infty \tag{11}$$

$$|u'(k)| < \infty \tag{12}$$

As for subsystem 2, it is also a stable closed-loop system. And by conditions 1) and 3) we know that equation (5) holds. Then we have

$$|y''(k)| = o(\|\phi_c(k)\|) \tag{13}$$

$$|u''(k)| = o(\|\phi_c(k)\|) \tag{14}$$

By (9) and (10), it is obvious that

$$y(k) = y'(k) + y''(k) = y'(k) + o(\|\phi_c(k)\|) \tag{15}$$

$$u(k) = u'(k) + u''(k) = u'(k) + o(\|\phi_c(k)\|) \tag{16}$$

From now on, we use reduction to absurdity to get our result. Suppose the virtual equivalent system is unstable, i.e. $y(k)$ and $u(k)$ are both unbounded. Then sequence $\|\phi_c(k)\|$ is also unbounded. There must exist a subsequence $\|\phi_c(l)\|$ that goes to infinity. For elements of $\phi_c(l)$, we have

$$y(l) = y'(l) + y''(l) = y'(l) + o(\|\phi_c(l)\|) \tag{17}$$

$$y(l-1) = y'(l-1) + y''(l-1) = y'(l-1) + o(\|\phi_c(l-1)\|) \tag{18}$$

$$y(l-2) = y'(l-2) + y''(l-2) = y'(l-2) + o(\|\phi_c(l-2)\|) \tag{19}$$

$$\vdots$$

$$u(l) = u'(l) + u''(l) = u'(l) + o(\|\phi_c(l)\|) \tag{20}$$

$$u(l-1) = u'(l-1) + u''(l-1) = u'(l-1) + o(\|\phi_c(l-1)\|) \tag{21}$$

$$u(l-2) = u'(l-2) + u''(l-2) = u'(l-2) + o(\|\phi_c(l-2)\|) \tag{22}$$

$$\vdots$$

From Lemma 1 (See Appendix), we know that

$$\|\phi_c(l-i)\| = O(\|\phi_c(l)\| + M), i = 1,2,...,I \quad 0 < M < \infty \tag{23}$$

where I is a limited integer. Then we obtain the following inequalities

$$|y(l-1)|^2 \le |y'(l-1)|^2 + o(\|\phi_c(l)\|^2) \tag{24}$$

$$\vdots$$

$$|u(l-1)|^2 \le |u'(l-1)|^2 + o(\|\phi_c(l)\|^2) \tag{25}$$

$$\vdots$$

$$|y_r(l)|^2 = |y_r(l)|^2 \tag{26}$$

$$|y_r(l-1)|^2 = |y_r(l-1)|^2 \tag{27}$$

$$\vdots$$

Making sums from (24) to (27) and taking (11)-(12) into account, we get

$$\frac{\|\phi_c(l)\|^2}{\|\phi_c(l)\|^2} \le 0 \tag{28}$$

That is obviously absurd. Then the assumption that the virtual equivalent system is unstable can not hold. It means the virtual equivalent system is stable, i.e.

$$|y(k)| < \infty \tag{29}$$

$$|u(k)| < \infty \tag{30}$$

Further we have

$$\|\phi_c(l)\| \le \infty \tag{31}$$

Then Equations (15) - (16) yield

$$y(k) \to y'(k) \tag{32}$$

$$u(k) \to u'(k) \tag{33}$$

That means the virtual equivalent system is convergent. Thus, the switching MMAC system is stable and convergent.

This completes the proof of theorem 1.

4.2 Parameter estimates converge to untrue values

Theorem 2

If a switching MMAC system has the following properties:

1) The parameter estimates of adaptive model converge to untrue values, $P_m(k)$ is uniformly controllable and the estimation error satisfies

$$y(k) - \phi^T(k-d)\hat{\theta}(k) = o(\alpha + \|\phi(k-d)\|) \tag{34}$$

2) After limited switching, the MMAC stops at adaptive controller;

3) The mapping from estimated parameters into controller parameters is continuous;

4) The controller is well defined such that $< C_0, P_0 >$ constitutes a stable closed-loop system

Then the switching MMAC system is stable and convergent.

Proof. Conditions 1), 2), 3) guarantee that the virtual equivalent system, as shown in Fig. 4, exists.

Decompose the virtual equivalent system (Fig. 4) into three subsystems; see Fig. 7, Fig.8 and Fig. 9.

First, by condition 4), we know that subsystem 1 as shown in Fig. 7 is stable.

In Fig. 8, condition 1) and condition 3) guarantee

$$\Delta u(k) = o(\|\phi_c(k)\|) \tag{35}$$

From the definition of $\phi_c^T(k)$ in equation (4), we have

$$\|\phi_c(k)\| = O(\|\phi(k-d)\|) + M, 0 < M < \infty \tag{36}$$

Then (35) and (36) indicate

$$\Delta u(k) = o(\alpha + \|\phi(k-d)\|), \alpha > 0 \tag{37}$$

And we also have (see Lemma 2 in Appendix A)

$$\|\phi(k-d-i)\| = O(\|\phi(k-d)\|), i = 1,2,...,I \tag{38}$$

where I is a limited integer.

In Fig. 9, from (34) of condition 1 and (6), we know that

$$e(k) = o(\alpha + \|\phi(k-d)\|), \alpha > 0 \tag{39}$$

Then we can develop the result of Theorem 2 following the similar procedures of the proof of Theorem 1. Details are omitted.

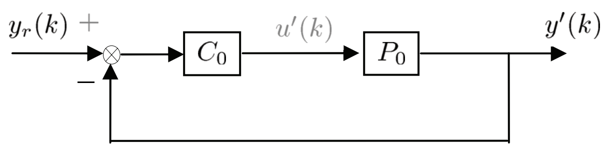

Fig. 7 Subsystem 1

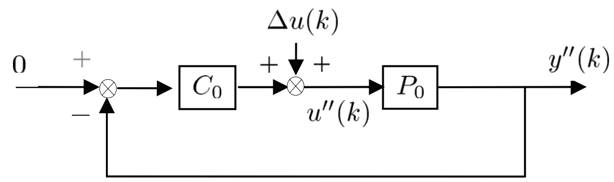

Fig. 8 Subsystem 2

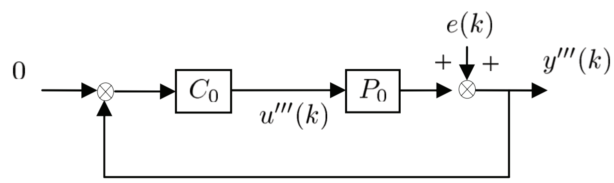

Fig. 9 Subsystem 3

4.3 Parameter estimates may not converge

Theorem 3

If a switching MMAC system with one-step-ahead adaptive controller has the following properties:

1) The parameter estimation error satisfies

$$y(k) - \phi^T(k-d)\hat{\theta}(k) = o(\alpha + \|\phi(k-d)\|), \alpha > 0 \quad (40)$$

2) $B(q^{-1})$ is Hurwitz and $b_0 \neq 0$;

3) The control signal $u(k)$ exists;

4) After limited switching, the MMAC stops at adaptive controller;

Then the switching MMAC system is stable and convergent.

Proof.

The virtual equivalent system and its decomposed sub-systems are shown in Fig. 3, Fig. 5 and Fig. 6.

First we introduce one-step-ahead adaptive control strategy (G. C. Goodwin, Sin Kwai Sang, 1984).

Rewrite equation (1) in prediction form

$$y(k+d) = \phi^T(k)\hat{\theta}(k) \quad (41)$$

One-step-ahead adaptive control signal $u(k)$ is decided by

$$\phi^T(k)\hat{\theta}(k) = y^*(k+d) \quad (42)$$

$y^*(k+d)$ is identical to $y_r(k+d)$. Equation (42) means

$$\begin{aligned} u(k) = \frac{1}{\hat{\theta}_{n+1}}[&-\hat{\theta}_1 y(k) - \hat{\theta}_2 y(k-1) - \cdots \\ &- \hat{\theta}_n y(k-n+1) - \hat{\theta}_{n+2} u(k-1) - \cdots \\ &- \hat{\theta}_{n+m+d} u(k-m-d+1) \quad + y^*(k+d)] \end{aligned} \quad (43)$$

Accordingly

$$\begin{aligned} u_0(k) = \frac{1}{\theta_{n+1}}[&-\theta_1 y(k) - \theta_2 y(k-1) - \cdots - \theta_n y(k-n+1) \\ &- \theta_{n+2} u(k-1) - \cdots - \theta_{n+m+d} u(k-m-d+1) \\ &+ y^*(k+d)] \end{aligned} \quad (44)$$

From (43) and (44), we have

$$\theta_{n+1}u_0(k) - \hat{\theta}_{n+1}u(k) = \theta_{n+1}u'(k) - \theta_{n+1}u(k) + (\theta_{n+1} - \hat{\theta}_{n+1})u(k) \quad (45)$$

Further, it is obvious that

$$\begin{aligned} \Delta u(k) &= \frac{1}{\vartheta_{n+1}}\left[\theta_{n+1}u_0(k) - \hat{\theta}_{n+1}u(k) + (\hat{\theta}_{n+1} - \theta_{n+1})u(k)\right] \\ &= \frac{1}{\vartheta_{n+1}}\left[\varphi^T(k-d)\theta - \varphi^T(k-d)\hat{\theta}(k)\right] \\ &= \frac{1}{\vartheta_{n+1}}\left[y(k) - \varphi^T(k-d)\hat{\theta}(k)\right] \end{aligned} \quad (46)$$

Here, θ_{n+1} is b_0 in (1).

Based on condition (1), we get

$$\Delta u(k) = o(\alpha + \|\phi(k-d)\|), \alpha > 0 \quad (47)$$

The remained procedures are similar to the proof of Theorem 1. Details are omitted.

5. CONCLUSIONS AND FUTURE WORK

Based on virtual equivalent system concept and methodology, we developed some general criteria for judging the stability and convergence of switching MMAC systems in which the adaptive control strategy and parameter estimation algorithm are arbitrary. Thus we argue that virtual equivalent system could provide a unified theoretical framework or a general theory for understanding and evaluating switching MMAC system.

In future work, we will investigate the stability and convergence of switching MMAC system of time-varying and/or stochastic plant. In addition, we will develop new adaptive strategy, such as hybrid or two time-scale adaptive control, to guarantee stability and convergence of switching MMAC without requiring the convergence of parameter estimates.

REFERENCES

A. S. Morse, D. Q. Mayne, and G. C. Goodwin (1992), Applications of hysteresis switching in parameter adaptive control, IEEE Trans. Automat. Contr., Vol. 37, pp. 1343–1354.

A.S. Morse (1993), Supervisory control of families of linear set point controllers, *Proc. IEEE Conf. Decision Contr.*, San Antonio, TX.

B. Martensson (1986), *Adaptive stabilization*, Ph.D. dissertation, Lund Institute of Technology, Lund, Sweden.

D. E. Miller and E. J. Davison (1989), An adaptive controller which provides Lyapunov stability, *IEEE Trans. Automat. Contr.*, Vol. 34, pp. 599–609.

D. E. Miller (1994), Adaptive stabilization using a nonlinear time-varying controller, *IEEE Trans. Automat. Contr.*, Vol. 39, pp. 1347–1359.

G. C. Goodwin (2000), Adaptive control: where to now, *Adaptive Systems for Signal Processing, Communications, and Control Symposium*.

G. C. Goodwin, Sin Kwai Sang (1984), *Adaptive Filtering, Prediction and Control*, NJ: Prentice Hall.

Ioannou P, Sun J (1996), Robust Adaptive Control, NJ: Prentice Hall.

K. J. Astrom, B. Wittenmark (1995), *Adaptive Control (2nd edn)*. MA: Addison-Wesley.

K. Poolla and S. J. Cusumano (1988), *A new approach to adaptive robust control—Parts I and II*, Coordinated Science Laboratory, Univ. Illinois, Urbana, Tech. Rep.

K.S. Narendra, and Osvaldo Driollet (2001), Stochastic Adaptive Control Using Multiple Estimation Models. *Int. J. Adapt. Control and Signal Process*, Vol.15, pp.287-317.

K. S. Narendra and C. Xiang (2000), Adaptive control of discrete-time systems using multiple models, *IEEE. Trans. Automat.Contr.*, Vol.45, No.9, pp.1669-1685.

K. S. Narendra and J. Balakrishnan (1997), Adaptive control using multiple models, *IEEE. Trans. Automat. Contr*, Vol.42, pp.171-187.

K. S. Narendra and S. Mukhopadhyay (1994), Intelligent control using neural networks, *Intelligent Control*, M. M. Gupta and N. K. Sinha, Eds.

K. S. Narendra and J. Balakrishnan (1994), Improving transient response of adaptive control systems using multiple models and switching, *IEEE. Trans. Automat.Contr*, Vol 40, pp.1861-1866.

Lingji Chen, K.S. Narendra (2001), Nonlinear adaptive control using neural networks and multiple models. *Automatica,* Vol.37, No. 8, pp.1245-1255.

M. Fu and B. R. Barmish (1986), Adaptive stabilization of linear systems via switching control, *IEEE Trans. Automat. Contr.*, Vol. 31, pp. 1097–1103.

R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne (1988), Design issues in adaptive control, IEEE Trans. Automat. Contr., Vol. AC-33, pp. 50–58.

S. R. Weller and G. C. Goodwin (1994), Hysteresis switching adaptive control of linear multivariable systems, *IEEE Trans. Automat. Contr.*, Vol. 39, pp. 1360–1375.

Sajjad Fekri, Michael Athans and Antonio Pascoal (2006), Issues, progress and new results in robust adaptive control, *Int. J. Adapt. Control and Signal Process*, Vol. 20(10), pp.519-579.

T. Autenrieth and E. Rogers (1999), Performance enhancements for a class of multiple model adaptive control schemes. *Int. J. Adapt. and Signal. Process*, Vol.13, pp.105-127.

Weicun Zhang, Tianguang Chu, Long Wang (2005), A new theoretical framework for self-tuning control, *International Journal of Information Technology*, Vol.11 (11), pp.123-139.

Weicun Zhang, Jin Young Choi (2007), On the Stability and Convergence of Self-Tuning Control Systems (I)--deterministic plant, *International Conference on Control, Automation and Systems*, Seoul, Korea.

Weicun Zhang, Jin Young Choi (2007), On the Stability and Convergence of Self-Tuning Control Systems (II) --stochastic plant, *International Conference on Control, Automation and Systems*, Seoul, Korea.

Xiaoli Li, Wei Wang, Shuning Wang (2001), Multiple model adaptive control for discrete time systems. *American Control Conference*, Arlington, Virginia, USA , pp.4820-4825.

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APPENDIX A

Lemma 1

$\|\phi_c(l-i)\| = O(\|\phi_c(l)\|) + M, 0 < M < \infty; i = 1,2,...,I$
where I is a limited integer.

Proof:
Suppose

$$\begin{aligned}\phi_c^T(k) =& [y(k), y(k-1), ...y(k-s_1), u(k-1) \\ & ,..., u(k-s_2), y_r(k), ...y_r(k-s_3)]\end{aligned}$$

Then we have

$$\begin{aligned}\phi_c^T(l) =& [y(l), y(l-1), ...y(l-s_1), u(l-1) \\ & ,...u(l-s_2), y_r(l), ...y_r(l-s_3)]\end{aligned}$$

$$\begin{aligned}\phi_c^T(l-1) =& [y(l-1), y(l-2), ...y(l-s_1-1), u(l-2), ... \\ & u(l-s_2-1), y_r(l-1), ...y_r(l-s_3-1)]\end{aligned}$$

By components comparison between $\phi_c^T(l)$ and $\phi_c^T(l-1)$, we obtain

$$\begin{aligned}&\|\phi_c(l-1)\|^2 \\ &= \|\phi_c(l)\|^2 - y^2(l-s_1-1) + y^2(l) - u^2(l-s_2-1) \\ &+ u^2(l-1) - y_r^2(l-s_3-1) + y_r^2(l)\end{aligned}$$

Due to the fact that $|y_r(k)| < M, 0 < M < \infty$, then we have

$$\begin{aligned}&\|\phi_c(l-1)\|^2 \\ &\le \|\phi_c(l)\|^2 + \|\phi_c(l)\|^2 - y^2(l-s_1-1) - u^2(l-s_2-1) + M \\ &\le \|\phi_c(l)\|^2 + \|\phi_c(l)\|^2 + y^2(l-s_1-1) + u^2(l-s_2-1) + M\end{aligned}$$

Generally speaking, we have

$$s_1 \le n, s_2 \le m$$

From (1), we know that

$$\begin{aligned}& y^2(l-s_1-1) + u^2(l-s_2-1) \\ &\le \frac{a_{l-s_1-1}^2 + b_{l-s_2-1}^2}{a_{l-s_1-1}^2 b_{l-s_2-1}^2}\{a_{l-s_1-1}^2 y^2(l-s_1-1) \\ &+ b_{l-s_2-1}^2 u^2(l-s_2-1)\} \\ &= O(\|\phi_c(l)\|^2))\end{aligned}$$

Then we have

$$\|\phi_c(l-1)\| = O(\|\phi_c(l)\| + M)$$

And similarly, for limited integer I, we can get

$$\|\phi_c(l-i)\| = O(\|\phi_c(l)\| + M), i = 1,2,...,I$$

That completes the proof of Lemma 1.

Lemma 2.

$$\|\phi(k-d-i)\| = O(\|\phi(k-d)\|), i = 1,2,...,I$$

where I is a limited integer.

Proof: Similar to the proof of Lemma 1.