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Linear Quadratic Regulation for Discrete-time Systems with Multiple Delays in Single Input Channel

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Abstract: This paper is concerned with the linear quadratic regulation (LQR) problem for linear discrete-time systems with multiple delays in a single input channel. Although the LQR problem for discrete-time systems with single delay in each of the multiple input channels has been studied in existing literature, the problem to be addressed in this paper is known to be very difficult and has not been well investigated. In this paper, we address the LQR problem for systems with multiple delays in a single input channel by first establishing a duality between the LQR problem and a smoothing problem for an associated stochastic backward system. An analytical solution to the LQR control is then derived by solving the smoothing problem and is given in terms of the solutions of Riccati difference equations of the same dimension as the plant (ignoring the delays). The infinite horizon LQR problem is also considered in this paper and the convergence and stability analysis of the LQR controller is provided.

1. INTRODUCTION

The research on delay systems has gained momentum since the 1960s due to their practical significance in various engineering systems such as industrial processes, communication systems and more recently networked control systems. A lot of studies have been focused on analysis and control problems of delay systems; see, e.g., Chyung [1969], Zhang et al. [2006] and the references therein.

For continuous-time delay systems, the control problem can be treated by the infinite-dimensional system theory Delfour and Karrakchou [1987]. An elegant solution to the H_∞ control of systems with single input delay has been presented in Tadmor [2000]. By converting the delay problem into a nested sequence of elementary problems, Meinsma and Mirkin [2005] gives a complete solution to the H_∞ control problem for systems with multiple I/O delays. On the other hand, there have been increasing interests in discrete-time delay systems due to applications in the fields of networked control and network congestion control. Delay problems associated with discrete-time systems can in principle be treated by a state augmentation approach. However, the augmentation leads to a higher state dimension and thus a higher computational cost. As such, there have been attempts in dealing with discrete delay problems via non-augmentation approaches. For example, the optimal tracking problem for discrete-time systems with single input delay has been studied in Pindyck [1972]. Zhang and Xie [2007] and Zhang et al. [2007] solve the LQR and H_∞ control problems for systems with a single delay in each of the multiple input channels by establishing a duality between the control problems and some smoothing problems for associated stochastic systems. The duality allows one to address the complicated multiple input delay problems via some elementary tools such as projection. It is, however, noted that control problems for systems with single input but multiple delays which contain the systems studied Zhang and Xie [2007] as a special case are very challenging as it becomes a constrained optimization problem if the approach of Zhang and Xie [2007] is applied directly. There have been few studies on such systems. Furthermore, we note that in Zhang and Xie [2007] only finite horizon control problems have been investigated.

In this paper, we investigate the LQR problem for discrete-time systems with multiple delays in a single input channel. Using a re-organized output approach, we first decouple the input signals at various delayed time instants and establish a duality between the LQR problem and a smoothing problem for an associated backward stochastic system. An analytical solution to the LQR problem is then derived and is given in terms of Riccati recursions of the same order of the plant (ignoring the delays). Compared with the augmentation approach, our result is computationally much more efficient. The infinite horizon LQR problem is also investigated. We show that under very mild conditions, the solution to the finite horizon LQR problem converges and the stability of the closed-loop system is guaranteed.

The contributions of the paper in relation to the work in Zhang et al. [2006] are two fold. First, we present a new and much simpler controller derivation approach as compared to Zhang et al. [2006]. This approach enables us to derive an analytical solution to the LQR problem for systems with single input multiple delays which cannot be addressed using the controller derivation method in Zhang et al. [2006] since it will involve a constrained optimization problem. Our approach, when specialized to the systems considered in Zhang et al. [2006], gives a much simpler controller derivation. Secondly, the new control design approach enables us to analyze the stability of the closed-loop system and thus solve the infinite horizon LQR problem for the delay system which has not been addressed in Zhang et al. [2006]. Our derivation in discrete-time can be extended to solve the even more challenging LQR problem for continuous-time systems with single input multiple delays.

Before closing the section, some remarks on the notation will be given. $\mathbf{R^n}$ denotes the n dimensional Euclidean space. Φ' is the transpose of the matrix Φ. $\langle a, b\rangle$ is the cross covariance between a and b, i.e., $\mathcal{E}\{(a-\mathcal{E}a)(b-\mathcal{E}b)'\}$, where $\mathcal{E}$ is the mathematical expectation. I_n denotes the identity matrix with dimension n. $\delta_{i,j}$ is the Kronecker delta function. We use bold letters to denote stochastic variables. $\chi_\Omega(t)$ is indicator function, i.e.

$$\chi_\Omega(t) = \begin{cases} 0, & t \notin \Omega, \\ 1, & t \in \Omega. \end{cases}$$

$\hat{\mathbf{x}}(i|j)$ represents the projection of $\mathbf{x}(i)$ onto the linear space $\mathcal{L}\{\bar{\mathbf{y}}(j), \cdots, \bar{\mathbf{y}}(N)\}$.

2. PROBLEM STATEMENT

We consider the following discrete linear system with multiple delays in a single input channel

$$x(k+1) = \Phi(k)x(k) + \sum_{i=0}^{l} \Gamma_{(i)}(k)u(k-d_i), x(0) = x_0, \quad (1)$$

where $x \in \mathbf{R}^n$ is the state, $u(k) \in \mathbf{R}^m$ is the control input with initial values $u(k) = \mu_k$ when $k < 0$, $\Phi(k)$ and $\Gamma_{(i)}(k)$ are time varying matrices with appropriate dimension. Without loss of generality, the delays are assumed to be of an increasing order: $0 = d_0 < d_1 < \cdots < d_l$. We consider the following quadratic performance index for the system (1):

$$J_N = \sum_{i=0}^{N} [u'(i)R_i u(i) + x'(i)Q_i x(i)] + x'_{N+1}P_{N+1}x_{N+1}, \quad (2)$$

where $N > d_l$ is an integer, x_{N+1} is the terminal state, i.e., $x_{N+1} = x(N+1)$, $P_{N+1} > 0$ is the penalty weighting matrix on the terminal state, $R_i > 0, Q_i \geq 0, i = 0, \ldots, N$ are weighting matrices on the input signal $u(i)$ and state $x(i)$, respectively.

The finite horizon LQR problem is to find the state feedback control $\{u^\star(s), 0 \leq s \leq N\}$ such that the cost function J_N is minimized. We also deal with the infinite horizon case, i.e. when $N \to \infty$.

Remark 1. It should be noted that the aforementioned LQR problem can be addressed by a state augmentation method together with the standard LQR solution for systems without delay. However, for large delays and/or high dimensional inputs, the augmented system will have a very high dimension, resulting in a high computational cost. More importantly, we aim to find a non-augmentation method that can be extended to continuous-time delay systems for which the augmentation approach is not applicable.

In the following, we shall present a non-augmentation approach to the LQR problem for the system (1) for both the finite and infinite horizon cases. We shall also establish stability and convergence properties of the LQR control.

3. PRELIMINARIES

In this section we shall convert the LQR problem into the optimal smoothing problem for an associated stochastic backward system.

First, we shall introduce some notation. $\forall k \geq s$, denote

$$u_s(k) \triangleq \begin{cases} col\{u(k), u(k-d_1), \cdots, u(k-d_i)\}, \\ \qquad d_i \leq k-s < d_{i+1}, \\ col\{u(k), u(k-d_1), \cdots, u(k-d_l)\}, \\ \qquad k-s \geq d_l, \end{cases} \quad (3)$$

$$\tilde{u}_s(k) \triangleq \sum_{j=0}^{l} \Gamma_{(j)}(k)u(k-d_j)\chi_{[-d_l,s)}(k-d_j), \quad (4)$$

$$\Gamma_s(k) \triangleq \begin{cases} \left[\Gamma_{(0)}(k)\,\Gamma_{(1)}(k)\cdots\Gamma_{(i)}(k)\right], \\ \qquad d_i \leq k-s < d_{i+1}, \\ \left[\Gamma_{(0)}(k)\,\Gamma_{(1)}(k)\cdots\Gamma_{(l)}(k)\right], \\ \qquad k-s \geq d_l, \end{cases} \quad (5)$$

$$R_s(k) \triangleq \begin{cases} diag\{R_{0,k}, \cdots, R_{i,k}\}, d_i \leq k-s < d_{i+1}, \\ diag\{R_{0,k}, \cdots, R_{l,k}\}, k-s \geq d_l, \end{cases} \quad (6)$$

$$R_{i,k} \triangleq \begin{cases} \dfrac{R_{k-d_i}}{l+1}, k-d_i \leq N-d_l, \\ \dfrac{R_{k-d_i}}{j+1}, N-d_{j+1} < k-d_i \leq N-d_j, \end{cases} \quad (7)$$

where $\chi_{[a,b]}(t)$ is indicator function defined in the introduction. Using the above notation, $\forall k \geq s$, the system (1) can be rewritten as

$$x(k+1) = \Phi(k)x(k) + \Gamma_s(k)u_s(k) + \tilde{u}_s(k). \quad (8)$$

The cost function (2) can be splitted into two terms,

$$J_N = J_N^s + \bar{J}_N^s, \quad (9)$$

where

$$\begin{aligned} J_N^s &= \sum_{i=s}^{N} u'(i)R_i u(i) + \sum_{i=s}^{N} x'(i)Q_i x(i) + x'_{N+1}P_{N+1}x_{N+1} \\ &= \sum_{i=s}^{N} u_s'(i)R_s(i)u_s(i) + \sum_{i=s}^{N} x'(i)Q_i x(i) \\ &\quad + x'_{N+1}P_{N+1}x_{N+1}, \end{aligned} \quad (10)$$

$$\bar{J}_N^s = \sum_{i=0}^{s-1} u'(i)R_i u(i) + \sum_{i=0}^{s-1} x'(i)Q_i x(i). \quad (11)$$

Note that $\{u(k), k \geq s\}$ has no effect on $\bar{J}_N^s$, so we can only design $\{u(k), k \geq s\}$ to minimize J_N^s. Now, we introduce the following backward stochastic state-space system associated with (8) and performance index (10):

$$\mathbf{x}(k) = \Phi'(k)\mathbf{x}(k+1) + \mathbf{q}(k), \quad (12)$$

$$\mathbf{y}(k) = \Gamma'_s(k)\mathbf{x}(k+1) + \mathbf{v}(k), s \le k \le N, \quad (13)$$

where $\mathbf{x}(N+1)$, $\mathbf{q}(k)$ and $\mathbf{v}(k)$ are uncorrelated white noises with zero means and covariances

$$\langle \mathbf{x}_{N+1}, \mathbf{x}_{N+1} \rangle = P_{N+1}, \quad \langle \mathbf{q}(i), \mathbf{q}(j) \rangle = Q_i \delta_{i,j},$$
$$\langle \mathbf{v}(i), \mathbf{v}(j) \rangle = R_s(i)\delta_{i,j}.$$

By introducing the notation,

$$u_s \triangleq col\{u_s(s), \cdots, u_s(N)\}, \quad (14)$$

$$\mathbf{y}_s \triangleq col\{\mathbf{y}(s), \cdots, \mathbf{y}(N)\}, \quad (15)$$

$$h_s = min\{d_l - 1, N - s\}, \quad (16)$$

we have the following result.

Lemma 1. By making use of the stochastic state-space model (12)-(13), J_N^s can be put in the following quadratic form

$$J_N^s = \begin{bmatrix} \xi_s \\ u_s \end{bmatrix}' \Pi_s \begin{bmatrix} \xi_s \\ u_s \end{bmatrix}, \quad (17)$$

where

$$\xi_s = \left[x'(s)\, \tilde{u}'_s(s) \cdots \tilde{u}'_s(s+h_s) \right]', \quad (18)$$

$$\Pi_s = \left\langle \begin{bmatrix} \mathbf{x}_s^0 \\ \mathbf{y}_s \end{bmatrix}, \begin{bmatrix} \mathbf{x}_s^0 \\ \mathbf{y}_s \end{bmatrix} \right\rangle \triangleq \begin{bmatrix} R_{\mathbf{x}_s^0} & R_{\mathbf{x}_s^0 \mathbf{y}_s} \\ R_{\mathbf{y}_s \mathbf{x}_s^0} & R_{\mathbf{y}_s} \end{bmatrix}, \quad (19)$$

$$\mathbf{x}_s^0 = col\{\mathbf{x}(s), \mathbf{x}(s+1), \cdots, \mathbf{x}(s+h_s+1)\}, \quad (20)$$

with $\tilde{u}_s(s+i)$, $i = 0, \ldots, h_s$ as defined in (4), $\langle \mathbf{x}_s^0, \mathbf{x}_s^0 \rangle = R_{\mathbf{x}_s^0}$, $\langle \mathbf{x}_s^0, \mathbf{y}_s \rangle = R_{\mathbf{x}_s^0 \mathbf{y}_s}$ and $\langle \mathbf{y}_s, \mathbf{y}_s \rangle = R_{\mathbf{y}_s}$.

Proof. The proof is similar to that in Zhang et al. [2006] and is omitted.

Due to the coupling of $u_s(k)$ at different delayed time instants, we apply a re-organization of measured output to decouple the u_s. First, decompose $\mathbf{y}(k)$ and $\mathbf{v}(k)$ of (13) as follows:

$$\mathbf{y}(k) = \begin{cases} col\left\{\mathbf{y}_0(k), \cdots, \mathbf{y}_i(k)\right\}, d_i \le k - s < d_{i+1}, \\ col\left\{\mathbf{y}_0(k), \cdots, \mathbf{y}_l(k)\right\}, k - s \ge d_l, \end{cases} \quad (21)$$

$$\mathbf{v}(k) = \begin{cases} col\left\{\mathbf{v}_0(k), \cdots, \mathbf{v}_i(k)\right\}, d_i \le k - s < d_{i+1}, \\ col\left\{\mathbf{v}_0(k), \cdots, \mathbf{v}_l(k)\right\}, k - s \ge d_l, \end{cases} \quad (22)$$

where $\mathbf{y}_i(k)$ and $\mathbf{v}_i(k), i = 0, \ldots, l$ satisfy

$$\mathbf{y}_i(k) = \Gamma'_{(i)}(k)\mathbf{x}(k+1) + \mathbf{v}_i(k), \quad (23)$$

with $\langle \mathbf{v}_i(k_1), \mathbf{v}_j(k_2) \rangle = R_{i,k_1}\delta_{i,j}\delta_{k_1,k_2}$. We can re-organize the outputs as follows:

$$\mathbf{y}_f(k) \triangleq \begin{cases} col\{\mathbf{y}_0(k), \cdots, \mathbf{y}_l(k+d_l)\}, d_l \le N - k, \\ col\{\mathbf{y}_0(k), \cdots, \mathbf{y}_i(k+d_i)\}, d_i \le N - k < d_{i+1}, \end{cases} \quad (24)$$

and $\mathbf{v}_f(k)$ has the same form as

$$\mathbf{v}_f(k) \triangleq \begin{cases} col\{\mathbf{v}_0(k), \cdots, \mathbf{v}_l(k+d_l)\}, d_l \le N - k, \\ col\{\mathbf{v}_0(k), \cdots, \mathbf{v}_i(k+d_i)\}, d_i \le N - k < d_{i+1}. \end{cases} \quad (25)$$

Now, we shall simplify the cost function J_N^s of (17). To this end, we denote, for $s \le k \le N - d_l$,

$$T(k) = \underbrace{\begin{bmatrix} I_m & I_m & \cdots & I_m \\ 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I_m \end{bmatrix}}_{l+1}, \quad (26)$$

and for $N - d_{i+1} < k \le N - d_i$,

$$T(k) = \underbrace{\begin{bmatrix} I_m & I_m & \cdots & I_m \\ 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I_m \end{bmatrix}}_{i+1}. \quad (27)$$

It is easy to write

$$T(k)\mathbf{y}_f(k) = \begin{cases} col\{\bar{\mathbf{y}}(k), -\mathbf{y}_1(k+d_1), \cdots, -\mathbf{y}_l(k+d_l)\}, \\ \qquad d_l \le N - k, \\ col\{\bar{\mathbf{y}}(k), -\mathbf{y}_1(k+d_1), \cdots, -\mathbf{y}_i(k+d_i)\}, \\ \qquad d_i \le N - k < d_{i+1}, \end{cases} \quad (28)$$

where

$$\bar{\mathbf{y}}(k) = \sum_{j=0}^{l} \mathbf{y}_j(k+d_j)\chi_{[k,N]}(k+d_j). \quad (29)$$

In view of (23), $\bar{\mathbf{y}}(k)$ can be rewritten as

$$\bar{\mathbf{y}}(k) = \sum_{j=0}^{l} \Gamma'_{(j)}(k+d_j)\mathbf{x}(k+d_j+1)\chi_{[k,N]}(k+d_j) + \bar{\mathbf{v}}(k), \quad (30)$$

where

$$\bar{\mathbf{v}}(k) = \sum_{j=0}^{l} \mathbf{v}_j(k+d_j)\chi_{[k,N]}(k+d_j). \quad (31)$$

It is easy to verify that $\bar{\mathbf{v}}(k), k = s, \ldots, N$ are uncorrelated white noises with zero means and covariances $\langle \bar{\mathbf{v}}(i), \bar{\mathbf{v}}(k) \rangle = R_i \delta_{i,k}$. Then we have the following result.

Lemma 2. Let

$$\bar{\mathbf{y}}_s = col\{\bar{\mathbf{y}}(s), \cdots, \bar{\mathbf{y}}(N)\}, \quad (32)$$

$$U_s = col\{u(s), \cdots, u(N)\}, \quad (33)$$

where $\bar{\mathbf{y}}(k)$ is as in (30) and $u(k)$ in (1). Then J_N^s of (10) can be further rewritten in the following quadratic form

$$J_N^s = \begin{bmatrix} \xi_s \\ U_s \end{bmatrix}' \bar{\Pi}_s \begin{bmatrix} \xi_s \\ U_s \end{bmatrix}, \quad (34)$$

where

$$\bar{\Pi}_s = \left\langle \begin{bmatrix} \mathbf{x}_s^0 \\ \bar{\mathbf{y}}_s \end{bmatrix}, \begin{bmatrix} \mathbf{x}_s^0 \\ \bar{\mathbf{y}}_s \end{bmatrix} \right\rangle \triangleq \begin{bmatrix} R_{\mathbf{x}_s^0} & R_{\mathbf{x}_s^0 \bar{\mathbf{y}}_s} \\ R_{\bar{\mathbf{y}}_s \mathbf{x}_s^0} & R_{\bar{\mathbf{y}}_s} \end{bmatrix}. \quad (35)$$

Furthermore, J_N^s can be simplified as

$$J_N^s = \xi_s' \mathcal{P}_s \xi_s + (U_s - U_s^\star)' R_{\bar{\mathbf{y}}_s} (U_s - U_s^\star), \quad (36)$$

where

$$U_s^\star = -R_{\bar{\mathbf{y}}_s}^{-1} R_{\bar{\mathbf{y}}_s \mathbf{x}_s^0} \xi_s, \tag{37}$$

$$\mathcal{P}_s = \left\langle \mathbf{x}_s^0 - \hat{\mathbf{x}}_s^0, \mathbf{x}_s^0 - \hat{\mathbf{x}}_s^0 \right\rangle, \tag{38}$$

and $\hat{\mathbf{x}}_s^0$ is the projection of $\mathbf{x}_s^0$ onto the linear space $\mathcal{L}\{\bar{\mathbf{y}}_s\}$. Therefore, by considering (33), the minimizing solution of J_N^s with respect to control input $u(k)$ is the $(k-s+1)$-th block of $U_s^\star$.

Proof. It can be proved similarly to Zhang and Xie [2007]. The detail is thus omitted.

Remark 2. As noted in Zhang et al. [2006], Lemma 2 reveals a duality between the LQR and the optimal smoothing problem for the stochastic backward system (12)-(13) in view of the fact that $R_{\bar{\mathbf{y}}_s}^{-1} R_{\bar{\mathbf{y}}_s \mathbf{x}_s^0}$ is in fact the transpose of the optimal smoothing gain of the backward system (12)-(13). The duality allows us to solve the LQR problem by some elementary tools such as projection.

4. SOLUTION TO THE FINITE HORIZON LQR PROBLEM

In view of the duality between the LQR problem and the smoothing problem, the controller gain matrix $-R_{\bar{\mathbf{y}}_s}^{-1} R_{\bar{\mathbf{y}}_s \mathbf{x}_0^s}$ in (37) can be obtained from the following projection problem:

$$\hat{\mathbf{x}}_s^0 = R_{\mathbf{x}_s^0 \bar{\mathbf{y}}_s} R_{\bar{\mathbf{y}}_s}^{-1} \bar{\mathbf{y}}_s. \tag{39}$$

So, we will derive the smoothing (filtering) gain matrix first.

Theorem 3. For system (12) and (30), the recursive optimal filter and smoother are given by

$$\hat{\mathbf{x}}(k|k) = \Phi'(k)\hat{\mathbf{x}}(k+1|k), \tag{40}$$

$$\hat{\mathbf{x}}(k+i|k) = \hat{\mathbf{x}}(k+i|k+1) + K(k+i,k)\mathbf{e}(k), \quad i = 1,\ldots,h_k+1, \tag{41}$$

$$\mathbf{e}(k) = \bar{\mathbf{y}}(k) - \sum_{j=0}^{l}\Big[\Gamma'_{(j)}(k+d_j)\hat{\mathbf{x}}(k+d_j+1|k+1) \times \chi_{[k,N]}(k+d_j)\Big], \tag{42}$$

$$K(k+i,k) = \sum_{j=0}^{l}\big[P(k+i,k+d_j+1,k+1)\Gamma_{(j)}(k+d_j) \times R_e^{-1}(k)\chi_{[k,N]}(k+d_j)\big], \tag{43}$$

$$K(k,k) = \Phi'(k)K(k+1,k), \tag{44}$$

$$R_e(k) = \sum_{i,j=0}^{l}\Big[\Gamma'_{(i)}(k+d_i)P(k+d_i+1,k+d_j+1,k+1) \times \Gamma_{(j)}(k+d_j)\chi_{[k,N]}(k+d_i)\chi_{[k,N]}(k+d_j)\Big] + R_k, \tag{45}$$

$$P(k+i,k+j,k) = P(k+i,k+j,k+1) - K(k+i,k) \times R_e(k)K'(k+j,k), i,j = 1,\cdots,h_k+1, \tag{46}$$

where $P(k+i,k+j,k) = \langle \mathbf{x}(k+i), \tilde{x}(k+j|k)\rangle$ satisfies

$$P(k,k+j,k) = \Phi'(k)P(k+1,k+j,k), \quad j = 1,\cdots,h_k+1, \tag{47}$$

$$P(k,k,k) = \Phi'(k)P(k+1,k+1,k)\Phi(k) + Q_k, \tag{48}$$

while h_k is defined in (16) with s replaced by k and $\mathbf{e}(k)$ is the innovation sequence with covariance $R_e(k)$. $P(k+i,k+j,k)$, satisfying

$$P(k+i,k+j,k) = P'(k+j,k+i,k),$$

is the cross covariance matrix between $\tilde{\mathbf{x}}(k+i|k)$ and $\tilde{\mathbf{x}}(k+j|k)$, where $\tilde{\mathbf{x}}(\cdot|k) = \mathbf{x}(\cdot) - \hat{\mathbf{x}}(\cdot|k)$.
The initial condition is:

$$\hat{\mathbf{x}}(N+1|N+1) = 0, \tag{49}$$

$$P(N+1,N+1,N+1) = P_{N+1}. \tag{50}$$

Proof. The proof is straightforward by applying projection theory.

Theorem 3 provides a way to calculate $\hat{\mathbf{x}}_s^0$, based on which we can obtain our main result of the finite horizon LQR as follows.

Theorem 4. Consider system (1) and cost function (2). The optimal LQR controller $u^\star(s)$ that minimizes (2) is calculated by

$$u^\star(s) = -K'(s,s)x(s) - \sum_{i=0}^{h_s} K'(s+i+1,s)\tilde{u}_s^\star(s+i), \tag{51}$$

where h_s is as defined in (16). $\tilde{u}_s^\star(k)$ is given in (4) with $u(k-d_i)$ replaced by $u^\star(k-d_i)$. $K(s+i,s), i = 0,\ldots,h_s+1$ can be calculated by Theorem 3.

Proof. From Theorem 3 we can write

$$\begin{bmatrix} \hat{\mathbf{x}}(s|s) \\ \hat{\mathbf{x}}(s+1|s) \\ \vdots \\ \hat{\mathbf{x}}(s+h_s+1|s) \end{bmatrix} = \begin{bmatrix} \Phi'(s) & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \times \begin{bmatrix} \hat{\mathbf{x}}(s+1|s+1) \\ \hat{\mathbf{x}}(s+2|s+1) \\ \vdots \\ \hat{\mathbf{x}}(s+h_s+2|s+1) \end{bmatrix} + \begin{bmatrix} K(s,s) \\ K(s+1,s) \\ \vdots \\ K(s+h_s+1,s) \end{bmatrix} \mathbf{e}(s), \tag{52}$$

where $\mathbf{e}(s) = \bar{\mathbf{y}}(s) - \hat{\bar{\mathbf{y}}}(s|s+1)$; $\hat{\mathbf{x}}(s+h_s+2|s+1) = 0$ when $s+h_s+2 \geq N+1$. Denote

$$K_s = \begin{bmatrix} K(s,s) \\ K(s+1,s) \\ \vdots \\ K(s+h_s+1,s) \end{bmatrix}, \tag{53}$$

$$\bar{K}_s \bar{\mathbf{y}}_{s+1} = \begin{bmatrix} \Phi'(s) & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(s+1|s+1) \\ \hat{\mathbf{x}}(s+2|s+1) \\ \vdots \\ \hat{\mathbf{x}}(s+h_s+2|s+1) \end{bmatrix} - K_s \hat{\bar{\mathbf{y}}}(s|s+1). \tag{54}$$

In the light of (39), the filter gain matrix $R_{\mathbf{x}_s^0\bar{\mathbf{y}}_s} R_{\bar{\mathbf{y}}_s}^{-1}$ can be derived as

$$\begin{aligned} R_{\mathbf{x}_s^0\bar{\mathbf{y}}_s} R_{\bar{\mathbf{y}}_s}^{-1} &= \langle \hat{\mathbf{x}}_s^0, \bar{\mathbf{y}}_s \rangle R_{\bar{\mathbf{y}}_s}^{-1} \\ &= \left\langle \begin{bmatrix} K_s \; \bar{K}_s \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}}(s) \\ \bar{\mathbf{y}}_{s+1} \end{bmatrix}, \begin{bmatrix} \bar{\mathbf{y}}(s) \\ \bar{\mathbf{y}}_{s+1} \end{bmatrix} \right\rangle R_{\bar{\mathbf{y}}_s}^{-1} \\ &= \begin{bmatrix} K_s \; \bar{K}_s \end{bmatrix}. \end{aligned} \tag{55}$$

In view of (37), the optimal controller can be given by

$$\begin{bmatrix} u^\star(s) \\ \vdots \\ u^\star(N) \end{bmatrix} = - \begin{bmatrix} K_s' \\ \bar{K}_s' \end{bmatrix} \begin{bmatrix} x(s) \\ \tilde{u}_s(s) \\ \vdots \\ \tilde{u}_s(s+h_s) \end{bmatrix}. \tag{56}$$

The first row block of (56) gives the optimal state feedback control law.

From Theorem 4, $u^\star(k)$ can be calculated step by step as follows:

Step 1: Calculate K_s in (53).

(1) set $s = N$;
(2) calculate $R_e(s)$ using (45);
(3) calculate $K(s+i, s), i = 0, \ldots, h_s+1$ using (43), (44);
(4) update $P(s+i+1, s+j+1, s+1), i, j = 0, \ldots, d_l$ to $P(s+i, s+j, s), i, j = 0, \ldots, d_l$ using (46), (47), (48);
(5) $s = s - 1$;
(6) if $s \neq 0$, goto (1); otherwise, end Step 1.

Step 2: Calculate $u^\star(k)$.

(1) set $s = 0$;
(2) calculate $\tilde{u}(s+i), i = 0, \ldots, h_s$ using (4);
(3) calculate $u(s)$ using (65);
(4) $s = s + 1$;
(5) if $s \neq N$, goto (1); otherwise, end Step 2.

Remark 3. Theorem 4 provides a solution to the finite horizon LQR problem of the delay system (1) via a non-augmentation approach. The computation of the optimal control law involves a two-dimensional recursion of a Riccati difference equation of the same dimension of the plant (ignoring the delays). This approach is of a significant computational advantage as compared with the augmentation method which is demonstrated below.

We now give a comparison of computational complexity between the augmentation method and the proposed one. Because the multiplications and divisions cost much more in computation than additions, we can use the number of multiplications and divisions as the operation count. Denote C_{aug} and C_{new} the the number of the multiplications and divisions for the augmentation method and the proposed one in one step, respectively. C_{aug} and C_{new} can be given as follows:

$$\begin{aligned} C_{aug} &= 2(n + d_l m)^3 + 4(n + d_l m)^2 m + 3(n + d_l m) m^2 \\ &\quad + m^3, \end{aligned} \tag{57}$$

$$\begin{aligned} C_{new} &= (d_l + 1)n^3 + \frac{1}{2}\left[(d_l + l + 1)(d_l + l + 2) + 2\right] n^2 m \\ &\quad + \frac{1}{2}\left[d_l(d_l + 3) + (l + 1)(l + 2)\right] nm^2 + m^3 + lnm. \end{aligned} \tag{58}$$

From (57) and (58), it is clear that the order of d_l in C_{aug} is 3, while the order of d_l in C_{new} is 2. So if d_l is large enough, $C_{aug} \gg C_{new}$.

5. SOLUTION TO THE INFINITE HORIZON LQR CONTROL PROBLEM

In this section, we shall study the LQR control problem for linear time-invariant systems in the infinite-horizon, i.e., we consider the system

$$x(k+1) = \Phi x(k) + \sum_{i=0}^{l} \Gamma_i u(k - d_i), \; x(0) = x_0, \tag{59}$$

and the following quadratic performance index:

$$J_\infty = \lim_{N \to \infty} \sum_{i=0}^{N} [u'(i) R u(i) + x'(i) Q x(i)], \tag{60}$$

where $R > 0$, $Q \geq 0$ are weighting matrices on input signal u and state x, respectively.

The infinite horizon LQR problem is to find a state feedback control $\{u^\star(s), s \geq 0\}$ such that the closed-loop system is asymptotically stable and the cost function J_∞ is minimized.

Different from the finite horizon case, for the infinite horizon case, we need to guarantee the asymptotic stability of the closed-loop system. So, before extending the above result to the infinite horizon case, the analysis of the stability of the closed-loop system should be provided. To this end, we first denote

$$\bar{\Phi} = \begin{bmatrix} \Phi & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \bar{H} = \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}', \bar{\Gamma} = \begin{bmatrix} \Gamma_0 \\ \vdots \\ \Gamma_1 \\ \vdots \\ \Gamma_l \end{bmatrix},$$

$$\bar{P}(k) = \begin{bmatrix} P(k,k,k) & \cdots & P(k, k+d_l, k) \\ \vdots & \ddots & \vdots \\ P(k+d_l, k, k) & \cdots & P(k+d_l, k+d_l, k) \end{bmatrix}.$$

Using the above notation, we can write the Riccati recursions (46)-(48) in a compact form:

$$\bar{P}(k) = \bar{\Phi}' \bar{P}(k+1) \bar{\Phi} + \bar{H}' Q \bar{H} - K(k) R_e(k) K'(k), \tag{61}$$

where

$K(k) = \bar{\Phi}' \bar{P}(k+1) \bar{\Gamma} R_e^{-1}(k), R_e(k) = \bar{\Gamma}' \bar{P}(k+1) \bar{\Gamma} + R.$

So we can analyze the standard Riccati difference equation (61) instead of (46)-(48).

We recall the following lemma, see Anderson and Moore [1979] and Chan et al. [1984].

Lemma 5. Consider the following algebraic Riccati equation (ARE):

$$P = F'PF - F'PG(G'PG + R)^{-1} G'PF + Q, \tag{62}$$

where $Q \geq 0$ and $R > 0$. If (F, G) is stabilizable and $(Q^{\frac{1}{2}}, F)$ is detectable, then there exists a non-negative definite stabilizing solution $\bar{P}$ to the ARE (62), i.e. there exists a non-negative definite solution $\bar{P}$ such that the matrix $F - G(G'\bar{P}G + R)^{-1}G'\bar{P}F$ has all its eigenvalues inside the unit disk. Furthermore, for any given $P(0) \geq 0$, the solution of the Riccati difference equation

$$P(k+1) = F'P(k)F + Q - F'P(k)G[G'P(k)G + R]^{-1}G'P(k)F \quad (63)$$

exists and converges to the stabilizing solution $\bar{P}$, i.e. $\lim_{k\to\infty} P(k) = \bar{P}$.

Now, we are in the position to give the conditions guaranteeing the convergence of the Riccati recursions (46)-(48) and the stability of the closed-loop system.

Theorem 6. For system (1) and cost function (2), if Φ is invertible, $(\Phi, \sum_{i=0}^{l} \Phi^{-d_i}\Gamma_i)$ is stabilizable and $(Q^{\frac{1}{2}}, \Phi)$ is detectable, then the solutions of Riccati recursions (46)-(48) converge and the closed-loop system of (1) with the converged control law given in Theorem 4 is asymptotically stable.

Proof. Due to the space limitation, the proof is omitted.

From Theorem 6 we know that the closed-loop system is stable, so we can design the optimal control using the result of finite horizon. The steady-state solutions of the Riccati difference equations (46)-(48) are denoted as

$$\lim_{k\to\infty} P(k+i, k+j, k) = P(i,j),\ i,j = 0,\ldots,d_l, \quad (64)$$

and the corresponding $K(k+i,k), i = 0,\ldots,d_l$ and $R_e(k)$ can be denoted as $K(i), i = 0,\ldots,d_l$ and R_e, respectively. Then, we have the following result.

Theorem 7. Consider system (59) and cost function (60). If Φ is invertible, $(\Phi, \sum_{i=0}^{l} \Phi^{-d_i}\Gamma_i)$ is stabilizable and $(Q^{\frac{1}{2}}, \Phi)$ is detectable, the infinite horizon LQR controller $u^\star(s)$ exists and is given by

$$u^\star(s) = -K'(0)x(s) - \sum_{i=0}^{d_l-1} K'(i+1)\tilde{u}_s^\star(s+i), \quad (65)$$

where $\tilde{u}_s^\star(k)$ is obtained from (4) with $u(k-d_i)$ replaced by $u^\star(k-d_i)$ and $K(i), i = 0,\ldots,d_l$ can be calculated by

$$K(i) = \sum_{j=0}^{l} P(i-1, d_j)\Gamma_j R_e^{-1}, i = 1,\ldots,d_l \quad (66)$$

$$K(0) = \Phi'K(1), \quad (67)$$

with

$$R_e = \sum_{i,j=0}^{l} \Gamma_i' P(d_i, d_j)\Gamma_j + R, \quad (68)$$

and $P(i,j)$ satisfying the following Riccati recursions,

$$P(i,j) = P(i-1, j-1) - K(i)R_e K'(j), \quad i,j = 1,\cdots,d_l, \quad (69)$$

$$P(0,j) = \Phi'P(1,j), j = 1,\cdots,d_l, \quad (70)$$

$$P(0,0) = \Phi'P(1,1)\Phi + Q. \quad (71)$$

Proof. The proof is straightforward by extending the finite horizon results and considering Theorem 6.

Remark 4. We note from Theorem 6 that under the standard stabilizability and detectability assumptions, the solutions to (69)-(71) can be computed by recursions.

6. CONCLUSIONS

We have investigated both the finite horizon and infinite horizon LQR control problems for discrete-time systems with multiple delays in single input channel. An explicit optimal controller is given in terms of the solutions of Riccati recursions or algebraic Riccati equations. The stability analysis has also been provided for the infinite horizon case. Our approach has an advantage in computation compared with the augmentation approach. It is worthy mentioning that the result in this paper has been extended to solve the control problem for continuous-time systems with the same type of input delays.

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