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Adaptive Control of a Scalar Linear Stochastic System with a Fractional Brownian Motion

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Abstract: In this paper, an adaptive control problem is formulated and solved for a scalar linear stochastic system perturbed by a fractional Brownian motion and an ergodic (or average cost per unit time) quadratic cost functional. The Hurst parameter for the fractional Brownian motion may take any value in $(1/2, 1)$.

1. INTRODUCTION

The adaptive control of a linear quadratic Gaussian system has a sizable history of research where the perturbations are white Gaussian noise. In this adaptive control problem, some parameters of the system are assumed to be unknown so that it is required to estimate the parameters and control the system simultaneously. These adaptive control problems have been studied in both discrete and continuous time. In this paper, the processes are in continuous time so it is most natural to associate the results with those in continuous time for white Gaussian noise (e.g. Caines [1992], Chen et al. [1996], Chen and Guo [1991], Duncan and Pasik-Duncan [1990], Kumar [1983]).

In this paper an adaptive control problem is solved for a linear quadratic Gaussian system where the perturbations are a fractional Gaussian noise with the Hurst parameter in the interval $(1/2, 1)$. By analogy with the fact that white Gaussian noise in continuous time is a "formal" process as the derivative of Brownian motion and one needs to consider Brownian motion, fractional Gaussian noise is a formal process as the derivative of a fractional Brownian motion and one needs to consider fractional Brownian motion.

A fractional Brownian motion with the Hurst parameter in $(1/2, 1)$ has a long range dependence that exhibits a bursty behaviour for the sample paths and seems to be a useful model for a variety of physical phenomena. A fractional Brownian motion was initially defined by Kolmogorov Kolmogorov [1940] and some statistics of it occurred in the study of rainfall in the Nile River valley by Hurst Hurst [1951]. Subsequently, Mandelbrot Mandelbrot [1963] noted its usefulness for modeling economic data, and Mandelbrot and van Ness Mandelbrot and Van Ness [1968] developed some of its properties. More recently, it has been proposed as a useful model in describing internet traffic in telecommunications and the occurrence of epileptic seizures.

In this paper, an adaptive control problem for a scalar linear stochastic control system perturbed by a fractional Brownian motion with the Hurst parameter H in $(1/2, 1)$ is solved. A necessary ingredient of a self-optimizing adaptive control is the corresponding optimal control for the known system. It seems that the optimal control problem has only been solved for a scalar system Kleptsyna et al. [2005]. In the solution of the adaptive control problem, a strongly consistent family of estimators of the unknown parameter are given and a certainty equivalence control is shown to be self-optimizing in an $L^2(P)$ sense. It seems that this paper is the initial work on the adaptive control of such systems.

2. PRELIMINARIES

Before formulating the adaptive control problem, it is useful to describe precisely a fractional Brownian motion and to review the result Kleptsyna et al. [2005] for the solution of the optimal control of a scalar system with a fractional Brownian motion and an ergodic (or average cost per unit time) quadratic cost functional.

A real-valued process $(B(t), t \geq 0)$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a (real-valued) standard fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$\begin{aligned} \mathbb{E}\left[B(t)\right] &= 0 \\ \mathbb{E}\left[B(s)B(t)\right] &= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \end{aligned} \tag{1}$$

for all $s, t \in \mathbb{R}_+$.

Since a (standard) fractional Brownian motion B with the Hurst parameter $H \neq 1/2$ is not a semimartingale, the stochastic calculus for a Brownian motion, or more generally for a continuous square integrable martingale, is not applicable. However, a stochastic calculus for a fractional Brownian motion particularly for $H \in (1/2, 1)$ has been developed (e.g. Alòs and Nualart [2003], Decreusefond and Üstünel [1999], Duncan et al. [2000, 2006], Nualart [2003]) which preserves some of the properties for the (Itô) stochastic calculus for Brownian motion. A few of

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the properties of this stochastic calculus for a fractional Brownian motion with $H \in (1/2, 1)$ are reviewed now.

Let $H \in (1/2, 1)$ be fixed and B be a fractional Brownian motion with Hurst parameter H. For the applications in this paper, only a few results from a stochastic calculus are necessary. Let $f\colon [0,T] \to \mathbb{R}$ be a Borel measurable function. If f satisfies

$$\begin{aligned} &|f|^2_{L^2_H} \\ &= \rho(H) \int_0^T \left(u_{1/2-H}(s) \left| I^{H-1/2}_{T-} \left(u_{H-1/2} f\right)(s) \right| \right)^2 ds \\ &\qquad < \infty \end{aligned}$$

then $f \in L^2_H$ and $\int_0^T f\, dB$ is a zero mean Gaussian random variable with second moment

$$\mathbb{E}\left[\left(\int_0^T f\, dB \right)^2 \right] = |f|^2_{L^2_H} \tag{2}$$

where $u_a(s) = s^a$ for $a > 0$ and $s \geq 0$, $I^{H-1/2}_{T-}$ is a fractional integral defined almost everywhere and given by

$$\left(I^{H-1/2}_{T-} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^T \frac{f(t)}{(t-x)^{3/2-H}}\, dt \tag{3}$$

for $x \in [0,T)$, $f \in L^1([0,T])$ and $\Gamma(\cdot)$ is the gamma function and

$$\rho(H) = \frac{H\Gamma(H+1/2)\Gamma(3/2-H)}{\Gamma(2-2H)}\ .$$

The inverse operator of the fractional integral $I^{H-1/2}$, is called the fractional derivative, $I^{1/2-H}$, and can be given in its Weyl representation as

$$\begin{aligned} (I^{1/2-H}_{T-} F)(x) = \frac{1}{\Gamma(3/2-H)} & \left(\frac{f(x)}{(T-x)^{H-1/2}} \right. \\ & \left. + (H-1/2) \int_x^T \frac{f(s)-f(x)}{(s-x)^{H+1/2}}\, ds \right)\ . \end{aligned} \tag{4}$$

A stochastic integral with respect to a fractional Brownian motion B for $H \in (1/2, 1)$ can also be defined for a stochastic integrand (e.g. Alòs and Nualart [2003], Decreusefond and Üstünel [1999], Duncan et al. [2000, 2006], Nualart [2003]). The integral is a zero mean random variable with an explicit expression for the second moment.

Now the linear-quadratic control problem is reviewed. Let $(X(t), t \geq 0)$ be the real-valued process that satisfies the stochastic differential equation

$$\begin{aligned} dX(t) &= \alpha_0 X(t)\, dt + bU(t)\, dt + dB(t) \\ X(t) &= X_0 \end{aligned} \tag{5}$$

where X_0 is a constant, $(B(t), t \geq 0)$ is a standard fractional Brownian motion with the Hurst parameter $H \in (1/2, 1)$, $\alpha_0 \in [a_1, a_2]$ where $a_2 < 0$, $b \in \mathbb{R} \setminus \{0\}$.

For $t \geq 0$, let $\mathcal{F}_t$ be the P-completion of the sub-σ-algebra $\sigma(B(u), 0 \leq u \leq t)$. The family of sub-$\sigma$-algebras $(\mathcal{F}_t, t \geq 0)$ is called the filtration associated with $(B(t), t \geq 0)$. Let $(U(t), t \geq 0)$ be a process adapted to $(\mathcal{F}_t, t \geq 0)$. It is known that the filtration generated by $(X(t), t \geq 0)$ is the same as the filtration generated by $(B(t), t \geq 0)$. The process U in (5) is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ such that (5) has one and only one solution.

Consider the optimal control problem where the state X satisfies (5) and the ergodic (or average cost per unit time) cost function J is

$$J(U) = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left(qX^2(t) + rU^2(t) \right) dt \tag{6}$$

where $q > 0$ and $r > 0$ are constants. The family $\mathcal{U}$ of admissible controls is all $(\mathcal{F}_t)$ adapted processes such that (5) has one and only one solution.

To introduce some notation, recall the well-known solution with $H = 1/2$, that is $(B(t), t \geq 0)$ is a standard Brownian motion. An optimal control is U^* given by

$$U^*(t) = -\frac{b}{r} \rho_0 X^*(t) \tag{7}$$

where $(X^*(t), t \geq 0)$ is the solution of (5) with the control U^*, ρ_0 is the unique positive solution of the scalar algebraic Riccati equation

$$\frac{b^2}{r} \rho^2 - 2a\rho - q = 0 \tag{8}$$

so

$$\rho_0 = \frac{r}{b^2} [\alpha_0 + \delta_0] \tag{9}$$

$$\delta_0 = \sqrt{\alpha_0^2 + \frac{b^2}{r} q}\ . \tag{10}$$

Furthermore,

$$J(U^*) = \rho_0 \qquad \text{a.s.} \tag{11}$$

The following result is given in Kleptsyna et al. [2005] and solves the analogous control problem for $H \in (1/2, 1)$.

Theorem 1. Let $(U^*(t), t \geq 0)$ be the control given by

$$U^*(t) = -\frac{b}{r} \rho_0 [X^*(t) + V^*(t)] \tag{12}$$

$$\begin{aligned} V^*(t) &= \int_0^t \delta_0 V^*(s)\, ds \\ &\quad + \int_0^t \left[\overline{k}(t,s) - 1 \right] \left(dX^*(s) - \alpha_0 X^*(s) - bU^*(s) \right) ds \\ &= \int_t^\infty e^{-\delta_0 (s-t)}\, dB(s \mid t) \end{aligned} \tag{13}$$

where $(X^*(t), t \geq 0)$ is the solution of (5) with the admissible control $(U^*(t), t \geq 0)$, ρ_0 and δ_0 are given in (9) and (10) respectively, and

$$\overline{k}(t,s) = -c_H^{-1} s^{1/2-H} \frac{d}{ds} \int_s^t (r-s)^{1/2-H} \gamma(r,r)\, dr$$

$$\gamma(t,s) = \delta e^{st} \int_t^\infty e^{\delta_0} K_H(\tau, s)\, d\tau$$

$$K_H(t,s) = H(2H-1) \int_s^\tau r^{H-1/2} (r-s)^{H-3/2}\, dr\ .$$

$$\begin{aligned} B(s \mid t) &= \mathbb{E}[B(s) \mid \mathcal{F}_t] \\ &= B(t) + \int_0^t u_{1/2-H} (I^{1/2-H}_{t-} (I^{H-1/2}_{s-} \mathbb{1}_{[t,s)}))\, dB \\ &= B(t) + \int_0^t u_{1/2-H} (I^{H-1/2}_{s-} u_{H-1/2} \mathbb{1}_{(t,s)})\, dW \end{aligned} \tag{14}$$

where c_H is a constant that only depends on H, $u_a(s) = s^a$ for $s \geq 0$, $I^{H-1/2}$ is the fractional integral (3), $I^{1/2-H}$ is the fractional derivative (4) and $(W(t), t \geq 0)$ is a standard Brownian motion (Wiener process) associated with $(B(t), t \geq 0)$ (e.g. Duncan [2006]).

Then the control U^* is optimal in $\mathcal{U}$ and the optimal cost is

$$J(U^*) = \lambda \qquad \text{a.s.} \tag{15}$$

where

$$\lambda = \frac{q\Gamma(2H+1)}{2\delta_0^{2H}}\left[1 + \frac{\delta_0 + \alpha_0}{\delta_0 - \alpha_0}\sin \pi H\right] . \tag{16}$$

If α_0 is unknown, then it is important to find a family of strongly consistent estimators of the unknown parameter α_0 in (5). A method is used in Duncan and Pasik-Duncan [2001, 2002] that is called pseudo-least squares because it uses the least squares estimate for α_0 assuming $H = 1/2$, that is, B is a standard Brownian motion in (5). It is shown in Duncan and Pasik-Duncan [2001, 2002] that the family of estimators $(\hat{\alpha}(t), t \geq 0)$ is strongly consistent for $H \in (1/2, 1)$ where

$$\hat{\alpha}(t) = \alpha_0 + \frac{\int_0^t X^0(s)\, dB(s)}{\int_0^t (X^0(s))^2\, ds} \tag{17}$$

where

$$\begin{aligned} dX^0(t) &= \alpha_0 X^0(t)\, dt + dB(t) \\ X^0(0) &= X_0 \end{aligned} \tag{18}$$

This family of estimators can be obtained from (5) by removing the control term. The family of estimators $\hat{\alpha}$ is modified here using the fact that $\alpha_0 \in [a_1, a_2]$ as

$$\begin{aligned} \alpha(t) = {} & \hat{\alpha}(t)\mathbb{1}_{[a_1,a_2]}(\hat{\alpha}(t)) \\ & + a_1\mathbb{1}_{(-\infty,a_1)}(\hat{\alpha}(t)) + a_2\mathbb{1}_{(a_2,\infty)}(\hat{\alpha}(t)) \end{aligned} \tag{19}$$

for $t \geq 0$. $\hat{\alpha}(0)$ is chosen arbitrarily in $[a_1, a_2]$.

The solution of the stochastic equation (5) is obtained by the usual variation of parameters method and is given by

$$X(t) = e^{\alpha_0 t}X_0 + \int_0^t e^{\alpha_0(t-s)}\left(U(s)\, ds + dB(s)\right) . \tag{20}$$

For the optimal control $(U^*(t), t \geq 0)$, the corresponding solution $(X^*(t), t \geq 0)$ can be expressed as

$$\begin{aligned} X^*(t) = {} & e^{-\delta_0 t}X_0 \\ & + \int_0^t e^{-\delta_0(t-s)}[-(\alpha_0 + \delta_0)V^*(s)\, ds + dB(s)] , \end{aligned}$$

where

$$\begin{aligned} dX^*(t) &= \alpha_0 X^*(t)\, dt - \frac{b^2}{r}\rho_0[X^*(t) + V^*(t)]\, dt + dB(t) \\ &= -\delta_0 X^*(t)\, dt - (\alpha_0 + \delta_0)V^*(t)\, dt + dB(t) . \end{aligned} \tag{21}$$

An adaptive control $(U^\wedge(t), t \geq 0)$, is obtained from the certainty equivalence principle, that is, at time t, the estimate $\alpha(t)$ is assumed to be the correct value for the parameter. Thus the stochastic equation for the system (5) with the control $U^\wedge$ is

$$\begin{aligned} dX^\wedge(t) &= (\alpha_0 - \alpha(t) - \delta(t))X^\wedge(t)\, dt \\ &\quad - \frac{b\rho(t)}{r}V^\wedge(t)\, dt + dB(t) \\ &= (-\alpha_0 - \alpha(t) - \delta(t))X^\wedge\, dt \\ &\quad - (\alpha(t) + \delta(t))V^\wedge(t)\, dt + dB(t) \\ X^\wedge(0) &= X_0 \end{aligned} \tag{22}$$

and

$$\delta(t) = \sqrt{\alpha^2(t) + \frac{b^2}{r}q} \tag{23}$$

$$U^\wedge(t) = -\frac{b\rho(t)}{r}\left[X^\wedge(t) + V^\wedge(t)\right] \tag{24}$$

$$\rho(t) = \frac{r}{b^2}\left[\alpha(t) + \delta(t)\right] \tag{25}$$

$$\begin{aligned} V^\wedge(t) &= \int_0^t \tilde{\delta}(s)V^\wedge(s)\, ds \\ &\quad + \int_0^t [\tilde{k}(t,s) - 1] \\ &\qquad [dX^\wedge(s) - \alpha(s)X^\wedge(s)\, ds - bU^\wedge(s)\, ds] \\ &= \int_0^t \tilde{\delta}(s)V^\wedge(s)\, ds \\ &\quad + \int_0^t [\tilde{k}(t,s) - 1][dB(s) + (\alpha_0 - \alpha(t))X^\wedge(s)\, ds] \end{aligned} \tag{26}$$

$$\tilde{\delta}(t) = \delta(t) + \alpha(t) - \alpha_0 \tag{27}$$

and $\tilde{k}$ denotes the use of $\tilde{\delta}$ instead of δ_0 in $\bar{k}$. Note that $\delta(t) \geq -\alpha(t) + c$ for some $c > 0$ and all $t \geq 0$ so that

$$\alpha_0 - \alpha(t) - \delta(t) < -c .$$

The solution of the stochastic equation (22) is

$$\begin{aligned} X^\wedge(t) = {} & e^{-\int_0^t \hat{\delta}}X_0 \\ & + \int_0^t e^{-\int_s^t \hat{\delta}}[-(\alpha(s) + \delta(s)]V^\wedge(s)\, ds + dB(s)] . \end{aligned}$$

The following result states that the adaptive control $(U^\wedge(t), t \geq 0)$ is self-optimizing in $L^2(P)$, that is, the family of average costs converge in $L^2(P)$ to the optimal average cost (15).

Theorem 2. Let the scalar-valued control system satisfy the equation (5). Let $(\alpha(t), t \geq 0)$ be the family of estimators of α_0 given by (19), let $(U^\wedge(t), t \geq 0)$ be the associated adaptive control in (24), and let $(X^\wedge(t), t \geq 0)$ be the solution of (5) with the control $U^\wedge$. Then

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left|U^*(s) - U^\wedge(s)\right|^2\, ds = 0 \tag{28}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left|X^*(s) - X^\wedge(s)\right|^2\, ds = 0 \tag{29}$$

so

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left(q(X^\wedge(s))^2 + r(U^\wedge(s))^2\right)\, ds = \lambda \tag{30}$$

where λ is given in (16).

Proof. Initially, it is verified that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left|U^*(s) - U^\wedge(s)\right|^2\, ds = 0 .$$

$$\begin{aligned} U^*(t) - U^\wedge(t) = {} & -(\alpha_0 + \delta_0))[X^*(t) + V^*(t)] \\ & + (\alpha(t) + \delta(t))[X^\wedge(t) + V^\wedge(t)] . \end{aligned} \tag{31}$$

Consider the following terms on the right-hand side of the (31):

$$
\begin{aligned}
-(\alpha_0+\delta_0)V^*(t)&+(\alpha(t)+\delta(t))V^\wedge(t)\\
&=(\alpha_0+\delta_0)(V^\wedge(t)-V^*(t))\\
&\quad+(\alpha(t)+\delta(t)-\alpha_0-\delta_0)V^\wedge(t) \qquad (32)\\
&=(\alpha(t)+\delta(t))(V^\wedge(t)-V^*(t))\\
&\quad+(\alpha(t)+\delta(t)-\alpha_0-\delta_0)V^*(t)\ .
\end{aligned}
$$

Using the Dominated Convergence Theorem, it is straightforward to verify that

$$
\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left|(\alpha(s)+\delta(s)-\alpha_0-\delta_0)V^*(s)\right|^2\,ds=0\ .
$$

For the other term that arises in (32), it suffices to consider

$$
\mathbb{E}\int_0^t\left|V^*(s)-V^\wedge(s)\right|^2\,ds\ . \qquad (33)
$$

Since α_0 is unknown, $dB(t)$ cannot be obtained from $dX^\wedge(t)$. However, there is the equality

$$
\begin{aligned}
dX^\wedge(t)-U^\wedge(t)\,dt-\alpha(t)X(t)\,dt&=dB(t)\\
&\quad+(\alpha_0-\alpha(t))X^\wedge(t)\,dt\ . \qquad (34)
\end{aligned}
$$

Let $\tilde\delta(t)$ be given by

$$
\tilde\delta(t)=\delta(t)+\alpha(t)-\alpha_0\ . \qquad (35)
$$

Using (34) and (35) it follows that (26) can be expressed as

$$
\begin{aligned}
V^\wedge(t)&=\int_t^\infty e^{-\int_t^\tau\hat\delta}\,dB(\tau\mid t)\\
&\quad+\int_t^\infty e^{-\int_t^\tau\tilde\delta}\mathbb{E}[(\alpha_0-\alpha(s))\hat X(s)\mid\mathcal{F}_t]\,ds\ . \qquad (36)
\end{aligned}
$$

Initially an upper bound is provided for the second term in the right hand side of (36).

$$
\begin{aligned}
&\left|\int_t^\infty e^{-\int_t^\tau\tilde\delta}\mathbb{E}[(\alpha_0-\alpha(s))\hat X(s)\mid\mathcal{F}_t]\,ds\right|\\
&-\left|\int_t^\infty e^{-\int_t^\tau\tilde\delta}\int_0^t u_{1/2-H}(I_{t-}^{1/2-H}(I_{s-}^{H-1/2}u_{H-1/2}\mathbb{1}_{[t,s)}))\right.\\
&\qquad\left.(\alpha_0-\alpha(s)X^\wedge(s))\,ds\right|\\
&\le\int_t^\infty e^{-\int_t^\tau\tilde\delta}\left|\int_0^t u_{1/2-H}(I_{t-}^{1/2-H}(I_{s-}^{H-1/2}u_{H-1/2}\mathbb{1}_{[t,s)}))\right.\\
&\qquad\left.(\alpha_0-\alpha(s)X^\wedge(s))\right|\,ds\\
&\le\int_t^\infty e^{-\int_t^\tau\tilde\delta}\left|\int_0^t u_{1/2-H}(I_{t-}^{1/2-H}(I_{s-}^{H-1/2}u_{H-1/2}\mathbb{1}_{[t,s)}))\right.\\
&\qquad\left.(\alpha_0-\alpha(s)X^\wedge(s))\right|^2\,ds\\
&\qquad\le\int_t^\infty e^{-\int_t^s\tilde\delta}\left|(\alpha_0-\alpha(s))X^\wedge(s)\right|^2\,ds\ .
\end{aligned}
$$

The next to the last inequality uses Jensen's inequality and the last inequality follows because the prediction (projection) of $((\alpha_0-\alpha(s)X(s),s\ge0)$ is replaced by the true process.

It can be shown by a comparison with another stochastic differential equation (e.g. (18)) that

$$
\mathbb{E}\left[(X^\wedge(t))^2\right]\le K
$$

for all $\tau\ge0$. Using the Dominated Convergence Theorem and Jensen's inequality, it follows that

$$
\mathbb{E}\int_s^\infty\left|e^{-c(\tau-s)}X^\wedge(\tau)\right|^2\,d\tau\le M\ .
$$

Thus

$$
\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left|\int_s^\infty e^{\int_s^\tau\tilde\delta}(\alpha_0-\alpha(\tau))X^\wedge(\tau)\,d\tau\right|^2\,ds=0\ .
$$

Now consider the other term that arises in (33), that is

$$
\mathbb{E}\int_0^t\left|\int_s^\infty\left(e^{-\int_s^\lambda\hat\delta}-e^{\delta_0(\tau-s)}\right)\,dB(\tau\mid s)\right|^2\,ds
$$

The term $I_{\tau-}^{H-1/2}$ is a fractional integral given by

$$
\begin{aligned}
&I_{\tau-}^{H-1/2}(u_{H-1/2}\mathbb{1}_{[s,\tau]})(r)\\
&\quad=\frac{1}{\Gamma(H-1/2)}\int_r^\tau\frac{q^{H-1/2}\mathbb{1}_{[s,\tau]}(q)}{(q-r)^{3/2-H}}\,dq\\
&\qquad=\frac{1}{\Gamma(H-1/2)}\int_s^\tau\frac{q^{H-1/2}}{(q-r)^{3/2-H}}\,dq\ . \qquad (37)
\end{aligned}
$$

An elementary upper bound for (37) is obtained as follows:

$$
\begin{aligned}
&\left|I_{\tau-}^{H-1/2}(u_{H-1/2}\mathbb{1}_{[s,\tau]})(r)\right|\\
&\quad\le\frac{1}{\Gamma(H-1/2)}\tau^{H-1/2}\int_s^\tau(q-r)^{H-3/2}\,dq\\
&=\frac{1}{\Gamma(H+1/2)}\frac{\tau^{H-1/2}}{H-1/2}\left[(\tau-r)^{H-\frac12}-(s-r)^{H-\frac12}\right]\ . \qquad (38)
\end{aligned}
$$

Now using (26),

$$
\begin{aligned}
&\int_0^\infty\left(e^{-\int_s^\lambda\hat\delta}-e^{\delta_0(\tau-s)}\right)\,dB(\tau\mid s)\\
&\qquad=\int_s^\infty\left(e^{-\int_s^\lambda\hat\delta}-e^{\delta_0(\tau-s)}\right)\\
&\quad\cdot\frac{\partial}{\partial\tau}\int_0^s r^{1/2-H}\left(I_{\tau-}^{H-1/2}u_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\,dW(r)\,d\tau\ . \qquad (39)
\end{aligned}
$$

Fix t and $s\in(0,t)$ and perform an integration by parts for the integration on the right-hand side of (39) to obtain

$$
\begin{aligned}
&\left(e^{-\int_s^\tau\tilde\delta}-e^{\delta_0(\tau-s)}\right)\\
&\cdot\left(\int_0^s r^{1/2-H}\left(I_{\tau-}^{H-1/2}\tilde\delta_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\,dW(r)\right)\Big|_{\tau=s}^{\tau=\infty}\\
&\quad-\int_s^\infty\left(e^{-\int_s^\tau\tilde\delta}(-\tilde\delta(\tau))-e^{\delta_0(\tau-s)}(-\delta_0)\right)\\
&\quad\cdot\int_0^s r^{1/2-H}\left(I_{\tau-}^{H-1/2}u_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\,dW(r)\,d\tau\\
&\quad=-\int_s^\infty\left(e^{-\int_s^\lambda\tilde\delta}(-\tilde\delta(\tau))-e^{\delta_0(\tau-s)}(-\delta_0)\right)\\
&\cdot\int_0^s r^{1/2-H}\left(I_{\tau-}^{H-1/2}u_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\,dW(r)\,d\tau\ . \qquad (40)
\end{aligned}
$$

The first two terms on the left-hand side of (40) are zero, resulting in this equality. The term for $\tau=\infty$ is easily verified to be zero by a limit computation. Use a stochastic Fubini theorem for an ordinary integral and a stochastic integral with respect to a standard Brownian motion to rewrite the order of integration on the right-hand side of (40) and compute the expectation as

$$\mathbb{E}\left|\int_0^s \int_s^\infty \left(-\delta(\tau)e^{-\int_s^\tau \tilde{\delta}} + \delta_0 e^{\delta_0(\tau-s)}\right) r^{1/2-H} \cdot \left(I_{\tau-}^{H-1/2} U_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\, d\tau\, dW(r)\right|^2$$
$$= \mathbb{E}\int_0^s \left|\int_s^\infty \left(-\tilde{\delta}(\tau)e^{-\int_s^\tau \tilde{\delta}} + \delta_0 e^{\delta_0(\tau-s)}\right) r^{1/2-H} \cdot \left(I_{\tau-}^{H-1/2} u_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\, d\tau\right|^2 dr \ . \quad (41)$$

Consider the integrand on the right-hand side for the integral with respect to r and let $A(\cdot)$ be this integrand

$$A(s) = \left|\int_s^\infty \left(-\tilde{\delta}(\tau)e^{-\int_s^\tau \tilde{\delta}} + \delta_0 e^{\delta_0(\tau-s)}\right) r^{1/2-H} \cdot \left(I_{\tau-}^{H-1/2} U_{H-1/2}\mathbb{1}_{[s,\tau]}\right)(r)\, d\tau\right|^2$$
$$\leq K \int_s^\infty \left|\left(-\tilde{\delta}(\tau)e^{-\int_s^\tau \tilde{\delta}} + \delta_0 e^{\delta_0(\tau-s)}\right) \cdot \left[(\tau-r)^{H-1/2} - (s-r)^{H-1/2}\right]\right| d\tau \leq M \quad (42)$$

where the inequality (38) is used and K and M are constants. By the Dominated Convergence Theorem,

$$\lim_{s\to\infty} A(s) = 0 \qquad \text{a.s.}$$

and

$$\lim_{s\to\infty} \mathbb{E}A(s) = 0 \ .$$

It follows directly that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left|\int_s^\infty \left(e^{-\int_s^\tau \tilde{\delta}} - e^{\delta_0(\tau-s)}\right) dB(\tau \mid s)\right|^2 ds = 0. \quad (43)$$

Now, consider the term

$$\mathbb{E}\int_0^t \left|X^\wedge(s) - X^*(s)\right|^2 ds \ . \quad (44)$$

A stochastic equation is obtained for $X^\wedge - X^*$ as

$$\begin{aligned} d(X^\wedge(t) - X^*(t)) &= (\alpha_0 - \alpha(t) - \delta(t))X^\wedge(t)\,dt \\ &\quad + \delta_0 X^*(t)\,dt - (\alpha(t)+\delta(t))V^\wedge(t)\,dt \\ &\quad + (\alpha_0+\delta_0)V^*(t)\,dt \\ &= (\alpha_0 - \alpha(t) - \delta(t))(X^\wedge(t) - X^*(t))\,dt \\ &\quad + (\alpha_0 - \alpha(t) - \delta(t) + \delta_0)X^*(t)\,dt \\ &\quad - (\alpha(t)+\delta(t))V^\wedge(t)\,dt + (\alpha_0+\delta_0)V^*(t)\,dt \ . \end{aligned} \quad (45)$$

Thus

$$\begin{aligned} X^\wedge(t) - X^*(t) &= \int_0^t e^{-\int_s^t \tilde{\delta}}(\alpha_0 - \alpha(t) - \delta(t) + \delta_0)X^*(s)\,ds \\ &- \int_0^t e^{-\int_s^t \tilde{\delta}}[(\alpha(s)+\delta(s))V^\wedge(s) - (\alpha_0+\delta_0)V^*(s)\,ds \ . \end{aligned} \quad (46)$$

Now consider (44). The first term on the right-hand side above substituted into (44) has limit zero by the inequality for $\tilde{\delta}$. Since

$$\begin{aligned} &(\alpha_0+\delta_0)V^*(t) - (\alpha(t)+\delta(t))V^\wedge(t) \\ &\quad = (\alpha_0+\delta_0-\alpha(t)-\delta(t))V^*(t) \\ &\qquad - (\alpha(t)+\delta(t))(V^\wedge(t) - V^*(t)) \ , \end{aligned}$$

the second term in (46) substituted in (44) has limit zero by the above results to verify that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left|V^\wedge(s) - V^*(s)\right|^2 ds = 0 \ .$$

This completes the proof of the equality (30).

Clearly it is important to solve a linear quadratic control problem for a multi-dimensional version of (5) and then to solve a corresponding adaptive control problem. Furthermore, even for this scalar problem it is desirable to replace the $L^2(P)$ convergence in (30) by almost sure convergence.

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