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A note on robust output feedback stabilization of nonlinear nonminimum phase systems [1]

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Abstract: This paper investigates a few issues related to the problem of robust output feedback stabilization of nonlinear non-minimum phase systems. In order to cope with (unstructured) parametric uncertainties we insist on well-known high-gain design principles which, however, are integrated with robust zero-assignment procedures to handle possible unstable zero dynamics. In this respect we propose two possible zero-assignment arguments. The first relies upon an output redesign obtained by means of a "feed-through" compensator. Interestingly enough, we show how existing results on this subject can be cast in terms of the resulting feed-through / high-gain design paradigm. In the second, by drawing inspiration from known results for linear systems which go under the name of *vibrational-feedback*, the output redesign is achieved by using time-varying periodic controllers. Remarkably, we show how the resulting design framework is able to deal also with severe uncertainties in the high-frequency gain of the controlled system by thus obtaining results which turn out to be interesting also in a linear setting.

Keywords: Output feedback, Robust Control, Nonminimum-phase, High-gain observers.

1. INTRODUCTION

We consider a smooth nonlinear system of the form

$$\dot{x} = f(x,u) \qquad y = h(x) \qquad x \in \mathbb{R}^n \tag{1}$$

with control input $u \in \mathbb{R}$, measurable output $y \in \mathbb{R}$ and initial condition in a compact set $X \subset \mathbb{R}^n$. For this system we address the problem of output feedback stabilization which amount to design a controller of the form

$$\dot{\chi} = \alpha(\chi, y) \qquad u = \beta(\chi, y) \tag{2}$$

such that the origin of the resulting closed-loop system is locally asymptotically stable with a domain of attraction whose projection in the x-space contains X. Implicit in the above description is the presence of possible uncertain parameters which affect the controlled dynamics whose values range in possibly large known compact sets.

It is a well-known fact that the problem at hand has a meaningful solution in the case the controlled system (1) has a well-defined uniform [2] relative degree, namely it is globally diffeomorphic to a system in normal form, and the system is minimum-phase, namely the associated zero dynamics are asymptotically stable with a proper domain of attraction. As a matter of fact, in this case, the theory in Teel and Praly (1995) shows how a combination of a robust high-gain observer and high-gain output feedback does the job. Roughly, the asymptotic stability of the zero dynamics represents the crucial property to have a system with an infinity gain margin which makes it possible the adoption of high gain control laws which, in turn, are crucial to offset the effect of (possibly large) parametric uncertainties.

In this note we explore possible solutions insisting on the high/gain paradigm in the case the system is non-minimum phase. The solution necessarily passes through a zeros assignment procedure which can be obtained by defining a new output (and possibly a new input) with respect to which the gain margin is infinity. In other words the idea is to figure out a way to identify a new input-output pair with respect to which the zeros dynamics are asymptotically stable. Then, the stabilization can be achieved by adopting the "usual" high-gain paradigm with respect to the new input-output pair. In these terms there are two possible ways to robustly assign zeros: the first is by means of *feed-through compensators* while the second is by means of *time-varying periodic fast controllers*. These two methods and related considerations are discussed in the next two subsections.

2. OUTPUT FEEDBACK STABILIZATION VIA FEED-THROUGH AND HIGH-GAIN

A natural way to deal with the robust output feedback stabilization of nonminimum-phase systems is the one relying upon a feed-through compensator of the form

$$\begin{aligned} \dot{\eta} &= \varphi_0(\eta) + \varphi_1(\eta) v \\ u &= \gamma_0(\eta) + \gamma_1(\eta) v \\ y_f &= r(\eta)\,. \end{aligned} \tag{3}$$

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[2] with respect to the uncertain parameter.

feeding the controlled system through the function $\gamma_0(\eta)+\gamma_1(\eta)v$ and generating a feed-through signal $r(\cdot)$ which, along with the measurable output y, induces a new output $\tilde{y} := y - y_f$. The objective of the feed-through compensator is to generate a system of the form

$$\begin{aligned} \dot{x} &= f(x, \gamma_0(\eta) + \gamma_1(\eta)v) \\ \dot{\eta} &= \varphi_0(\eta) + \varphi_1(\eta)v \\ \tilde{y} &= h(x) - r(\eta) \end{aligned} \tag{4}$$

which, with respect to the input v and output $\tilde{y}$, is minimum-phase so that conventional high-gain techniques can be adopted to close the loop between the two signals $\tilde{y}$ and v.

Proposition 1. Suppose that system (4) is affine in the input v and that there exists a feed-through compensator of the form (3) such that system (4) with input v and output $\tilde{y}$ has a well-defined relative degree and the origin of the associated zero dynamics are asymptotically stable (locally exponentially [3]) with a a domain of attraction $\mathcal{M} \times \mathcal{X}$ with $X \subset \mathcal{X}$. Then there exists a dynamic output feedback stabilizer of the form (2) solving the problem at hand.

Proof. The result immediately follows by the results of Teel and Praly (1995). In particular if the relative degree of the system (1)-(3) with input v and output $\tilde{y}$ is one, it follows that there exists a positive $\kappa^\star$ such that for all possible $\kappa \geq \kappa^\star$ the output feedback controller

$$\begin{aligned} \dot{\eta} &= \varphi_0(\eta) + \varphi_1(\eta)v \\ u &= \gamma_0(\eta) + \gamma_1(\eta)v \\ v &= \kappa[y - r(\eta)] \end{aligned}$$

solves the problem at hand. In the case of higher relative degree, the use of high-gain dirty derivatives observers makes it possible to solve the problem as proposed in Teel and Praly (1995).

Interestingly enough, it is possibly to show that the existence of a feed-through compensator of the form (3) is not only a sufficient condition to design an output feedback stabilizer (as claimed by the previous theorem) but, indeed, also necessary. In other words the existence of an output feedback stabilizer for (1) necessarily implies the existence of a feed-through compensator of the form (3) yielding a minimum phase system (4).

Proposition 2. Assume that there exists a smooth stabilizer of the form

$$\begin{aligned} \dot{\chi} &= L(\chi) + M(\chi)y \\ u &= N(\chi) + d(\chi)y \end{aligned} \tag{5}$$

with $d(\chi) \neq 0$ such that the closed-loop system (1), (5) is asymptotically (locally exponentially) stable with domain of attraction $\mathcal{X} \times \mathcal{C}$ with $X \subset \mathcal{X}$. Then there exists a feed-through system of the form [4]

$$\begin{aligned} \dot{\eta}_1 &= v \\ \dot{\eta}_2 &= L(\eta_2) - \frac{1}{d(\eta_2)} M(\eta_2)N(\eta_2) + \frac{1}{d(\eta_2)} M(\eta_2)\eta_1 \\ u &= \eta_1 \\ y_f &= \frac{1}{d(\eta_2)}[-N(\eta_2) + \eta_1] \end{aligned} \tag{6}$$

such that system (1), (6), with input v and output $\tilde{y} = y - y_f$, has relative degree one and the associated zero dynamics coincide with the closed-loop dynamics of (1), (5).

The proof is by direct check.

Remark. It is worth noting that the previous result relies upon the somehow restrictive assumption that the stabilizer (5) is affine in the measured output y. Furthermore we note that it has been assumed a stabilizer (5) with zero relative degree ($d(\chi) \neq 0$). Indeed, it can be proved that the result holds also in the case the output feedback stabilizer of the form

$$\begin{aligned} \dot{\chi} &= L(\chi) + M(\chi)y \\ u &= N(\chi)\,. \end{aligned} \tag{7}$$

In fact, consider the regulator

$$\begin{aligned} \dot{\chi} &= L(\chi) + M(\chi)y \\ u &= N(\chi) + \epsilon y \end{aligned} \tag{8}$$

with ϵ a positive parameter. If the closed-loop of (1) with (7) is asymptotically stable with a domain of attraction $\mathcal{X} \times \mathcal{M}$ with $X \subset \mathcal{X}$, Lyapunov arguments can be used to prove that there exists an $\epsilon^\star$ such that for all $\epsilon \leq \epsilon^\star$ also the closed-loop of (1) with (8) is asymptotically stable with a domain of attraction $\mathcal{X}' \times \mathcal{M}'$ with $X \subset \mathcal{X}'$. From this the previous proposition yields that a feed-through compensator of the form (6) with $d(\chi)$ replaced by ϵ has the desired properties.

Clearly, the previous result about the existence of the feed-through compensator is useless for design purposes as it starts from the knowledge of an output feedback stabilizer. Thus the previous framework to stabilize systems via "feed-through" and "high-gain" seems useful for design purposes if one finds a way to stabilize the zero dynamics of system (4) without relying upon the knowledge of a system stabilizer. In the following we present a few developments along this direction which allow us to frame, in the previous "feed-through" and "high gain" paradigm, the approach proposed in Isidori (2000). To this purpose we restrict our attention on the class of systems (1) which are affine in the input and characterized by a well-defined unitary relative-degree so that they can be transformed in the normal form

$$\begin{aligned} \dot{z} &= f(z, y) \\ \dot{y} &= q(z, y) + b(z, y)u \end{aligned} \tag{9}$$

with $b(z, y) \neq 0$ the high-frequency gain. In the case the relative degree is larger than one (namely if y in (9) is not the measured output but one of its time derivative) the techniques in Teel and Praly (1995) allow one to extend in a straightforward way the results we are going to present. Coherently with the state partition in (9) we denote by

[3] the local exponential stability requirement can be weakened without adding, however, conceptual value to the result

[4] see Misra and Patel (1988) for linear systems.

$Z \times Y$ the compact set of initial condition where the initial state of (9) is supposed to range.

Motivated by the framework in Isidori (2000) we associate to system (9) the "auxiliary system"

$$\begin{aligned} \dot{z} &= f(z, u_a) \\ y_a &= q(z, u_a) \end{aligned} \tag{10}$$

with auxiliary input u_a and output y_a and we make the following stabilizability assumption (which, as shown in Isidori (2000), is necessary for the problem at hand to be solvable in case of linear systems):

Assumption AUX. There exists an "auxiliary controller" of the form

$$\begin{aligned} \dot{\eta} &= \varphi_0(\eta) + \varphi_1(\eta) y_a \\ u_a &= r(\eta) \end{aligned} \tag{11}$$

such that the closed auxiliary loop (10)-(11) is locally exponentially stable with a domain of attraction which projected in the z-space contains the set Z.

Under this assumption it is possible to show that, if the high-frequency gain $b(z, y)$ of system (9) is known, i.e. $b(z, y) = b(y)$ with $b(y)$ not affected by uncertainties, a feed-through compensator of the form (3) can be designed so that the zero dynamics of system (9)-(3) with respect to the input v and output $\tilde{y} = y - y_f$ are asymptotically stable. As a matter of fact, the constraints $\tilde{y} = 0$ and $\dot{\tilde{y}} = 0$ yield that, along the zero dynamics of (9)-(3), the following holds

$$\begin{aligned} &\frac{\partial r}{\partial \eta} [\varphi_0(\eta) + \varphi_1(\eta) v] = \\ &q(z, r(\eta)) + b(r(\eta)) [\gamma_0(\eta) + \gamma_1(\eta) v] \end{aligned} \tag{12}$$

This expression suggests the choice

$$\begin{aligned} \gamma_0(\eta) &= \frac{1}{b(r(\eta))} \frac{\partial r}{\partial \eta} \varphi_0(\eta) \\ \gamma_1(\eta) &= \frac{1}{b(r(\eta))} \left[\frac{\partial r}{\partial \eta} \varphi_1(\eta) - 1 \right] \end{aligned} \tag{13}$$

yielding a zero dynamics

$$\begin{aligned} \dot{z} &= f(z, r(\eta)) \\ \dot{\eta} &= \varphi_0(\eta) + \varphi_1(\eta) q(z, r(\eta)) \end{aligned} \tag{14}$$

which, by assumption, is asymptotically stable. From this it turns out, by following the arguments of Proposition 1 and by observing that, by the previous choice, the relative degree between the output $\tilde{y}$ and the input v is one, that there exists a $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the controller

$$\begin{aligned} \dot{\eta} &= \varphi_0(\eta) + \varphi_1(\eta) v \\ u &= \frac{1}{b(r(\eta))} \left[\frac{\partial r}{\partial \eta} \varphi_0(\eta) + \left(\frac{\partial r}{\partial \eta} \varphi_1(\eta) - 1 \right) v \right] \\ v &= \kappa (y - r(\eta)) \end{aligned} \tag{15}$$

solves the problem at hand. It is interesting to note that the final controller so-obtained is the same as the one proposed [5] in Isidori (2000) which, as a consequence, can be seen as a particular way of designing output feedback controller via feed-through and high-gain.

It is worth noting how the knowledge of the high frequency gain plays a crucial role in the previous control structure. In the following part we make a few attempts to show how the not perfect knowledge of the term $b(z, y)$ can be tolerated in the previous paradigm. In particular we show how the most natural choice which one would make to extend the previous results in presence of an uncertain $b(z, y)$ leads to the considerations and results reported in Isidori (1999).

The same computations carried out before to compute the zero dynamics of system (9), (3) lead to the relation (12) with $b(r(\eta))$ replaced by $b(z, r(\eta))$. This, inspired by the choice (13), suggests to take

$$\gamma_0(\eta) = \frac{1}{b_0} \frac{\partial r}{\partial \eta} \varphi_0(\eta) \quad \gamma_1(\eta) = \frac{1}{b_0} \left[\frac{\partial r}{\partial \eta} \varphi_1(\eta) - 1 \right]$$

with b_0 being the nominal value of the high frequency gain at the equilibrium, yielding the following relation

$$\begin{aligned} v &= q(z, N(\eta)) - \Delta w \\ w &= \left[\frac{\partial r}{\partial \eta} \varphi_1(\eta) v - v + \frac{\partial r}{\partial \eta} \varphi_0(\eta) \right] \end{aligned} \tag{16}$$

with $\Delta = \frac{b_0 - b(z, r(\eta))}{b_0}$. Note that if $b(z, r(\eta)) = b(r(\eta))$ were known, the choice $b_0 = b(r(\eta))$ would yield $\Delta = 0$ and, in turn, the same ideal controller obtained before. The overall zero dynamics are thus given by the feedback composition of the auxiliary system (10) with the system

$$\begin{aligned} \dot{\eta} &= \varphi_0(\eta) + \varphi_(\eta) v \\ u_a &= r(\eta) \\ w &= \frac{\partial r}{\partial \eta} \varphi_1(\eta) v - v + \frac{\partial r}{\partial \eta} \varphi_0(\eta) \end{aligned}$$

with v given by $v = y_a - \Delta w$ (see (16)). According to this the zero dynamics have the desired asymptotic properties if the assumption AUX before is strengthened by asking that the "ideal" interconnection (14) remains asymptotically stable under the effect of the extra perturbation Δw which, in turn, requires the term Δ to be "sufficiently small". These are precisely the same considerations and conclusions drawn in Isidori (1999).

The previous considerations highlights a structural limitation in the stabilization paradigm based on robust zeros assignment via feed-through to deal with (severe) uncertainties in the high frequency. In the next section we show how this limitation can be overcome, for a particular class of nonlinear systems, by using time-varying controllers.

3. ZERO-DYNAMICS ROBUST STABILIZATION VIA VIBRATIONAL FEEDBACK

It is a well-known fact for linear systems that fast periodic controllers have the ability to assign zeros (see Lee et al. (1987)). In this part we show how the same can be obtained in a nonlinear framework and, above all, how severe uncertainties on the high-frequency gain can be dealt with in this framework.

We consider the special class of nonlinear systems described by

[5] Indeed the framework of Isidori (2000) considered an auxiliary controller (11) and a final controller (15) with an "input map" $\varphi_1(\cdot)$ not state dependent ($\varphi_1(\cdot) = M$). However this limitation could be easily removed even in Isidori (2000).

$$\begin{aligned} \dot{z} &= f(z,y,\mu) \\ \dot{y} &= q(z,y,\mu) + b(\mu)u \end{aligned} \tag{17}$$

with input u, measured output y, initial condition in a known compact set $\mathcal{Z} \times \mathcal{Y}$, in which μ is a vector of uncertain parameters ranging in a known compact set, $b(\mu)$ is the high-frequency gain assumed constant, and $f(\cdot)$, $q(\cdot)$ are functions which are assumed affine in the output y, namely

$$\begin{aligned} f(z,y,\mu) &= f_0(z,\mu) + f_1(z,\mu)y \\ q(z,y,\mu) &= q_0(z,\mu) + q_1(z,\mu)y\,. \end{aligned} \tag{18}$$

The main goal is to design a semiglobal output feedback stabilizer which does not depend explicitly on the high-frequency gain as in the existing stabilization frameworks.

The main assumption regards the existence of a *linear* output feedback stabilizer for the *auxiliary system* (see Isidori (2000))

$$\begin{aligned} \dot{z} &= f(z,u_a',\mu) \\ y_a &= q(z,u_a',\mu) \end{aligned} \tag{19}$$

with input u_a' and output y_a. More specifically it is assumed the following:

Assumption VIB. *There exists a triplet (F,G,N), such that (19) in closed-loop with*

$$\begin{aligned} \dot{\eta} &= F\eta + Gy_a \\ u_a' &= N\eta \end{aligned} \tag{20}$$

is globally asymptotically (locally exponentially) stable [6].

The proposed controller is of the form

$$\begin{aligned} \dot{\eta} &= F\eta + B(\frac{t}{\epsilon})\kappa y \\ u &= \dot{\beta}(\frac{t}{\epsilon})\eta + \beta(\frac{t}{\epsilon})\left[F\eta + B(\frac{t}{\epsilon})\kappa y\right] + u_a(\frac{t}{\epsilon}) \end{aligned} \tag{21}$$

where ϵ and κ are design parameters, $u_a(t/\epsilon)$ is an auxiliary control input yet to be chosen, $\beta(t/\epsilon)$ and $B(t/\epsilon)$ are time-varying (row and column) vectors, with $B(\cdot)$ of the form $B(t/\epsilon) = B_0 k(t/\epsilon)$ with B_0 a column vector and $k(t/\epsilon)$ a scalar function, to be chosen. The time varying entries of the control law are supposed to be functions with a well-defined averaged description (for instance periodic functions).

Consider the change of variable

$$y \mapsto \tilde{y} := y - b(\mu)\beta(\frac{t}{\epsilon})\eta$$

which transforms the closed-loop system as

[6] Global Stability can be weaken to local asymptotic stability with a suitable domain of attraction. Furthermore, without loss of generality, we assume $G^{\mathrm{T}}G = 1$.

$$\begin{aligned} \dot{z} &= f(z,\tilde{y} + b(\mu)\beta(\frac{t}{\epsilon})\eta,\mu) \\ \dot{\tilde{y}} &= q(z,\tilde{y} + b(\mu)\beta(\frac{t}{\epsilon})\eta,\mu) + b(\mu)u_a(\frac{t}{\epsilon}) \\ \dot{\eta} &= F\eta + B_o k(\frac{t}{\epsilon})\kappa\left[\tilde{y} + b(\mu)\beta(\frac{t}{\epsilon})\eta\right] \end{aligned} \tag{22}$$

In this system the parameter ϵ, which will be eventually chosen small, represents an averaging parameter which is used to simplify the closed-loop analysis. In particular, by re-scaling time as $\tau = t/\epsilon$ and denoting $x' = dx/d\tau$, system (22) reads as

$$\begin{aligned} z' &= \epsilon f(z,\tilde{y} + b(\mu)\beta(\tau)\eta,\mu) \\ \tilde{y}' &= \epsilon q(z,\tilde{y} + b(\mu)\beta(\tau)\eta,\mu) + \epsilon b(\mu)u_a(\tau) \\ \eta' &= \epsilon F\eta + \epsilon B_o k(\tau)\kappa\left[\tilde{y} + b(\mu)\beta(\tau)\eta\right] \end{aligned} \tag{23}$$

whose asymptotic properties can be studied by considering the averaged system. Denoting by $\bar{k}$, $\bar{\beta}$, $\overline{k\beta}$ and $\bar{u}_a$ the averaged descriptions of the functions $k(\tau)$, $\beta(\tau)$, $k(\tau)\beta(\tau)$ and $u_a(\tau)$ respectively, and by taking advantage of (18), it turns out that the averaged description of (23) is given by

$$\begin{aligned} \frac{1}{\epsilon}z' &= f(z,\tilde{y} + b(\mu)\,\bar{\beta}\,\eta,\mu) \\ \frac{1}{\epsilon}\tilde{y}' &= q(z,\tilde{y} + b(\mu)\,\bar{\beta}\,\eta,\mu) + b(\mu)\bar{u}_a \\ \frac{1}{\epsilon}\eta' &= F\eta + B_o\kappa\left[\bar{k}\,\tilde{y} + b(\mu)\,\overline{k\beta}\,\eta\right]\,. \end{aligned} \tag{24}$$

Proposition 3. Under the assumption VIB formulated before, let the degree-of-freedom of the controller (21) be chosen so that

$$-\frac{\overline{k\beta}}{\bar{k}} + \bar{\beta} = N\,, \qquad B_o\left(-\frac{\bar{k}}{\overline{k\beta}B_0}\right) = G \tag{25}$$

and choose u_a so that

$$\bar{u}_a = -\frac{\overline{k\beta}}{\bar{k}}F\eta\,, \tag{26}$$

Then for any compact set $\mathcal{Z} \subset \mathbb{R}^{\dim_z}$, $\tilde{\mathcal{Y}} \subset \mathbb{R}$ and $\mathcal{M} \subset \mathbb{R}^{\dim_\eta}$ there exists a $\kappa^\star > 0$ (not dependent on ϵ) such that for any $\kappa \leq -\kappa^\star$ system (24) is asymptotically (locally exponentially) stable with domain of attraction containing $\mathcal{Z} \times \tilde{\mathcal{Y}} \times \mathcal{M}$.

Proof. Consider system

$$\begin{aligned} \dot{z} &= f(z,\tilde{y} + b(\mu)\,\bar{\beta}\,\eta,\mu) \\ \dot{\tilde{y}} &= q(z,\tilde{y} + b(\mu)\,\bar{\beta}\,\eta,\mu) + b(\mu)\bar{u}_a \\ \dot{\eta} &= F\eta + B_o\kappa\left[\bar{k}\,\tilde{y} + b(\mu)\,\overline{k\beta}\,\eta\right] \end{aligned} \tag{27}$$

which, somewhere, will be written as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} = \mathrm{col}(z,\tilde{y},\eta)$. Note that this system can be regarded as the system

$$\begin{aligned} \dot{z} &= f(z,\tilde{y} + b(\mu)\,\bar{\beta}\,\eta,\mu) \\ \dot{\tilde{y}} &= q(z,\tilde{y} + b(\mu)\,\bar{\beta}\,\eta,\mu) + b(\mu)\bar{u}_a \\ \dot{\eta} &= F\eta + B_o v \\ w &= \bar{k}\,\tilde{y} + b(\mu)\,\overline{k\beta}\,\eta \end{aligned} \tag{28}$$

with input v and output w, under the static positive feedback

$$v = \kappa w\,.$$

Moreover, if $\overline{k\beta} B_0 \neq 0$, it turns out that system (28) has a well-defined unitary relative degree. We show that the choices (25), (26) make the zero dynamics of system (28) with respect to the input v and the output w asymptotically stable and, in particular, coincident with the auxiliary loop characterizing the main assumption formulated before. To compute the zero dynamics, note that

$$w = 0 \qquad \Rightarrow \qquad \tilde{y} = -b(\mu)\frac{\overline{k\beta}}{\bar{k}}\eta$$

and that, in the case $\overline{k\beta} B_0 \neq 0$ and $u_a(\tau)$ is chosen so that its averaged description satisfies (26) the friend v associated to the zero dynamics (computable by the constraint $w'/\epsilon \equiv 0$) is

$$v = -\frac{\bar{k}}{\overline{k\beta} B_0}\,\frac{q(z, b(\mu)\,L\eta, \mu)}{b(\mu)}$$

with $L := -\frac{\overline{k\beta}}{\bar{k}} + \bar{\beta}$. From this it turns out that the zero dynamics are described by

$$\begin{aligned}
\frac{1}{\epsilon} z' \;&=\; f(z, b(\mu) L\eta, \mu)\\
\frac{1}{\epsilon}\eta' \;&=\; F\eta + B_o\left(-\frac{\bar{k}}{\overline{k\beta} B_0}\right)\frac{q(z, b(\mu)\,L\eta, \mu)}{b(\mu)}
\end{aligned}$$

namely, by re-scaling the η variable as $\eta_b := b(\mu)\eta$,

$$\begin{aligned}
z' \;&=\; f(z, L\eta_b, \mu)\\
\eta_b' \;&=\; F\eta_b + B_o\left(-\frac{\bar{k}}{\overline{k\beta} B_0}\right) q(z, L\eta_b, \mu)
\end{aligned} \tag{29}$$

Remarkably the dynamics (29) do not depend on $b(\mu)$. Furthermore, according to the choices (25) and to assumption VIB, system (29) is globally (locally exponentially) stable. From this standard results (using the fact that $b(\mu)$ has a well defined sign assumed positive) yield that for any compact sets $\mathcal{Z}$, $\tilde{\mathcal{Y}}$ and $\mathcal{M}$, there exists a $\kappa^\star > 0$ such that for any $\kappa \leq -\kappa^\star$ system (27) is asymptotically (locally exponentially) stable with domain of attraction containing $\mathcal{Z} \times \tilde{\mathcal{Y}} \times \mathcal{M}$. In particular there exist an open set $\mathcal{D} \supset \mathcal{Z} \times \tilde{\mathcal{Y}} \times \mathcal{M}$, class-$\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha} : \mathcal{D} \to \mathbb{R}$, a class-$\mathcal{K}$ function $\alpha(\cdot) : \mathcal{D} \to \mathbb{R}$ and a differentiable function $V : \mathcal{D} \to \mathbb{R}$ satisfying

$$\underline{\alpha}(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|)$$

and

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) \leq -\alpha(|\mathbf{x}|)\,. \tag{30}$$

Furthermore, from local exponential stability, $\underline{\alpha}$, $\overline{\alpha}$ and α can be taken locally linear. From (30) it follows that

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\epsilon\mathbf{f}(\mathbf{x}) \leq -\epsilon\alpha(|\mathbf{x}|)$$

which, rewriting (24) as $\dot{\mathbf{x}} = \epsilon\mathbf{f}(\mathbf{x})$, yields the desired result.

Indeed, a possible choice which makes (25) fulfilled is given by

$$\begin{aligned}
B_0 \;&=\; -G\,, & \beta(\tau) \;&=\; N - G^T + N\cos\tau\\
k(\tau) \;&=\; 1 + k_1\cos\tau\,, & k_1 \;&=\; \frac{-1}{[\cos^2\tau]_{\mathrm{av}}} = -2\,.
\end{aligned}$$

From this, with κ fixed and not dependent on ϵ, standard averaging results (not repeated for reasons of space) guarantee the existence of an $\epsilon^\star$ such that also the origin of (22) is asymptotically stable with a domain of attraction containing $\mathcal{Z} \times \tilde{\mathcal{Y}} \times \mathcal{M}$.

The overall result can be summarized as follows.

Proposition 4. Consider system (17) (under the assumption 18) and the controller (21) with $\beta(t)$, $k(t)$, $u_a(t)$ so that (26) and (25) are satisfied. Let $\mathcal{Z}$, $\mathcal{Y}$ and $\mathcal{M}$ be arbitrary compact sets of initial conditions for (17) and (21) and let μ be ranging in a compact set. There exist a $\kappa^\star > 0$ and, for all $\kappa \leq -\kappa^\star$, a $\epsilon^\star > 0$ such that for all positive $\epsilon \leq \epsilon^\star$ the closed-loop system (17), (21) is asymptotically stable with a domain of attraction containing $\mathcal{Z} \times \mathcal{Y} \times \mathcal{M}$.

Remark. We stress that the previous result holds under the limitative condition (18).

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