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Tracking Control with Saturating Actuators: A Method Based on State-Dependent Gain-Scheduling and Reference Management

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Abstract: In this paper, we consider tracking control problems in the presence of actuator saturation. We first show a control law that internally stabilizes the closed-loop system and the tracking error converges to zero in the case where a reference signal is generated by a certain dynamics. The control law is based on the recently developed state dependent gain-scheduling algorithm and makes it possible to achieve large region of attraction and fast convergence of tracking error. Then we extend this result to the cases where the reference signal is an arbitrary time-varying signal.

1. INTRODUCTION

Recently, various attempts have been made to construct a control law for constrained control problems. The scheduling scheme is one of effective methods for dealing with the problems (see e.g., Lin [1998], Megretski [1996], Saberi et al. [2000], Teel [1995]). In this scheme, a control law which has a structure that a high-gain control law and a low-gain control law are interpolated by a scheduling parameter is utilized. The scheduling parameter is determined by solving an optimization problem on-line. It is shown in Lin [1998], Megretski [1996], Saberi et al. [2000], Teel [1995] that, by applying this approach, both high control performance near the origin and global asymptotic stability can be achieved in the case where the plant is null controllable with bounded control.

On the other hand, when the plant has exponentially unstable poles, it is impossible to globally stabilize the plant by using any bounded control law. In such a case, local asymptotic stability in a neighborhood of the origin can be only achieved. In Hu and Lin [2001], a novel polytopic representation of a saturation function is proposed and an analysis condition of the region of attraction based on the representation is derived. Further, a synthesis condition of a controller which guarantees local asymptotic stability is derived. This representation has several remarkable features. By using the polytopic representation, a necessary and sufficient condition for estimating region of attraction in the case where the system is single input and the estimation is performed by a single quadratic Lyapunov function can be derived. Also, in multivariable case, a less conservative analysis condition as compared with the circle criterion can be derived. Further, both an analysis condition and a synthesis condition can be reduced to complete Linear Matrix Inequality (LMI) conditions.

In Wada and Saeki [2007a], a scheduling control algorithm for locally stabilizing discrete-time linear systems with input saturation is proposed. In the paper, the problem of computing a scheduling parameter is reduced to an optimization problem with an LMI constraint and the problem can be solved as a simple eigenvalue problem very efficiently. The gain-scheduled feedback control law is construct based on the polytopic representation of a saturation function of Hu and Lin [2001]. As a result, the control scheme achieves large region of attraction even if the plant is unstable. Moreover, based on a parameter dependent Lyapunov function, a less conservative scheduling control algorithm is derived in Wada and Saeki [2007b]. However, since the control laws of Wada and Saeki [2007a,b] only guarantee asymptotic stability of the closed-loop system, it can not be applied to tracking control problems.

In this paper, we consider tracking control problems in the presence of input saturation. Firstly, based on Wada and Saeki [2007a], we show a design method of a controller that guarantees closed-loop stability and asymptotic convergence of tracking errors. Then, we extend this result to the case where the reference signal is an arbitrary time-varying signal. To guarantee feasibility of the control algorithm, we introduce a reference management mechanism. This class of control law can be applied to manual control problems (see e.g., Åkesson and Åström [2005], Bemporad [1998], Kogiso and Hirata [2007], Pachter and Miller [1998]). The effectiveness of the proposed methods are shown through numerical examples.

Notations: For a vector $u \in \mathcal{R}^m$, we define the standard multivariable saturation function as $\Phi(u) := (\phi(u_1), \cdots, \phi(u_m))^T$, where

$$\phi(u_i) := \begin{cases} \mathrm{sgn}(u_i), & |u_i| > 1 \\ u_i, & |u_i| \le 1 \end{cases}$$

For a vector $v \in \mathcal{R}^n$, we denote its Euclidean norm as $\|v\|_2 := (v^T v)^{1/2}$. For a positive definite matrix $P \in \mathcal{R}^{n\times n}$, we denote $\mathcal{E}(P,\eta) := \{x \in \mathcal{R}^n : x^T P x \le \eta\}$. For $F \in \mathcal{R}^{m\times n}$, we denote the ith row of F as $F^{(i)}$. Furthermore, we define $\mathcal{L}(F,\rho) := \{x \in \mathcal{R}^n : |F^{(l)}x| \le \rho_l, l = 1,\cdots,m\}$, where $\rho = \mathrm{diag}[\rho_1,\cdots,\rho_m]$.

2. PRELIMINARY

In this section, we introduce a polytopic model of a saturation function of Hu and Lin [2001]. Let $\mathcal{V}$ be the set of $m \times m$ diagonal matrices whose elements are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as $\mathbf{E}_j, j = 1,2,\cdots,2^m$, and denote $\mathbf{E}_j^- := I - \mathbf{E}_j$. Clearly, $\mathbf{E}_j^-$ is also an element of $\mathcal{V}$.

Lemma 1. (Hu and Lin [2001]) Let $u, v \in \mathcal{R}^m$. Suppose that $|v_j| \le 1, \forall j \in [1,m]$, then $\Phi(u)$ can be represented as $\Phi(u) = \sum_{j=1}^{2^m} \lambda_j(\mathbf{E}_j u + \mathbf{E}_j^- v)$, where $0 \le \lambda_j \le 1$, $\sum_{j=1}^{2^m}\lambda_j = 1$.

3. PROBLEM FORMULATION

Let us consider the system described by

$$x(t+1) = Ax(t) + B\Phi(u(t)) + Ew(t) \tag{1}$$

$$z(t) = Cx(t) + D\Phi(u(t)) + D_w w(t) \tag{2}$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}^m, w \in \mathcal{R}^p, z \in \mathcal{R}^q$. $w(t)$ represents a reference signal. $z(t)$ represents a tracking error.

In Section 4, we first consider the following problem.

Problem 1. Consider the system (1) and (2). Suppose that $w(t)$ is generated by

$$r(t+1) = Sr(t) \tag{3}$$

$$w(t) = r(t) \tag{4}$$

Further, we assume that the system (3) is neutrally stable and $\|r(t)\|_2 \le r_{\max}, \forall t \ge 0$. Design a feedback control law

$$u(t) = F(t)x(t) + M(t)w(t) \tag{5}$$

that achieves fast convergence of the signal $z(t)$ and large region of attraction.

In Section 4, we show a state-dependent gain-scheduled feedback control law that achieves the control objectives of Problem 1. Then, in Section 5, we extend the results of Section 4 to the case where $r(t)$ is an arbitrary time-varying signal.

4. TRACKING CONTROL FOR A REFERENCE SIGNAL GENERATED BY AN EXO-SYSTEM

4.1 Controller Design

We initially introduce the following theorem.

Theorem 2. Consider the system (1)–(4). We suppose that there exist matrices $\Pi \in \mathcal{R}^{n\times p}, \Gamma \in \mathcal{R}^{m\times p}$ that satisfy

$$\Pi S = A\Pi + B\Gamma + E \tag{6}$$

$$0 = C\Pi + D\Gamma + D_w \tag{7}$$

Further, we suppose that $\max_{t\ge 0}|\Gamma^{(l)}r(t)| < 1, \forall l \in [1,m]$. For given positive scalars η, γ_0, γ_1 such that $\gamma_0 < \gamma_1$ and a matrix $\mathbf{R} > 0$, assume that there exist matrices $Q_i, Y_i, Z_i, (i = 0,1)$ that satisfy

$$\begin{bmatrix} Q_i & * & * \\ \begin{bmatrix} CQ_i \\ \mathbf{R}^{\frac{1}{2}}Y_i \end{bmatrix} + \mathbf{D}(\mathbf{E}_jY_i + \mathbf{E}_j^- Z_i) & \gamma_i I & * \\ AQ_i + B(\mathbf{E}_jY_i + \mathbf{E}_j^- Z_i) & 0 & Q_i \end{bmatrix} > 0 \quad \forall i \in [0,1],\ \forall j \in [1,2^m] \tag{8}$$

$$\begin{bmatrix} Q_i & * \\ Z_i^{(l)} & \dfrac{\rho_l^2}{\eta} \end{bmatrix} \ge 0, \qquad \forall i \in [0,1],\ \forall l \in [1,m] \tag{9}$$

$$Q_0 < Q_1 \tag{10}$$

where $\rho_l := 1 - \max_{t\ge 0}|\Gamma^{(l)}r(t)|$, $\mathbf{D} := [D^T, 0]^T$ and the symbol $*$ stands for symmetric block in matrix inequalities. Further, for some constant $\alpha \in [0,1]$, we suppose that $\xi(0) \in \mathcal{E}(P(\alpha),\eta)$ where $P(\alpha) := Q(\alpha)^{-1}, Q(\alpha) := (1-\alpha)Q_0 + \alpha Q_1$, $\xi := x - \Pi w$. Then, by applying the feedback control law

$$u(t) = F(\alpha)x(t) + M(\alpha)w(t) \tag{11}$$

where $F(\alpha) = Y(\alpha)Q(\alpha)^{-1}$, $Y(\alpha) := (1-\alpha)Y_0 + \alpha Y_1$ and $M(\alpha) = \Gamma - F(\alpha)\Pi$ to the system (1)–(4), the relations $\xi(t) \in \mathcal{E}(P(\alpha),\eta), \forall t \ge 0$, $\lim_{t\to\infty} z(t) = 0$ and $J := \sum_{t=0}^{\infty}\|\mathbf{z}(t)\|_2^2 < \gamma(\alpha)\eta$, where $\mathbf{z} := [z^T, u_e^T\mathbf{R}^{1/2}]^T$, $u_e := u - \Gamma w$, $\gamma(\alpha) := (1-\alpha)\gamma_0 + \alpha\gamma_1$ hold.

Proof. From (1), (3), (4), (6), (11), we obtain

$$\xi(t+1) = A\xi(t) + B\Psi(F(\alpha)\xi(t)) \tag{12}$$

where $\Psi(F(\alpha)\xi) := \Phi(F(\alpha)\xi + \Gamma w) - \Gamma w$. In the following, we first show that if $\xi \in \mathcal{L}(H(\alpha),\rho)$ and $\max_{t\ge 0}|\Gamma^{(l)}r(t)| < 1$, $\forall l \in [1,m]$, $\Psi(F(\alpha)\xi)$ can be represented as $\Psi(F(\alpha)\xi) = \sum_{j=1}^{2^m}\lambda_j\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^- H(\alpha)\}\xi$, where $H(\alpha) := Z(\alpha)Q(\alpha)^{-1}$, $Z(\alpha) := (1-\alpha)Z_0 + \alpha Z_1$ and $\rho := \mathrm{diag}[\rho_1,\cdots,\rho_m]$. If $\xi \in \mathcal{L}(H(\alpha),\rho)$ and $\max_{t\ge 0}|\Gamma^{(l)}r(t)| < 1$, $\forall l \in [1,m]$, then $|H(\alpha)^{(l)}\xi + \Gamma^{(l)}w| \le 1$, $\forall l \in [1,m]$. Hence, in this case, the relation $\Phi(F(\alpha)\xi + \Gamma w) = \sum_{j=1}^{2^m}\lambda_j\{\mathbf{E}_j(F(\alpha)\xi + \Gamma w) + \mathbf{E}_j^-(H(\alpha)\xi + \Gamma w)\}$ holds (see Lemma 1 in Appendix). Therefore, we can show that $\Psi(F(\alpha)\xi) = \sum_{j=1}^{2^m}\lambda_j\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^- H(\alpha)\}\xi$. By using this relation, if $\xi(t) \in \mathcal{L}(H(\alpha),\rho)$ and $\max_{t\ge 0}|\Gamma^{(l)}r(t)| < 1$, $\forall l \in [1,m]$, the close-loop system (1), (3) and (11) can be rewritten as

$$\xi(t+1) = \mathcal{A}(\lambda(t))\xi(t) \tag{13}$$

where $\mathcal{A}(\lambda) := \sum_{j=1}^{2^m}\lambda_j\mathcal{A}_j$, $\mathcal{A}_j := A + B\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^- H(\alpha)\}$. On the other hand, from the definition, the signal $\mathbf{z}$ can be rewritten as

$$\mathbf{z} = \mathbf{C}x + \mathbf{D}\Phi(u) + \mathbf{D_w}w \tag{14}$$

where $\mathbf{C} := [C^T, F(\alpha)^T\mathbf{R}^{\frac{1}{2}}]^T$, $\mathbf{D_w} := [D_w^T, (M(\alpha) - \Gamma)^T\mathbf{R}^{\frac{1}{2}}]^T$. It can be verified that the matrices Π and Γ that satisfy (6) and (7) also satisfy

$$\mathbf{C}\Pi + \mathbf{D}\Gamma + \mathbf{D_w} = 0. \quad (15)$$

From (14), (15) and (11) if $\xi(t) \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0} |\Gamma^{(l)} r(t)| < 1,\ \forall l \in [1, m]$, the signal $\mathbf{z}(t)$ can be represented as

$$\mathbf{z}(t) = \mathcal{C}(\lambda(t))\xi(t) \quad (16)$$

where $\mathcal{C}(\lambda) := \sum_{j=1}^{2^m} \lambda_j \mathcal{C}_j,\ \mathcal{C}_j := \mathbf{C} + \mathbf{D}\{\mathbf{E}_j F(\alpha) + \mathbf{E}_j^- H(\alpha)\}$. From (13) and (16), in the ξ-coordinate system, the feedback system can be regarded as the system without exogenous input. In the following, we prove Theorem 1 based on this representation.

In the following, we first show that the condition (9) implies that $\mathcal{E}(P(\alpha), \eta) \subseteq \mathcal{L}(H(\alpha), \rho)$. From (9), we have

$$\begin{bmatrix} Q(\alpha) & * \\ Z(\alpha)^{(l)} & \dfrac{\rho_l^2}{\eta} \end{bmatrix} \geq 0,\ \forall l \in [1, m] \quad (17)$$

Then, by substituting $Z(\alpha)^{(l)} = H(\alpha)^{(l)} Q(\alpha)$ for (17) and performing a congruence transformation with block-diag$[Q(\alpha)^{-1}, 1]$ and substituting $Q(\alpha)^{-1} = P(\alpha)$, and applying Schur complement, we have

$$\frac{1}{\rho_l^2} H(\alpha)^{(l)T} H(\alpha)^{(l)} \leq \frac{1}{\eta} P(\alpha),\ \forall l \in [1, m] \quad (18)$$

Equation (18) implies that $\mathcal{E}(P(\alpha), \eta) \subseteq \mathcal{L}(H(\alpha), \rho)$.

Then, we show that the relations $\xi(t) \in \mathcal{E}(P(\alpha), \eta), \forall t \geq 0$ and $\lim_{t\to\infty} z(t) = 0$ and $J < \gamma(\alpha)\eta$ hold. From (8), we obtain

$$\begin{bmatrix} Q(\alpha) & * & * \\ \begin{bmatrix} CQ(\alpha) \\ \mathbf{R}^{\frac{1}{2}} Y(\alpha) \end{bmatrix} + \mathbf{D}(\mathbf{E}_j Y(\alpha) + \mathbf{E}_j^- Z(\alpha)) & \gamma(\alpha) I & * \\ AQ(\alpha) + B(\mathbf{E}_j Y(\alpha) + \mathbf{E}_j^- Z(\alpha)) & 0 & Q(\alpha) \end{bmatrix} > 0,\ \ \forall j \in [1, 2^m] \quad (19)$$

By substituting $Z(\alpha) = H(\alpha)Q(\alpha)$ and $Y(\alpha) = F(\alpha)Q(\alpha)$ for (19) and performing a congruence transformation with block-diag$[Q(\alpha)^{-1}, I, I]$, and multiplying the resulting inequality by $\lambda_j(t)$, and summing them up for $j = 1, \cdots, 2^m$, we have

$$\begin{bmatrix} P(\alpha) & * & * \\ \mathcal{C}(\lambda(t)) & \gamma(\alpha) I & * \\ \mathcal{A}(\lambda(t)) & 0 & P(\alpha)^{-1} \end{bmatrix} > 0 \quad (20)$$

By applying Schur complement to (20), and multiplying the resulting inequality from the left by $\xi(t)^T$ and from the right by $\xi(t)$, and using (13) and (16), we have

$$V(\xi(t+1)) - V(\xi(t)) < -\frac{1}{\gamma(\alpha)} \|\mathbf{z}(t)\|_2^2 \quad (21)$$

where $V(\xi) := \xi^T P(\alpha)\xi$. From (21), we can conclude that if $\xi(0) \in \mathcal{E}(P(\alpha), \eta)$ then

$$V(\xi(t)) < V(\xi(0)) \leq \eta,\ \ \forall t \geq 0 \quad (22)$$

Equation (22) implies that $\xi(t) \in \mathcal{E}(P(\alpha), \eta),\ \ \forall t \geq 0$. On the other hand, the nonlinearity $\Psi(F(\alpha)\xi(t))$ can be represented as $\Psi(F(\alpha)\xi(t)) = \sum_{j=1}^{2^m} \lambda_j(t)\{\mathbf{E}_j F(\alpha) + \mathbf{E}_j^- H(\alpha)\}\xi(t)$ if $\xi(t) \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0} |\Gamma^{(l)} r(t)| < 1,\ \forall l \in [1, m]$. From (18) and (22), we can state that if the conditions in Theorem 2 hold, the relation $\xi(t) \in \mathcal{L}(H(\alpha), \rho),\ \forall t \geq 0$ holds. From (21), since $\xi(t) \to 0, (t \to \infty)$, $z(t) \to 0, (t \to \infty)$ holds. Moreover, from (21) and (22), $\sum_{t=0}^{\infty} \|\mathbf{z}(t)\|_2^2 < \gamma(\alpha)\eta$ holds.

Remark 1. Equations (6) and (7) are the conditions for the output regulation problem is solvable in the case of linear systems (see e.g., Francis [1975], Saberi et al. [2000]).

In this paper, based on Theorem 2, we design a gain $F(1) = Y_1 Q_1^{-1}$ which makes the region $\mathcal{E}(P(1), \eta)$ large and a gain $F(0) = Y_0 Q_0^{-1}$ which achieves fast convergence of the state variable in $\mathcal{E}(P(0), \eta)$ by suitably choosing the parameters γ_0, γ_1 and $\mathbf{R}$. Then we construct a control law (11) by interpolating the obtained gains.

4.2 Scheduling Algorithm

In this section, we show a gain-scheduling algorithm of the control law (11) which achieves fast convergence of $z(t)$.

Algorithm 1.

Step 0: Set $t = 0$.
Step 1: Measure $x(t)$ and $w(t)$.
Step 2: Solve $\min_{\alpha \in [0,1]} \alpha$, s.t. $\xi(t)^T Q(\alpha)^{-1} \xi(t) \leq \eta$.
Then, set $\alpha(t) = \alpha$.
Step 3: Apply $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ to the plant (1), (2).
Step 4: $t \leftarrow t + 1$ and go to Step 1.

The optimization problem of Step 2 in Algorithm 1 is an LMI optimization problem (see e.g., Boyd et al. [1994]). Hence, the problem can be solved by the interior point method. Alternatively, the problem can be solved as a simpler eigenvalue problem as follows. By the Schur complement, $\xi(t)^T Q(\alpha)^{-1} \xi(t) \leq \eta$ is equivalent to $Q(\alpha) - \frac{1}{\eta}\xi(t)\xi(t)^T \geq 0$. Further, This condition can be rewritten as $\alpha I \geq \mathbf{Q}(\xi(t))$ where $\mathbf{Q}(\xi(t)) := Q_h^{-1/2} \left[\frac{1}{\eta}\xi(t)\xi(t)^T - Q_0\right] Q_h^{-1/2}$ and $Q_h := Q_1 - Q_0$. Hence, with considering $\alpha \geq 0$, the solution of the optimization problem of Step 2 can be obtained as $\alpha = \max[0,\ \lambda_{\max}(\mathbf{Q}(\xi(t)))]$.

4.3 Feasibility and Stability

The following theorem can be stated.

Theorem 3. Consider the system (1), (2). Assume that there exist matrices Π and Γ that satisfy (6) and (7). Further, assume that $\max_{t\geq 0} |\Gamma^{(l)} r(t)| < 1, \forall l \in [1, m]$. Moreover, for given positive scalars η, γ_0 and γ_1, assume that there exist matrices Q_i, Y_i, Z_i that satisfy (8)–(10). Moreover, assume that $\xi(0) \in \mathcal{E}(P(1), \eta)$. Then by applying Algorithm 1 to the system (1), (2), $z(t)$ converges to zero as $t \to \infty$.

Proof. In the following, we initially show that by applying Algorithm 1 $\alpha(t)$ monotonically decreases until the condition $\alpha(t) \leq \epsilon$ holds. We assume that at time t the optimization problem of Step 2 in Algorithm 1 is feasible. In this case, it is clear that $\xi(t) \in \mathcal{E}(P(\alpha(t)), \eta)$ holds.

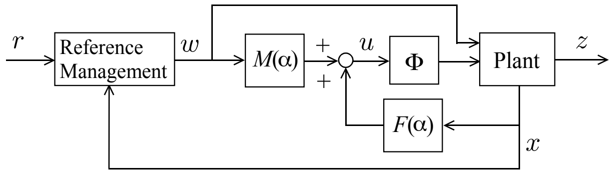

Fig. 1. Feedback System with Reference Management

When the control signal $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ is applied to the system (1), $\xi(t)^T P(\alpha(t))\xi(t) > \xi(t+1)^T P(\alpha(t))\xi(t+1)$ holds from Theorem 2. Hence, for some scalar $\kappa < 1$, $\xi(t+1) \in \mathcal{E}(P(\alpha(t))/\kappa, \eta)$ holds. In the following, we show that the relation $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t)), \eta)$ holds for a scalar β such that $\kappa\alpha(t) < \beta < \alpha(t)$.

- $\mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t)), \eta)$:
  Since $Q_0 < Q_1$ and $\beta < \alpha(t)$ hold from the assumption, we obtain $0 < (\alpha(t) - \beta)(Q_1 - Q_0)$. This implies that $\mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t), \eta)$.
- $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta)$:
  From the assumption, $\kappa\alpha(t) < \beta < \alpha(t)$ holds. Further, since $\kappa < 1$ and $\alpha(t) \le 1$, $(1-\kappa)\alpha(t) \le (1-\kappa)$ holds. Hence, $\alpha(t) \le \kappa\alpha(t) + (1-\kappa)$ holds. Therefore, we obtain $\kappa\alpha(t) < \beta < \kappa\alpha(t) + (1-\kappa)$. From this relation and $Q_0 < Q_1$, we have $0 < [(1-\kappa) - (\beta - \kappa\alpha(t))]Q_0 + [\beta - \kappa\alpha(t)]Q_1$. This implies that $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta)$.

From the above discussion, we can conclude that for a scalar β such that $\kappa\alpha(t) < \beta < \alpha(t)$, the relation $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t)), \eta)$ holds.

Then we set $\alpha(t+1) = \beta$. In this case, it is clear that $\xi(t+1) \in \mathcal{E}(P(\alpha(t+1)), \eta)$ holds. Namely, the optimization problem of Step 3 in Algorithm 1 is feasible at $t+1$, and the solution $\alpha(t+1)$ satisfies $\alpha(t+1) < \alpha(t)$. The same arguments also hold for $t+2, t+3, \cdots$. Therefore, $\alpha(t)$ decreases monotonically. Further, $\alpha(t)$ is bounded from below by zero. Hence, there exists some time T such that the condition $\alpha(T) = 0$ holds. It can be verified that a contradiction occurs if there is not such a time T. After the time T, the control law $u(t) = F(\alpha(T))x(t) + M(\alpha(T))w(t)$ is applied to the system (1), (2). In this case, from Theorem 2, $\xi(t)$ converges to zero as $t \to \infty$. As a result, $z(t)$ converges to zero as $t \to \infty$

5. TRACKING CONTROL FOR ARBITRARY TIME VARYING REFERENCE SIGNALS

As we have shown that by applying Algorithm 1 to the system, if the reference signal is generated by the dynamics (3), (4), both feasibility of the algorithm and stability of the closed-loop system are guaranteed. However, if this control algorithm is used in the case where the signal $r(t)$ is an arbitrary time-varying signal, feasibility of the algorithm and closed-loop stability may not be guaranteed. Hence, in this section, we extend the previous control algorithm so that any time-varying reference signal can be applied. In this case, it is difficult to guarantee strict asymptotic convergence of the tracking error. Hence, in this section, we show a control algorithm that makes the tracking error as small as possible at each time and guarantees asymptotic convergence in the case where the reference signal becomes constant after a finite time. In order to guarantee that the error signal converges to zero when the reference signal is constant, we make the following assumption.

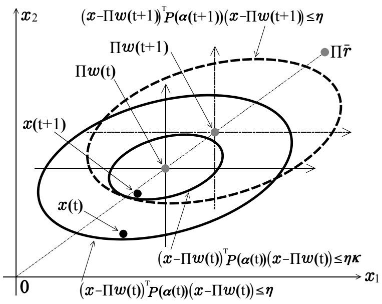

Fig. 2. Graphical Interpretation of Algorithm 2 in case of $n = 2$ and $p = 1$

Assumption 1. For the system (1), (2) and $S = I$, there exist $\Pi, Q_i, Y_i, Z_i, \gamma_i$ that satisfy the conditions (6)–(10) and $\Gamma = 0$.

The constraint $\Gamma = 0$ is satisfied if the plant has an integrator (see Numerical Example).

5.1 Scheduling Algorithm

In this section, we assume that $r(t) \in \mathcal{R}^p$ is an arbitrary time-varying signal. If we simply set $w(t) = r(t)$ and apply Algorithm 1 to the system, feasibility of the algorithm and stability of the closed-loop system may not be guaranteed. To avoid such a situation, we introduce a reference management mechanism that computes a modified reference signal $w(t)$ from the signal $r(t)$ (see Fig.1). In the following, we show a control algorithm that includes the reference management and the state-dependent gain-scheduling.

Algorithm 2.

Step 0: Set $t = 0$ and $\alpha(-1) = 1$.
Step 1: Measure $x(t)$ and $r(t)$.
Step 2: If $x(t) - \Pi r(t) \in \mathcal{E}(P(\alpha(t-1)), \eta)$, then set $w(t) = r(t)$ and go to Step 4. Otherwise, go to Step 3.
Step 3: Solve $\min_{\tilde{w} \in \mathcal{R}^p} \|r(t) - \tilde{w}\|_2^2$, s.t.

$$\begin{bmatrix} \eta & * \\ x(t) - \Pi\tilde{w} & Q(\alpha(t-1)) \end{bmatrix} \ge 0 \tag{23}$$

Then, set $\alpha(t) = \alpha(t-1)$, $w(t) = \tilde{w}$ and go to Step 5.
Step 4: Solve $\min_{\alpha \in [0,1]} \alpha$, s.t. $\xi(t)^T Q(\alpha)^{-1} \xi(t) \le \eta$. Then, set $\alpha(t) = \alpha$.
Step 5: Apply $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ to the plant (1), (2).
Step 6: $t \leftarrow t+1$ and go to Step 1.

In the above algorithm, Step2 and Step 3 represent the reference management mechanism that compute the modified reference signal $w(t)$ from the original reference signal $r(t)$.

Remark 2. The optimization problem of Step 3 in Algorithm 2 is a quadratic optimization problem with respect to $\tilde{w}$. Hence, this problem can be easily solved.

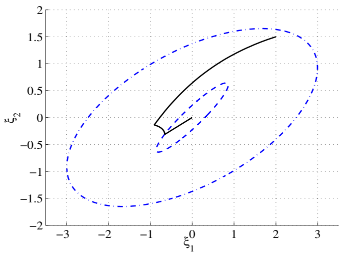

Fig. 3. $\mathcal{E}(P(1),1)$ (dash-dot), $\mathcal{E}(P(0),1)$ (dashed), ξ (solid)

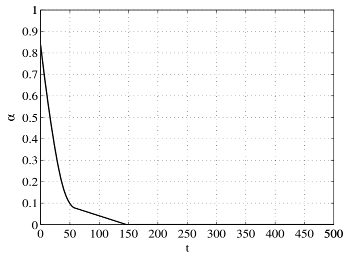

Fig. 4. Scheduling parameter α

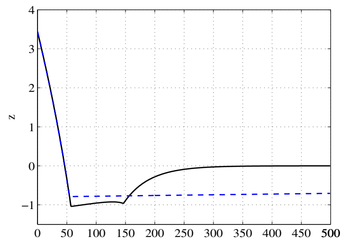

Fig. 5. z (solid: gain-scheduled, dashed: constant feedback)

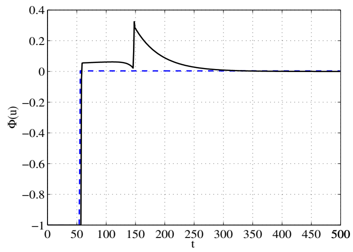

Fig. 6. $\Phi(u)$ (solid: gain-scheduled, dashed: constant feedback)

5.2 Feasibility and Stability

Theorem 4. Consider the system (1), (2). Assume that there exists $\tilde{w} \in \mathcal{R}^p$ that satisfies $x(0) - \Pi\tilde{w} \in \mathcal{E}(P(1),\eta)$. Then by applying Algorithm 2 to the system (1), (2), feasibility of the algorithm is guaranteed for all times. Moreover, if $r(t) = \bar{r}, \forall t \geq T_r$, $\lim_{t\to\infty} w(t) = \bar{r}$ and $\lim_{t\to\infty} z(t) = 0$ hold.

Proof. We first show that feasibility of Algorithm 2 is guaranteed for all time. In Algorithm 2, if the condition $x(0) - \Pi r(0) \in \mathcal{E}(P(1),\eta)$ holds, the optimization problem of Step 4 is solved to update α. It is clear that there exists a solution α that satisfies $\alpha \leq 1$. Otherwise, if the condition $x(0) - \Pi r(0) \in \mathcal{E}(P(1),\eta)$ does not hold, the optimization problem of Step 3 is solved to compute the modified reference signal $\tilde{w}$. The existence of the solution $\tilde{w}$ is guaranteed from the assumption. Hence, we can conclude that there exists a pair of solutions $\alpha(0)$ and $w(0)$. By applying $u(0) = F(\alpha(0))x(0) + M(\alpha(0))w(0)$ with $\alpha(0), w(0)$ obtained from Step3 or Step4 to (1), (2), the inequality $(x(1) - \Pi w(0))^T P(\alpha(0))(x(1) - \Pi w(0)) \leq (x(0) - \Pi w(0))^T P(\alpha(0))(x(0) - \Pi w(0))$ holds from Theorem 3. Therefore, $x(1) - \Pi w(0) \in \mathcal{E}(P(\alpha(0)),\eta)$ holds. This implies that there exists a pair of solutions $\alpha(1), w(1)$ at $t = 1$. The same arguments also hold for $t \geq 2$. Therefore, we can conclude that feasibility of Algorithm 2 is guaranteed for all time.

Then we show that if $r(t) = \bar{r}, \forall t \geq T_r$, the relations $\lim_{t\to\infty} w(t) = \bar{r}$ and $\lim_{t\to\infty} z(t) = 0$ hold. Let us consider the case where the condition $x(t) - \Pi w(t) \in \mathcal{E}(P(\alpha(t)),\eta)$ holds but the equality $w(t) = \bar{r}$ does not hold for a time $t \geq T_r$ (Note that $z(t)$ converges to zero if the equality $w(t) = \bar{r}$ holds from Theorem 3). In this case, at Step3, a modified reference signal $w(t) \neq \bar{r}$ is computed and the scheduling parameter is chosen as $\alpha(t) = \alpha(t-1)$. By applying $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ to the system, the relation $x(t+1) - \Pi w(t) \in \mathcal{E}(P(\alpha(t)),\eta\kappa)$ holds for some positive scalar $\kappa < 1$ (see Fig.2). Then, at time $t+1$, if the equality $w(t+1) = \bar{r}$ does not hold, a modified reference signal $w(t+1) \neq \bar{r}$ is computed and the scheduling parameter is chosen as $\alpha(t+1) = \alpha(t)$ at Step 3. In this case, since $x(t+1) - \Pi w(t) \in \mathcal{E}(P(\alpha(t)),\eta\kappa)$, $\Pi w(t+1)$ can be chosen so that $\|\Pi\bar{r} - \Pi w(t)\|_2$ decreases (see Fig.2). Hence, by repeating this process, $x(t) - \Pi\bar{r} \in \mathcal{E}(P(\alpha(t)),\eta)$ holds for some time. As a result, $\lim_{t\to\infty} w(t) = \bar{r}$ and $\lim_{t\to\infty} z(t) = 0$ hold.

6. NUMERICAL EXAMPLE

Let us consider the following system.

$$x(t+1) = Ax(t) + B\Phi(u(t))$$
$$z(t) = y(t) - w(t),\ y(t) = Cx(t)$$

where $A = \mathrm{diag}[1.0, 1.0253]$, $B = [0.050, 0.0506]^T$, $C = [1, 1]$. Note that the matrix A has an exponentially unstable eigenvalue. For this system, we solve the linear equations (6), (7) with $S = 1$ and obtain $\Pi = [1, 0]^T, \Gamma = 0$. Further, we solve the feasibility problem with LMI constraints in Theorem 2 with $\gamma_0 = 40$, $\gamma_1 = 51412$, $\mathbf{R} = 10^{-3}$, $\eta = 1$, $\mathbf{E}_1 = 1$, $\mathbf{E}_2 = 0$. In Fig.3, the dash-dot line shows $\mathcal{E}(P(1),1)$ and the dashed line shows $\mathcal{E}(P(0),1)$. We can confirm that $\mathcal{E}(P(0),1) \subset \mathcal{E}(P(1),1)$.

The solid line in Fig.3 shows the state trajectory $\xi(t)$ for $x(0) = [3, 1.5]^T$ and $w(0) = 1$ with Algorithm 1. We can confirm that $\xi(t)$ converges to zero as $t \to \infty$. Fig.4 shows responses of $\alpha(t)$. We can see that the scheduling parameter $\alpha(t)$ monotonically decreases. Figs.5 and 6 show responses of $z(t)$ and $\Phi(u(t))$. In these figures, the solid lines show the responses of the system with the proposed gain-scheduled control law and the dashed lines show those with the constant feedback control $u(t) = F(1)x(t) + M(1)w(t)$. From these figures, we can see that when the constant feedback control law is utilized the transient response is quite slow.

Then, we apply Algorithm 2 to the above system. The reference signal is given by

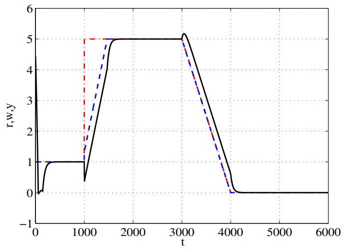

Fig. 7. r (dash-dot), w (dashed), y (solid): gain-scheduled

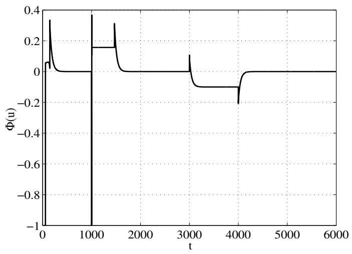

Fig. 8. $\Phi(u)$: gain-scheduled

$$r(t) = \begin{cases} 1, & 0 \le t < 1000 \\ 5, & 1000 \le t < 3000 \\ -\dfrac{1}{200}(t-4000), & 3000 \le t < 4000 \\ 0, & 4000 \le t \end{cases}$$

Fig.7–Fig.10 show the responses of the system for $x(0) = [3, 1.5]^T$. Fig.7 and Fig.8 show the responses of the system with the proposed control algorithm. In this case, the controlled output $y(t)$ tracks the reference signal $r(t)$ with relatively small tracking error. Fig.9 and Fig.10 show the responses of the system with the constant control law $u(t) = F(1)x(t) + M(1)w(t)$ with the reference management of Algorithm 2. In this case, the tracking error $r - y$ becomes large.

7. CONCLUSIONS

We have considered tracking control problems in the presence of actuator saturation. We have first considered the case where the reference signal is generated by an exo-system. The proposed control law is based on the state dependent gain-scheduling algorithm. This property makes it possible to achieve large region of attraction and fast convergence of the tracking error. The scheduling parameter is determined by solving a maximum eigenvalue problem. Then we have extended this result to the cases where the reference signal is an arbitrary time-varying signal. To guarantee feasibility of the control algorithm, we have introduced a reference management mechanism. The modified reference signal is calculated by solving a quadratic convex optimization problem. The effectiveness of the proposed methods is shown through numerical examples.

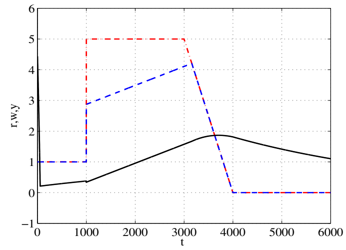

Fig. 9. r (dash-dot), w (dashed), y (solid): constant feedback

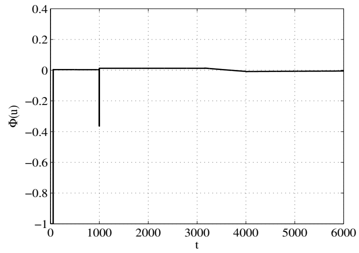

Fig. 10. $\Phi(u)$: constant feedback

REFERENCES

J.Åkesson and K.J.Åström. Manual control and stabilization of an inverted pendulum. *Proc. IFAC World Congress*, 2005.

A.Bemporad. Predictive Control of Teleoperated Constrained Systems with Unbounded Communication Delays. *Proc. IEEE Conf. Decision & Control*, pages 2133–2138, 1998.

D.S.Bernstein and A.N.Michel. A chronological bibliography on saturating actuators. *Int. J. Robust & Nonlinear Control*, volume 5, pages 375-380, 1995.

S.Boyd, L.E.Ghaoui, E.Feron and V.Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

B.A.Francis. The linear multivariable regulator problem. *SIAM J. Control & Optimization*, volume 15, pages 480-505, 1975.

T.Hu and Z.Lin. *Control Systems with Actuator Saturation Analysis and Design*, Birkhäuser, 2001.

K.Kogiso and K.Hirata. Static Controller Switching Approach to Constraint Fulfillment for Time-varying References and Its Experimental Validations. *Trans. JSME*, volume C-73, no.726, pages 422-427, 2007.

Z.Lin. Global Control of Linear Systems with Saturating Actuators. *Automatica*, volume 34, no.7, pages 897-905, 1998.

A.Megretski. L_2 BIBO output feedback stabilization with saturated control. *Proc. IFAC World Congress*, pages 435-440, 1996.

M.Pachter and R.B.Miller. Manual Flight Control with Saturating Actuators. *IEEE Control Systems*, February, 1998.

A.Saberi, A.A.Stoorvogel and P.Sannuti. *Control of Linear Systems with Regulation and Input Constraints*, Springer, 2000.

A.R.Teel. Linear Systems with input nonlinearities: Global stabilization by scheduling a family of $\mathcal{H}_\infty$-type controllers. *Int. J. Robust & Nonlinear Control*, volume 5, pages 399-411, 1995.

N.Wada and M.Saeki. An LMI Based Scheduling Algorithm for Constrained Stabilization Problem. *Systems & Control Letters*, volume 57, pages 255-261, 2008.

N.Wada and M.Saeki. A Scheduling Algorithm for Constrained Control Systems: An Approach Based on a Parameter Dependent Lyapunov Function. *Proc. American Control Conference*, 2007.