Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

On consensus in multi-agent systems with linear high-order agents

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Abstract: Consensus of a group of agents in a multi-agent system is considered. All agents are modeled by identical linear nth order dynamical systems and the interconnection topology between the agents is modeled as a directed weighted graph. We provide answers to the questions whether the group converges to consensus and what consensus value it eventually reaches. Furthermore, we give a necessary and sufficient condition for convergence to consensus in the double integrator case and propose an LMI-based design for group consensus in the general case. An example is used to illustrate the results.

Keywords: Multi-agent systems; Consensus; Directed graph; LMI.

1. INTRODUCTION

Recently, the consensus problem among multi-agent systems (MAS) has received a lot of attention in the literature. The interest in this problem is mainly motivated by the huge variety of applications in various areas, e.g. unmanned aerial vehicles, mobile robots, satellites, formation control, and sensor networks, to name only a few. For a nice overview of recent results on the topic, see Ren et al. (2007); Olfati-Saber et al. (2007); Tanner et al. (2007) and the references therein.

Numerous results have been proposed on consensus for MASs, most of which consider agents that are modeled by single or double integrators. However, in some applications, agents of higher dynamical order are required if consensus of more than two variables is aimed at. While the consensus problem for agents modeled as integrator chains of length greater than two was reported quite recently (Ren et al., 2006), only little attention has been paid to the inherent instability of integrator chains of length greater than one. To overcome this problem and for full generality, it is interesting to investigate the consensus problem considering agents modeled by general linear time-invariant (LTI) single-input systems.

It is well known that the interconnection topology among agents plays a pivotal role in reaching group consensus in a MAS. Graphs are commonly employed in order to model the interconnection topology in a MAS (cf. de Gennaro and Jadbabaie (2006); Ren et al. (2006); Ren and Atkins (2005); Olfati-Saber (2006)). In the most general setup, directed and weighted graphs are used. A graph can be completely characterized by its Laplacian matrix L, hence, it is important to fully understand the properties of this matrix, most notably its eigenstructure which is crucial for the consensus problem.

In the past, research was mainly focused on the *analysis* of MAS consensus: Given the interconnection topology and some consensus algorithm, the question is answered, whether the states of all agents converge to some common value, the consensus. For agents modeled as single integrator with arbitrary interconnection topology or double integrator with an interconnection topology admitting a real Laplacian spectrum, connectedness of the MAS is necessary and sufficient for convergence to consensus. In the general case however, it was reported recently (Ren and Atkins, 2005) that connectedness of the MAS is only a necessary condition for convergence to consensus. Whether or not a sufficiently connected MAS actually converges to consensus depends on the gains in the consensus algorithm. Therefore, it is required to develop a systematic method to determine the gains or to find tight bounds on the gains in the consensus algorithm.

Considering the status of recent consensus research, firstly, this paper presents conditions for consensus of MASs with agents modeled as general LTI systems. These conditions explicitly depend on the gains in the consensus algorithm, the parameters of the agent model, and the interconnection topology. To obtain these conditions, we derive the characteristic equation of the closed-loop consisting of all agents and a given consensus algorithm. The characteristic equation enables us to derive conditions on the gains and to unveil the dynamic evolution of the consensus state.

Secondly, an implication of the left-eigenvector of the Laplacian matrix corresponding to the zero eigenvalue is elucidated: Non-zero elements in the left-eigenvector correspond to agents that can be chosen as the root of a spanning tree in the graph describing the interconnection topology of the MAS. This implies that if the ith component in the left-eigenvector is non-zero, the ith agent has some influence on the consensus value. This influence is quantified by the components of the left-eigenvector.

Thirdly, a systematic way to choose the gains in the consensus algorithm is proposed for agents modeled as double integrators and general LTI systems. In the case of double integrator agents, we present exact convergence bounds that are tighter than previously proposed bounds. For agents modeled as general LTI systems, an LMI based design method for the consensus algorithm is proposed. Unlike in previous results, in our design methods a desired convergence rate can be specified.

The remainder of this paper is organized as follows: We start by presenting some basic facts from graph theory in Section 2. Section 3 exposes some results on analysis of consensus algorithms for agents modeled by LTI systems followed by the presentation of design methods for the double integrator and the general LTI system case in

Section 4. The results are illustrated on an example in Section 5 before Section 6 concludes the paper.

2. PRELIMINARIES

2.1 Notation

Throughout this paper, we write $\mathbf{1}$ and $\mathbf{0}$ for the all ones and all zero vector of appropriate dimension. The ith unit vector in $\mathbb{R}^n$ is denoted $\boldsymbol{e}_i$. The imaginary unit is written as $\mathrm{j} := \sqrt{-1}$.

2.2 Basic graph theory

To make the paper self-contained, we start by an overview of concepts from graph theory. We consider weighted and directed graphs given as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$ where

- $\mathcal{V} = \{1, \dots, N\}$ is a nonempty finite vertex set with each vertex i representing one agent and the number of vertices denoted as $|\mathcal{V}| = N$,
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of ordered pairs of vertices called edges satisfying $(i, j) \in \mathcal{E}$ if and only if there is a link from vertex i called the parent to vertex j called the child, i.e. information flows from agent i to agent j,
- $W : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ is a mapping that assigns positive weights to the edges of $\mathcal{G}$ and satisfies $W(i, j) \neq 0$ if and only if $(i, j) \in \mathcal{E}$.

The graph $\mathcal{G}$ is undirected if $W(i, j) = W(j, i)$ for all $i, j = 1, \dots, N$, it is balanced if $\sum_{j=1}^{N} W(i, j) = \sum_{j=1}^{N} W(j, i)$ for $i = 1, \dots, N$. We assume that there are no selfloops, i.e. $(i, i) \notin \mathcal{E}$, $i = 1, \dots, N$.

A directed path p in $\mathcal{G}$ is a finite sequence of vertices $p = \{i_1, i_2, \dots, i_l\}$ satisfying $(i_k, i_{k+1}) \in \mathcal{E}$ for $k = 1, \dots, l-1$. Vertex i is an ancestor of vertex j and vertex j is a descendant of vertex i if there is a directed path from i to j. We define the sets $\mathcal{P}_i \subseteq \mathcal{V}$ as the set of parents of i, $\mathcal{C}_i \subseteq \mathcal{V}$ as the set of children of i, $\mathcal{A}_i \subseteq \mathcal{V}$ as the set of ancestors of i, and $\mathcal{D}_i \subseteq \mathcal{V}$ as the set of descendants of i. A directed tree is a directed graph where every vertex except for one distinct vertex, called the root of the tree, has exactly one parent. Define the restriction $W|_{\tilde{\mathcal{E}}} : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ of the mapping $W : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ to a subset $\tilde{\mathcal{E}} \subseteq \mathcal{E}$ as

$$W|_{\tilde{\mathcal{E}}}(i, j) := \begin{cases} W(i, j) & \text{if } (i, j) \in \tilde{\mathcal{E}}, \\ 0 & \text{if } (i, j) \notin \tilde{\mathcal{E}}. \end{cases}$$

The graph $\mathcal{G}$ is said to contain a spanning tree if there exists a subset $\tilde{\mathcal{E}} \subseteq \mathcal{E}$ such that the graph $\tilde{\mathcal{G}} = \{\mathcal{V}, \tilde{\mathcal{E}}, W|_{\tilde{\mathcal{E}}}\}$ is a directed tree. A graph $\mathcal{G}$ is strongly connected if any vertex in $\mathcal{V}$ is the root of a spanning tree of $\mathcal{G}$.

The graph $\mathcal{G}$ can be completely characterized by its Laplacian matrix $L = D - A$ where $D = [d_{ij}]$ is the diagonal matrix of the vertex in-degrees, i.e.

$$d_{ii} = \sum_{j=1}^{N} W(j, i), \quad d_{ij} = 0 \text{ if } i \neq j,$$

and $A = [a_{ij}]$ is the adjacency matrix defined as

$$a_{ii} = 0, \qquad a_{ij} = W(j, i) \text{ if } i \neq j.$$

As all row-sums of L vanish, the Laplacian matrix has always at least one zero eigenvalue with corresponding right-eigenvector $\mathbf{1}$.

The properties of the Laplacian matrix L of an undirected graph are very well understood (Godsil and Royle, 2004; Fiedler, 1973). In particular, the connectedness of a graph can be expressed in terms of the second smallest eigenvalue $\lambda_2(L)$ of the Laplacian, which equals the algebraic connectivity

$$a(\mathcal{G}) := \min_{\substack{x^T \mathbf{1} = 0 \\ x \neq \mathbf{0}}} \frac{x^T L x}{x^T x}$$

of an undirected graph. Adding edges to an undirected graph, $a(\mathcal{G})$ cannot decrease. In a general directed graph, however, $a(\mathcal{G})$ cannot be related to graph connectivity without further knowledge of the topology. Instead, it was shown recently (Ren et al., 2004) that the eigenvalue with the second smallest real part can be used to characterize connectedness of a directed graph. Unfortunately, the second smallest real part may decrease when new edges are added to the graph.

The following facts are used throughout the paper:

Fact 1. The zero eigenvalue of L is simple and all the other eigenvalues have positive real part if and only if $\mathcal{G}$ contains a spanning tree (cf. Ren et al. (2004)).

Fact 2. If $\mathcal{G}$ is balanced, the algebraic connectivity satisfies $0 \leq a(\mathcal{G}) \leq \mathrm{Re}(\lambda_2(L))$, where $\lambda_2(L)$ is the eigenvalue of L with the second smallest real part, and $a(\mathcal{G}) > 0$ if and only if $\mathcal{G}$ is connected (cf. Wu (2005)).

In the remainder of this paper, we denote the eigenvalues of L as $\lambda_1(L), \dots, \lambda_N(L)$ with $\mathrm{Re}(\lambda_i(L)) \leq \mathrm{Re}(\lambda_{i+1}(L))$, $i = 1, \dots, N-1$ and $\lambda_1(L) = 0$. Let $\boldsymbol{p}^T$ be a left-eigenvector of L corresponding to the eigenvalue $\lambda_1(L) = 0$ and satisfying $\boldsymbol{p}^T \mathbf{1} = 1$. If $\mathcal{G}$ is balanced, then $\boldsymbol{p}^T = \frac{\mathbf{1}^T}{N}$. If $\mathcal{G}$ contains a spanning tree, $\boldsymbol{p}^T$ is uniquely defined because the zero eigenvalue is simple.

3. CONSENSUS ANALYSIS

We consider a MAS of N linear agents with dynamics

$$\xi_i^{(n)} + \alpha_{n-1} \xi_i^{(n-1)} + \dots + \alpha_0 \xi_i = u_i, \qquad i = 1, \dots, N, \quad (1)$$

where $\xi_i^{(k)}$ denotes the kth derivative of ξ_i and u_i is an input. Every controllable single-input linear system can be written as (1).

3.1 The consensus dynamics

In this section we analyze consensus in the group of agents defined by (1). The group of agents is said to reach consensus if the following requirement is satisfied:

(I) $\xi_i^{(k)}(t) - \xi_j^{(k)}(t) \to 0$ as $t \to \infty$ for all $i, j = 1, \dots, N$, $i \neq j$ and all $k = 0, \dots, n-1$.

In many applications, an additional requirement is that

(II) $\xi_i^{(k)}(t) < \infty$ for all $t > 0$, $i = 1, \dots, N$, and $k = 0, \dots, n-1$.

A consensus algorithm frequently found in literature is given as

$$u_i = \sum_{j=1}^{N} \sum_{k=0}^{n-1} \beta_k \gamma_i W(j, i) (\xi_j^{(k)} - \xi_i^{(k)}) \quad (2)$$

for $i = 1, \dots, N$, where $\beta_k > 0$, $k = 0, \dots, n-1$ and $\gamma_i > 0$, $i = 1, \dots, N$ are design parameters to be determined, while the weights $W(j, i)$, $i, j = 1, \dots, N$ are assumed to be given by the interconnection topology. The values γ_i, $i = 1, \dots, N$ can be interpreted as positive scalings of the weights of the edges incident to vertex i. Algorithm (2) uses only state information of agents j satisfying $j \in \mathcal{P}_i$ to determine the input of agent i, i.e. algorithm (2) is local in nature. Using the definitions

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \quad \Gamma = \text{diag}\left[\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_N \end{pmatrix}\right],$$

algorithm (2) can be written in matrix form as

$$\boldsymbol{u} = -\sum_{k=0}^{n-1} \beta_k \Gamma L \boldsymbol{\xi}^{(k)}. \tag{3}$$

Note that $\tilde{L} := \Gamma L$ is a Laplacian matrix corresponding to the graph $\tilde{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}, \tilde{W}\}$ with $\tilde{W}(i,j) := \gamma_j W(i,j)$. If $\mathcal{G}$ is strongly connected, γ_i, $i = 1, \ldots, N$ can be chosen such that $\tilde{\mathcal{G}}$ is balanced.

Theorem 3. Given the MAS (1) with algorithm (3), Requirement (I) is satisfied if and only if the polynomial

$$\prod_{j=2}^{N} \underbrace{\left(s^n + \sum_{k=0}^{n-1} (\alpha_k + \beta_k \lambda_j(\Gamma L)) s^k\right)}_{=: p_j(s)} \tag{4}$$

is Hurwitz. In that case all agents converge to a consensus state, i.e. $\xi_i^{(k)}(t) \to \zeta^{(k)}(t)$, $i = 1, \ldots, N$, $k = 0, \ldots, n-1$ for $t \to \infty$ and the consensus state $\zeta(t)$ evolves according to

$$\zeta^{(n)} + \alpha_{n-1}\zeta^{(n-1)} + \cdots + \alpha_0 \zeta = 0 \tag{5}$$

with initial condition given as the weighted average $\zeta^{(k)}(0) = \tilde{\boldsymbol{p}}^T \boldsymbol{\xi}^{(k)}(0)$, $k = 0, \ldots, n-1$ where $\tilde{\boldsymbol{p}} = \frac{\Gamma^{-1}\boldsymbol{p}}{\boldsymbol{p}^T \Gamma^{-1} \mathbf{1}}$.

To prove Theorem 3, we need the following Lemma, the proof of which is omitted due to space limitations:

Lemma 4. Given an $(N \times N)$ matrix L and constants α_k, β_k, $k = 0, \ldots, n-1$,

$$\begin{aligned} &\det\big(s^n I + \sum_{k=0}^{n-1} s^k (\alpha_k I + \beta_k L)\big) \\ &= \prod_{i=1}^{N} \Big(s^n + \sum_{k=0}^{n-1} s^k \big(\alpha_k + \beta_k \lambda_i(L)\big)\Big). \end{aligned}$$

Proof. The closed loop consisting of N agents described by (1) and the consensus algorithm (3) reads

$$\boldsymbol{\xi}^{(n)} = -\sum_{k=0}^{n-1} (\alpha_k I + \beta_k \Gamma L) \boldsymbol{\xi}^{(k)}.$$

Define $\zeta(t) = \tilde{\boldsymbol{p}}^T \boldsymbol{\xi}(t)$, $\eta_i(t) = \xi_i(t) - \xi_1(t)$, $i = 2, \ldots, N$, and $\boldsymbol{\eta} = (\eta_2, \ldots, \eta_N)^T$, i.e.

$$\begin{pmatrix} \zeta \\ \boldsymbol{\eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \tilde{\boldsymbol{p}}^T \\ 0 \end{pmatrix}}_{=: S_1} \boldsymbol{\xi} + \underbrace{\begin{pmatrix} 0 & \mathbf{0}^T \\ -\mathbf{1} & I \end{pmatrix}}_{=: S_2} \boldsymbol{\xi} = S\boldsymbol{\xi},$$

with $S = S_1 + S_2$ and

$$S^{-1} = T = \underbrace{\begin{pmatrix} 1 & -\tilde{p}_2 & \ldots & -\tilde{p}_N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -\tilde{p}_2 & \ldots & -\tilde{p}_N \end{pmatrix}}_{=: T_1} + \underbrace{\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & I \end{pmatrix}}_{=: T_2}.$$

Note that $S\Gamma L T = S_2 \Gamma L T_2$. Consequently, we obtain the transformed dynamics

$$\zeta^{(n)} = -\sum_{k=0}^{n-1} \alpha_k \zeta^{(k)}, \tag{6}$$

$$\boldsymbol{\eta}^{(n)} = -\sum_{k=0}^{n-1} \left(\alpha_k I + \beta_k \begin{pmatrix} -\mathbf{1} & I \end{pmatrix} \Gamma L \begin{pmatrix} \mathbf{0}^T \\ I \end{pmatrix}\right) \boldsymbol{\eta}^{(k)}. \tag{7}$$

Requirement (I) is satisfied if and only if $\boldsymbol{\eta}^{(k)}(t) \to \mathbf{0}$, $k = 1, \ldots, n-1$ for $t \to \infty$ in which case $\xi_i^{(k)}(t) \to \zeta^{(k)}(t)$, $i = 1, \ldots, N$, $k = 0, \ldots, n-1$ and $\zeta(t)$ evolves according to (5). It remains to verify that the assertions given in the theorem are satisfied if and only if the $\boldsymbol{\eta}$-dynamics (7) is stable, i.e. the characteristic polynomial of the $\boldsymbol{\eta}$-dynamics is given by (4). To that end, let $K_k = -\alpha_k I - \beta_k \Gamma L$, $k = 1, \ldots, N$ and define $\boldsymbol{x} = \left(\boldsymbol{\xi}^T, \dot{\boldsymbol{\xi}}^T, \ldots, (\boldsymbol{\xi}^{(n-1)})^T\right)^T$ to write the closed loop dynamics in state-space form as

$$\dot{\boldsymbol{x}} = \underbrace{\left(\begin{array}{cccc|c} 0 & I & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & I & 0 \\ & & & 0 & I \\ \hline K_0 & \cdots & \cdots & K_{n-2} & K_{n-1} \end{array}\right)}_{=: M} \boldsymbol{x}.$$

The characteristic equation of the dynamic matrix M can be computed using the Schur complement as

$$\begin{aligned} \det(sI - M) &= s^{(n-1)N} \det\left(sI - \sum_{k=0}^{n-1} \frac{K_k}{s^{n-1-k}}\right) \\ &= \det\left(s^n I - \sum_{k=0}^{n-1} s^k K_k\right) = 0. \end{aligned}$$

Using Lemma 4 and the fact that the first eigenvalue of ΓL is $\lambda_1(\Gamma L) = 0$, the characteristic polynomial can be rewritten as

$$\underbrace{\left(s^n + \sum_{k=0}^{n-1} \alpha_k s^k\right)}_{=: p_1(s)} \prod_{j=2}^{N} p_j(s) \tag{8}$$

where $p_1(s)$ corresponds to the ζ-dynamics (6) and hence the remaining part, which equals to (4), corresponds to the $\boldsymbol{\eta}$-dynamics (7). □

Corollary 5. Given the MAS (1) with algorithm (3), Requirement (II) is satisfied if and only if (5) is stable, i.e. if the dynamics of the individual agent given by (1) with input $u_i \equiv 0$ is stable.

Proof. By Theorem 3, $\xi_i^{(k)}(t) \to \zeta^{(k)}(t)$, $i = 1, \ldots, N$, $k = 0, \ldots, n-1$, i.e. $\xi_i^{(k)}(t) < \infty$ iff $\zeta^{(k)}(t) < \infty$. □

Remark 6. Polynomial (4) has real coefficients even if L has eigenvalues with non-zero imaginary part: If $\lambda_{i_+}(\tilde{L})$ and $\lambda_{i_-}(\tilde{L})$ are complex conjugate eigenvalues of $\tilde{L}$, then $p_{i_+}(s) \cdot p_{i_-}(s)$ has real coefficients.

Remark 7. By Theorem 3, the consensus state $\zeta(t)$ evolves independently of the interconnection topology given by L but the initial condition does depend on the interconnection topology.

3.2 The consensus state

It was shown in Theorem 3 that the initial value of the consensus state depends on the left-eigenvector $\boldsymbol{p}^T$ of L

corresponding to the eigenvalue zero. More precisely, agent i's initial state has some influence on the consensus state if and only if $\boldsymbol{p}^T \boldsymbol{e}_i \neq 0$. The theorem given below enlightens this fact from an algebraic graph theory point of view. To that end, we need the following definitions and lemma (a direct corollary to the Perron-Frobenius Theorem):

A matrix A is said to be non-negative (positive), denoted $A \geq 0$ ($A > 0$), if all its elements are non-negative (positive). Equivalently, a vector $\boldsymbol{x}$ is said to be non-negative (positive), denoted $\boldsymbol{x} \geq 0$ ($\boldsymbol{x} > 0$), if all its components are non-negative (positive). A matrix $A \geq 0$ is said to be stochastic if $A\mathbf{1} = \mathbf{1}$; the spectral radius of a stochastic matrix A is $\rho(A) = 1$. The graph induced by an $(N \times N)$ matrix $A = [a_{ij}]$ is the graph $\mathcal{G}(A)$ on N vertices that contains an edge from vertex j to vertex i if and only if $a_{ij} \neq 0$. A is irreducible if and only if $\mathcal{G}(A)$ is strongly connected.

Lemma 8. (adapted from Horn and Johnson (1985)). If A is an irreducible stochastic matrix, then $\lambda = 1 = \rho(A)$ is a simple eigenvalue, $A\mathbf{1} = \mathbf{1}$ and $\exists \boldsymbol{q} > 0 : \boldsymbol{q}^T A = \boldsymbol{q}^T$.

Theorem 9. Given a graph $\mathcal{G}$ with Laplacian matrix L and a vector $\boldsymbol{p}$ satisfying $\boldsymbol{p}^T L = \mathbf{0}^T$ and $\boldsymbol{p}^T \mathbf{1} = 1$. Assume $\mathcal{G}$ contains a spanning tree. Then $\boldsymbol{p} \geq 0$ and $\boldsymbol{p}^T \boldsymbol{e}_i > 0$ if and only if the ith vertex of $\mathcal{G}$ can be chosen as the root of a spanning tree in $\mathcal{G}$.

Proof. Denote the set of roots of spanning trees in $\mathcal{G}$ as $\mathcal{V}_0 = \{i \in \mathcal{V} | \{i\} \bigcup \mathcal{D}_i = \mathcal{V}\} \subseteq \mathcal{V}$ with $N_0 = |\mathcal{V}_0| > 0$ and define $\mathcal{V}_1 = \mathcal{V} \setminus \mathcal{V}_0$ with $N_1 = |\mathcal{V}_1| \geq 0$. There exists no directed path from any element of $\mathcal{V}_1$ to any element of $\mathcal{V}_0$. Thus, without loss of generality, we assume

$$L = \begin{pmatrix} L_0 & 0 \\ L_\times & L_1 \end{pmatrix}$$

where L_0 is a $N_0 \times N_0$ Laplacian matrix corresponding to the connections between the elements of $\mathcal{V}_0$. Let D_0 be the matrix of vertex in-degrees and A_0 the adjacency matrix such that $L_0 = D_0 - A_0$; L_0 corresponds to a strongly connected graph $\mathcal{G}_0$, hence D_0^{-1} exists and A_0 is irreducible. Define $\tilde{L}_0 = D_0^{-1} L_0 = I - D_0^{-1} A_0$. Let $\tilde{A}_0 = D_0^{-1} A_0$ which is again an irreducible matrix. From $\tilde{L}_0 \mathbf{1} = \mathbf{0}$ we deduce $\tilde{A}_0 \mathbf{1} = \mathbf{1}$, hence $\tilde{A}_0$ is stochastic.

By Lemma 8, there is a vector $\tilde{\boldsymbol{q}}_0 > 0$ such that $\tilde{\boldsymbol{q}}_0^T \tilde{A}_0 = \tilde{\boldsymbol{q}}_0^T$, which implies $\tilde{\boldsymbol{q}}_0^T \tilde{L}_0 = \mathbf{0}^T$. It follows that $\boldsymbol{q}_0 = D_0^{-1} \tilde{\boldsymbol{q}}_0 > 0$ and $\boldsymbol{q}_0^T L_0 = \mathbf{0}^T$. Hence $\boldsymbol{q}^T = (\boldsymbol{q}_0^T, \mathbf{0}^T) \geq 0$ is a left-eigenvector of L with $\boldsymbol{q}^T \boldsymbol{e}_i > 0$ if $i \in \mathcal{V}_0$, $\boldsymbol{q}^T \boldsymbol{e}_i = 0$ if $i \in \mathcal{V}_1$, and $\boldsymbol{q}^T L = \mathbf{0}^T$. Finally $\boldsymbol{p} = \frac{\boldsymbol{q}}{\boldsymbol{q}^T \mathbf{1}}$ satisfies $\boldsymbol{p}^T \boldsymbol{e}_i > 0$ if and only if $i \in \mathcal{V}_0$ ($\boldsymbol{p}^T \boldsymbol{e}_i = 0$ if $i \in \mathcal{V}_1$) and $\boldsymbol{p}$ is the unique vector satisfying $\boldsymbol{p}^T L = \mathbf{0}^T$ and $\boldsymbol{p}^T \mathbf{1} = 1$. □

Theorem 9 conveys that the consensus state is influenced by the initial state of agent i iff there is a directed path from vertex i to every other vertex in the graph. Intuitively, as all agents converge to the consensus state, it can only depend on information that is available to all agents at least indirectly. Observe that

$$\zeta^{(k)}(0) = \tilde{\boldsymbol{p}}^T \boldsymbol{\xi}^{(k)}(0) = \frac{\sum_{i \in \mathcal{V}_0} \frac{p_i}{\gamma_i} \xi_i^{(k)}(0)}{\sum_{i \in \mathcal{V}_0} \frac{p_i}{\gamma_i}}$$

is a weighted average of the initial states of agents corresponding to the vertices in $\mathcal{V}_0$ where the weights can be adjusted arbitrarily by appropriately choosing the values $\gamma_i, i \in \mathcal{V}_0$. The remaining values $\gamma_i, i \notin \mathcal{V}_0$ do not affect the consensus state but may affect the eigenvalues of $\tilde{L}$.

4. CONSENSUS DESIGN

So far we reduced the problem of checking convergence to consensus to investigate whether a given polynomial is Hurwitz. The coefficients of the polynomial depend on the interconnection topology, the single agent dynamics, and the consensus algorithm, namely the values β_k, $k = 0, \ldots, n-1$. In this section, we derive methods to determine the gains β_k, $k = 0, \ldots, n-1$ in the consensus algorithm, such that convergence to consensus is guaranteed. In the sequel we assume $\gamma_i = 1$, $i = 1, \ldots, N$, i.e. $\Gamma = I$ for simplicity. If $\gamma_i \neq 1$ for some i, all results remain valid substituting L by $\tilde{L}$.

4.1 Double integrator agents

We first consider the case of double-integrator dynamics

$$\ddot{\xi}_i = u_i, \quad i = 1, \ldots, N \tag{9}$$

which received a lot of attention in consensus literature (de Gennaro and Jadbabaie, 2006; Olfati-Saber, 2006; Ren and Atkins, 2005). Applying the consensus algorithm

$$u_i = \beta_0 W(j,i)(\xi_j - \xi_i) + \beta_1 W(j,i)(\dot{\xi}_j - \dot{\xi}_i), \tag{10}$$

we know that the consensus state grows unbounded since (9) is unstable. For problems where this unboundedness is acceptable, the following theorem gives a necessary and sufficient condition for convergence to consensus.

Theorem 10. Assume the group of agents defined by (9) with consensus algorithm (10) and interconnection topology $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$. The agents reach consensus if and only if $\mathcal{G}$ contains a spanning tree, $\beta_0 > 0$, $\beta_1 > 0$, and

$$\frac{\beta_1^2}{\beta_0} > \max_{i=2,\ldots,N} \frac{1}{\mathrm{Re}(\lambda_i(L))} \left(\frac{|\mathrm{Im}(\lambda_i(L))|}{|\lambda_i(L)|} \right)^2. \tag{11}$$

Proof. Using Theorem 3 we need to check that

$$\prod_{j=2}^{N} \underbrace{\left(s^2 + \beta_1 \lambda_j(L) s + \beta_0 \lambda_j(L)\right)}_{=: \, p_j(s)} \tag{12}$$

is Hurwitz if and only if $\beta_0 > 0$, $\beta_1 > 0$, (11) holds, and $\mathcal{G}$ contains a spanning tree. Polynomial (12) is Hurwitz if and only if all real factors of (12) are Hurwitz.

In the first part of the proof, we shall show that (12) is not Hurwitz if $\mathcal{G}$ does not contain a spanning tree. In the second part of the proof we show that, under the assumption that $\mathcal{G}$ contains a spanning tree, (12) is Hurwitz if and only if $\beta_0 > 0$, $\beta_1 > 0$, and (11) holds.

Part 1: If $\mathcal{G}$ does not contain a spanning tree, $\lambda_2(L) = 0$, thus $p_2(s) = s^2$ is not Hurwitz.

Part 2: In the sequel, we assume that $\mathcal{G}$ contains a spanning tree, thus $\mathrm{Re}(\lambda_i(L)) > 0$, $i = 2, \ldots, N$. Assume $\mathrm{Im}(\lambda_i(L)) = 0$ for some i. Then the correpsonding $p_i(s)$ is Hurwitz if and only if $\beta_0 \lambda_i(L) > 0$ and $\beta_1 \lambda_i(L) > 0$ which is equivalent to $\beta_0 > 0$ and $\beta_1 > 0$.

If $\mathrm{Im}(\lambda_{i_+}(L)) \neq 0$ for some i_+, $\exists i_-$ such that $\lambda_{i_+}(L)$ and $\lambda_{i_-}(L)$ are complex conjugate. In that case $p_{i_+}(s) \cdot p_{i_-}(s)$ has real coefficients. Let $\lambda_{i_\pm}(L) = \sigma_i \pm \jmath\omega_i$, then

$$p_{i+}(s) \cdot p_{i-}(s) = \begin{pmatrix} s^4 \\ s^3 \\ s^2 \\ s \\ 1 \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ 2\beta_1 \sigma_i \\ 2\beta_0 \sigma_i + \beta_1^2(\sigma_i^2 + \omega_i^2) \\ 2\beta_0 \beta_1 (\sigma_i^2 + \omega_i^2) \\ \beta_0^2 (\sigma_i^2 + \omega_i^2) \end{pmatrix}$$

which is Hurwitz if and only if $\beta_0 > 0$, $\beta_1 > 0$, and

$$\frac{\beta_1^2}{\beta_0} > \frac{1}{\sigma_i}\left(\frac{|\omega_i|}{|\sigma_i + \mathrm{j}\omega_i|}\right)^2. \qquad (13)$$

For $\omega_i = 0$, condition (13) reduces to $\beta_1^2/\beta_0 > 0$ which is satisfied for any $\beta_0 > 0$, $\beta_1 > 0$. Consequently, (13) is satisfied for $i = 2, \ldots, N$ iff (11) is satisfied. Hence, if $\mathcal{G}$ contains a spanning tree, (12) is Hurwitz iff $\beta_0 > 0$, $\beta_1 > 0$ and (11) hold. □

Corollary 11. Assume the group of agents as in Theorem 10 and $\mathcal{G}$ balanced. If $\beta_1 > 0$ and

$$0 < \frac{\beta_0}{\beta_1^2} \leq a(\mathcal{G}),$$

then $\xi_i^{(k)}(t) \to \zeta^{(k)}(t)$, $i = 1, \ldots, N$, $k = 0, \ldots, n-1$.

Proof. Corollary 11 is a direct consequence of Theorem 10 and Fact 2.

Remark 12. Choosing $\beta_0 = 1$, condition (11) of Theorem 10 reduces to

$$\beta_1 > \max_{i=2,\ldots,N} \sqrt{\frac{1}{\mathrm{Re}(\lambda_i(L))}\frac{|\mathrm{Im}(\lambda_i(L))|}{|\lambda_i(L)|}}$$

As this bound on β_1 is necessary and sufficient for consensus, it needs to be tighter than the sufficient bound

$$\beta_1 > \max_{i=2,\ldots,N} \sqrt{\frac{2}{|\lambda_i(L)|\cos(\frac{\pi}{2} - \tan^{-1}\frac{\mathrm{Re}(\lambda_i(L))}{\mathrm{Im}(\lambda_i(L))})}}$$

of Theorem IV.2 in Ren and Atkins (2005). This fact can be seen observing that

$$\sqrt{\frac{2}{|\lambda_i(L)|\cos(\frac{\pi}{2} - \tan^{-1}\frac{\mathrm{Re}(\lambda_i(L))}{\mathrm{Im}(\lambda_i(L))})}} = \sqrt{\frac{2}{\mathrm{Re}(\lambda_i(L))}}.$$

In fact, given some $r > 0$, if $\beta_1 > 0$, $\beta_0 = r\beta_1 > 0$, and

$$\frac{\beta_1^2}{\beta_0} = \frac{\beta_1}{r} \geq \max_{i=2,\ldots,N} \frac{2}{\mathrm{Re}(\lambda_i(L))} \qquad (14)$$

it can be observed after some computation that all roots s_i of polynomial (12) satisfy $\mathrm{Re}(s_i) \leq -\frac{\beta_0}{\beta_1} = -r < 0$.

4.2 General LTI single-input systems

For the general case of agents modeled as LTI system (1) with consensus algorithm (2), an LMI-based method to find values β_k, $k = 0, \ldots, n-1$ that guarantee convergence to consensus is given in the following theorem.

Theorem 13. Assume the group of agents defined by (1) with consensus algorithm (3) using $\Gamma = I$ and interconnection topology $\mathcal{G}$ containing a spanning tree. Define

$$M_0 = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix}.$$

If, for some $\nu \geq 0$, there exist a matrix $Q \in \mathbb{R}^{N\times N}$, $Q = Q^T \succ 0$ and a vector $\boldsymbol{\kappa} \in \mathbb{R}^N$ such that the LMIs

$$C_0 + \mathrm{Re}(\lambda_i(L))C_R + \mathrm{Im}(\lambda_i(L))C_I \prec 0 \qquad (15)$$

with

$$C_0 = \begin{pmatrix} QM_0^T + M_0Q + 2\nu Q & 0 \\ 0 & QM_0^T + M_0Q + 2\nu Q \end{pmatrix},$$

$$C_R = -\begin{pmatrix} \boldsymbol{e}_n\boldsymbol{\kappa}^T + \boldsymbol{\kappa}\boldsymbol{e}_n^T & 0 \\ 0 & \boldsymbol{e}_n\boldsymbol{\kappa}^T + \boldsymbol{\kappa}\boldsymbol{e}_n^T \end{pmatrix},$$

$$C_I = \begin{pmatrix} 0 & \boldsymbol{\kappa}\boldsymbol{e}_n^T - \boldsymbol{e}_n\boldsymbol{\kappa}^T \\ \boldsymbol{e}_n\boldsymbol{\kappa}^T - \boldsymbol{\kappa}\boldsymbol{e}_n^T & 0 \end{pmatrix}$$

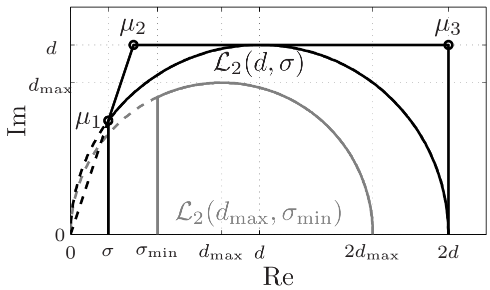

Fig. 1. Sets containing the non-zero eigenvalues of L.

are satisfied for $i = 2, \ldots, N$, the agents reach consensus choosing $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{n-1})^T = Q^{-1}\boldsymbol{\kappa}$, and the roots s_j of polynomial (4) satisfy $\mathrm{Re}(s_j) < -\nu$, $j = 1, \ldots, nN$.

Proof. Assume $\lambda_i(L) = \sigma_i + \mathrm{j}\omega_i$, $i = 2, \ldots N$. The characteristic polynomial of

$$\dot{\boldsymbol{w}} = \underbrace{\begin{pmatrix} M_0 & 0 \\ 0 & M_0 \end{pmatrix}}_{=:\tilde{M}}\boldsymbol{w} - \underbrace{\begin{pmatrix} \sigma_i\boldsymbol{e}_n & \omega_i\boldsymbol{e}_n \\ -\omega_i\boldsymbol{e}_n & \sigma_i\boldsymbol{e}_n \end{pmatrix}}_{=:\tilde{B}}\underbrace{\begin{pmatrix} \boldsymbol{\beta}^T & 0 \\ 0 & \boldsymbol{\beta}^T \end{pmatrix}}_{=:\tilde{K}}\boldsymbol{w}$$

is given by $p_i^2(s)$ if $\lambda_i = \sigma_i$ ($\omega_i = 0$) or $p_{i_+}(s) \cdot p_{i_-}(s)$ if $\lambda_{i_\pm} = \sigma_i \pm \mathrm{j}\omega_i$. Hence, if there exist a constant $\nu \geq 0$ and a matrix $\tilde{P} = \tilde{P}^T \succ 0$ such that

$$\tilde{M}^T\tilde{P} + \tilde{P}\tilde{M} - \tilde{K}^T\tilde{B}^T\tilde{P} - \tilde{P}\tilde{B}\tilde{K} \prec -2\nu\tilde{P}, \qquad (16)$$

polynomials $p_i^2(s)$ as well as $p_{i_+}(s) \cdot p_{i_-}(s)$ are Hurwitz with roots s_j satisfying $\mathrm{Re}(s_j) < -\nu$, $j = 1, \ldots, 2n$. With

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad Q = P^{-1} \succ 0, \quad \boldsymbol{\kappa} = Q\boldsymbol{\beta},$$

LMI condition (16) reduces to (15). □

The design proposed in Theorem 13 contains some sources of conservatism due to the fixed block-diagonal structure of $\tilde{P}$ and most notably because the same P is used for all $\lambda_i(L)$, $i = 2, \ldots, N$. Allowing different matrices P_i for different eigenvalues $\lambda_i(L)$ would result in a nonlinear matrix inequality due to the terms $P_i\boldsymbol{e}_n\boldsymbol{\beta}$.

Exploiting convexity, the LMI conditions of Theorem 13 can be relaxed to obtain conditions that are easier to check. Denote $\mathcal{L}_1(L) = \mathrm{conv}(\{\lambda_2(L), \ldots, \lambda_N(L)\})$ where $\mathrm{conv}(\cdot)$ is the convex hull. Observe that $C(\lambda) = C_0 + \mathrm{Re}(\lambda)C_R + \mathrm{Im}(\lambda)C_i$ is affine in λ and $C(\lambda) \prec 0$ if and only if $C(\overline{\lambda}) \prec 0$ where $\overline{\lambda}$ denotes the complex conjugate of λ. Hence, given values $\mu_i \in \mathbb{C}, i = 1, \ldots, q$ such that $\mathcal{L}_1(L) \subseteq \mathrm{conv}(\{\mu_1, \ldots, \mu_q, \overline{\mu}_1, \ldots, \overline{\mu}_q\})$, $C(\mu_i) \prec 0$, $i = 1, \ldots, q$ is a sufficient condition for LMI (15) being satisfied for $i = 2, \ldots, N$.

By Gershgorin's disk theorem (see Horn and Johnson (1985)) we know that $\lambda_i(L) \in \mathcal{B}(d_{\max}) := \{\lambda \in \mathbb{C} : \|\lambda - d_{\max}\| \leq d_{\max}\}$ for $i = 1, \ldots, N$ where $d_{\max} = \max_i \sum_{j\in\mathcal{P}_i} W(j,i)$ is the maximum vertex in-degree. Hence, given any $\sigma \leq \sigma_{\min} := \mathrm{Re}(\lambda_2(L))$ and any $d \geq d_{\max}$, one obtains $\mathcal{L}_1(L) \subseteq \mathcal{L}_2(d,\sigma) := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq \sigma\} \bigcap \mathcal{B}(d)$. Let $\mu_1 = \sigma + \mathrm{j}\sigma\sqrt{\frac{2d-\sigma}{\sigma}}$, $\mu_2 = d\sqrt{\frac{\sigma}{2d-\sigma}} + \mathrm{j}d$, and $\mu_3 = 2d + \mathrm{j}d$ to obtain a simple polygon containing $\mathcal{L}_2(d,\sigma)$, namely $\mathrm{conv}(\{\mu_1, \mu_2, \mu_3, \overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3\}) \supseteq \mathcal{L}_2(d,\sigma) \supseteq \mathcal{L}_1(L)$. Figure 1 illustrates the sets $\mathcal{L}_2(d,\sigma)$, $\mathcal{L}_2(d_{\max},\sigma_{\min})$, and the values μ_i, $i = 1, \ldots, 3$. A sufficient condition for LMI (15) being satisfied for $i = 2, \ldots N$ is

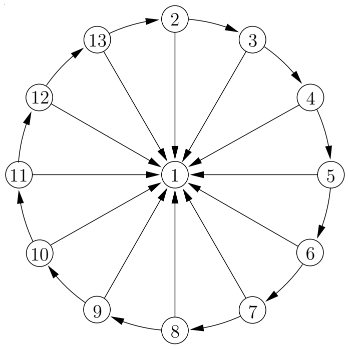

Fig. 2. Graph $\mathcal{G}$ of an imploding star with a directed circle.

given by $C(\mu_i) \prec 0$, $i = 1, \ldots, 3$. If the interconnection topology is balanced, one may choose $\sigma = a(\mathcal{G})$.

5. EXAMPLE

As an example we consider again individual agents modeled as (9). We assume the interconnection topology given by Figure 2 with all edge weights equal to one. All vertices on the circle can be chosen as the root of a spanning tree. The spectrum of the Laplacian L of $\mathcal{G}$ reads

$$\lambda(L) \in \Big\{0, \frac{(2-\sqrt{3}) \pm \mathrm{j}}{2}, \frac{1 \pm \mathrm{j}\sqrt{3}}{2}, 1 \pm \mathrm{j}, \frac{3 \pm \mathrm{j}\sqrt{3}}{2}, \frac{(2+\sqrt{3}) \pm \mathrm{j}}{2}, 2, 12\Big\}.$$

In accordance with Theorem 9, one obtains the vector $\boldsymbol{p} = (0, \frac{1}{12}, \ldots, \frac{1}{12})^T$, i.e. all nodes on the circle have the same influence on the consensus state while the node in the center does not influence the consensus state. As graph $\mathcal{G}$ is not balanced, the algebraic connectivity $a(\mathcal{G})$ is meaningless in this example (in fact $a(\mathcal{G}) < 0$). Conditions (11) and (14) evaluate to

$$\frac{\beta_1^2}{\beta_0} > \frac{7+4\sqrt{3}}{2} \approx 6.96, \quad \frac{\beta_1^2}{\beta_0} = \frac{\beta_1}{r} > \frac{4}{2-\sqrt{3}} \approx 14.93,$$

respectively. Using the LMI of Theorem 13 with $\nu = 2$, one obtains $\beta_0 \approx 916.49$ and $\beta_1 \approx 245.05$ as a solution with $r = \frac{\beta_0}{\beta_1} \approx 3.740$ and $\frac{\beta_1}{r} \approx 65.52 > 14.93$. Hence, we know that $\mathrm{Re}(s_i) \leq -r$ and $\mathrm{Re}(s_i) \leq -\nu$ by Remark 12 and Theorem 13, respectively; here $r > \nu$, i.e. Remark 12 gives a better estimate of the convergence rate than Theorem 13. In fact, $\max_i \mathrm{Re}(s_i) \approx -3.745$ which is very close to r. Figure 3 shows simulation results for some initial conditions satisfying $\boldsymbol{p}^T\boldsymbol{\xi}(0) = \zeta(0) = 10$ and $\boldsymbol{p}^T\dot{\boldsymbol{\xi}}(0) = \dot{\zeta}(0) = -1$. All agents converge to consensus exponentially with convergence rate larger than r. The consensus state evolves according to $\zeta(t) = \zeta(0) + \dot{\zeta}(0)t = 10 - t$, $\dot{\zeta}(t) = \dot{\zeta}(0) = -1$ as expected.

6. SUMMARY

This paper was concerned with the consensus problem for MASs with agents modeled as general LTI systems. Motivated by the fact that there exist only few results on the problem, the present paper generalized several existing results on consensus with agents modeled as a single or double integrator. The characteristic equation of the whole MAS was derived explicitly, which was then used to obtain a condition on the consensus algorithm ensuring convergence. In addition, a meaningful interpretation for the role of the left-eigenvector of the Laplacian corresponding to its zero eigenvalue was given. Namely this eigenvector quantifies the influence of each agent on the consensus state. As far as design is concerned, a systematic method was proposed to choose the gains in the consensus algorithm such that the MAS reaches consensus asymptotically with prespecified convergence rate.

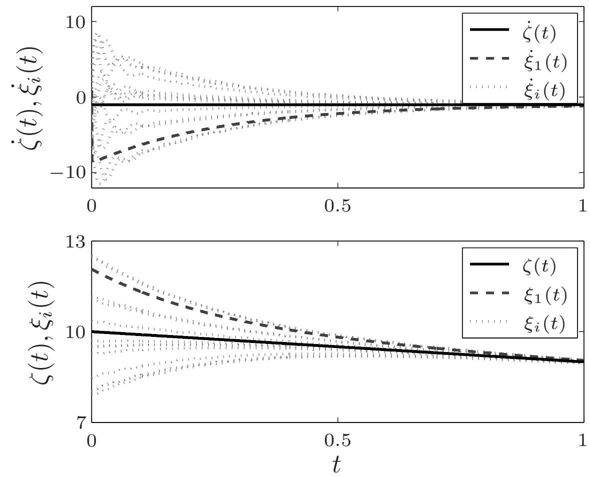

Fig. 3. Simulation results for topology given in Figure 2.

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