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Well-posed bimodal piecewise linear systems do not exhibit Zeno behavior

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Abstract: The phenomenon of infinitely mode transitions in a finite time interval is called Zeno behavior in hybrid systems literature. It plays a critical role in the study of numerical methods and fundamental system and control theoretic properties of hybrid systems. This paper studies Zeno behavior for bimodal piecewise linear systems with possibly discontinuous dynamics. Our treatment is inspired by the work of Imura and Van der Schaft on the well-posedness of the same type of systems. The main contribution of the paper is two folded. Firstly, we show that Imura-Van der Schaft conditions for well-posedness guarantee that Filippov solutions have certain local properties. Secondly, we employ these in order to prove that bimodal piecewise linear systems do not exhibit Zeno behavior.

Keywords: Hybrid systems, piecewise linear systems, bimodal systems, Filippov solutions, Zeno behavior.

1. INTRODUCTION

Piecewise linear systems form a subclass of hybrid systems that are encountered in various application areas. As these systems exhibit 'nonsmooth' behavior, their study diverges from the mainstream nonlinear systems and control theory which is, generally speaking, based on certain 'smoothness' assumptions. In fact, analysis and design of such systems require to deal with differential equations with discontinuous right hand sides (see Filippov [1988]).

Zeno behavior is a very curious phenomenon in hybrid systems. It corresponds to the accumulation of event times, i.e. times for which the system undergoes mode transition. If a system exhibits Zeno behavior, the active mode changes infinitely many times in a finite time interval. This, very often, complicates the analysis of the system behavior. In the literature, this type of behavior was already studied in different contexts. More than two decades ago, the papers Brunovsky [1980] and Sussmann [1982] addressed the switching behavior of piecewise analytic systems. In the hybrid systems context, the study of Zeno behavior has gained a considerable attention. In the hybrid automata framework, the papers Johansson et al. [1999], Zhang et al. [2000], Zhang et al. [2001] and Simic et al. [2005] investigates Zeno phenomenon and related issues.

For subclass of piecewise linear systems, the papers Camlibel and Schumacher [2001], Shen and Pang [2005] and Camlibel et al. [2006] proved absence of this critical behavior under certain conditions. The work Pang and Shen [2007] extended the results in Shen and Pang [2005] even to the nonlinear case. However, all these papers consider systems with continuous dynamics.

This paper investigates the simplest possible piecewise linear systems, namely bimodal systems. These systems constitute of two linear subsystems separated by a hyperplane. The dynamics along the hyperplane is, in general, discontinuous. As such, one of the very immediate issue is well-posedness in the sense of existence and uniqueness of solution. In Imura and Van der Schaft [2000], the authors addressed the issue of well-posedness and derived certain conditions under which the system has a unique forward Carathéodory [1] solution starting from any initial state. This solution concept rules out the possibility of left accumulation of event times by definition. A more natural solution concept is Filippov solutions (see Filippov [1988]) which does not rule out (left/right) Zeno phenomenon. The first contribution of this paper is to show that a Filippov solution for a bimodal system is necessarily a forward Carathéodory solution under the conditions presented in Imura and Van der Schaft [2000]. This automatically means that these conditions guarantee the absence

[1] In fact, the authors used the terminology 'extended Carathéodory' solutions. In this paper, we will call such solutions 'forward Carathéodory' solutions in Imura and Van der Schaft [2000]. As it will become clear later (see Section 2), we prefer the current terminology.

of left accumulation of events. Further, we will prove the absence of right accumulation of events by using similar arguments backward in time. Combining left/right non-Zenoness property, we will reach the ultimate conclusion: well-posed bimodal systems do not exhibit Zeno behavior.

The organization of the paper is as follows. We introduce different solution concepts, illustrate their differences, and state the main results in Section 2. Section 3 serves in summarizing and translating these results into the terms of the Zeno behavior. The paper will be closed by conclusions in Section 4.

2. SOLUTIONS OF BIMODAL SYSTEMS

Consider the bimodal piecewise linear system

$$\dot{x}(t) \in \begin{cases} A_1 x(t) & \text{if } y(t) \leqslant 0 \\ A_2 x(t) & \text{if } y(t) \geqslant 0 \end{cases} \tag{1a}$$

$$y(t) = c^T x(t) \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$, all matrices are of appropriate sizes, and $c \neq 0$.

We say that an absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is

- a *Carathéodory solution* for the initial state x_0 if x satisfies (1) for almost all $t \geqslant 0$ with $x(0) = x_0$.
- a *forward Carathéodory solution* for the initial state x_0 if it is a Carathéodory solution for the initial state x_0 and for each $t^* \geqslant 0$ there exists $\varepsilon_{t^*} > 0$ such that either
$$\dot{x}(t) = A_1 x(t) \text{ and } c^T x(t) \leqslant 0 \tag{2}$$
or
$$\dot{x}(t) = A_2 x(t) \text{ and } c^T x(t) \geqslant 0 \tag{3}$$
for all $t \in (t^*, t^* + \varepsilon_{t^*})$.
- a *backward Carathéodory solution* for the initial state x_0 if it is a Carathéodory solution for the initial state x_0 and for each $t^* > 0$ there exists $\varepsilon_{t^*} > 0$ such that either (2) or (3) holds for all $t \in (t^* - \varepsilon_{t^*}, t^*)$.
- a *Filippov solution* if x satisfies
$$\dot{x}(t) \in F(x(t))$$
for almost all $t \geqslant 0$ where the set-valued function F is given by
$$F(x) = \begin{cases} \{A_1 x\} & \text{if } c^T x < 0 \\ \operatorname{conv}(\{A_1 x, A_2 x\}) & \text{if } c^T x = 0 \\ \{A_2 x\} & \text{if } c^T x > 0 \end{cases}$$
where $\operatorname{conv}(S)$ denotes the convex hull of the set S.

To make the dependence on the initial state explicit, we write x^{x_0} for a solution with the initial state x_0 and y^{x_0} for the corresponding y.

Roughly speaking, forward (backward) solutions do not have left (right) accumulation of 'mode transitions' (or 'event times'). Later, we will elaborate on the mode transitions in a precise manner.

Clearly, a (forward/backward) Carathéodory solution is a Filippov solution. As illustrated in the following example, the reverse implication does not hold in general.

Example 2.1. [Pogromsky et al., 2003, Rem. 2] Consider the bimodal system (1) where

$$A_1 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \\ 0&0&0&0 \end{bmatrix}, \quad c^T = \begin{bmatrix} 1&0&0&0 \end{bmatrix}$$

and $x_0 = \operatorname{col}(0,0,0,1)$. As a consequence of [Pogromsky et al., 2003, Thm. 4], there are infinitely many Filippov solutions for this initial state whereas there does not exist any forward Carathéodory solution.

Filippov solutions seem to be more natural than forward/backward solutions in a number of ways. First of all, the definition of forward/backward solutions is asymmetric in time. Secondly, Filippov solutions have two crucial properties that are not possessed by forward/backward solutions:

- the set of Filippov solutions defined on a time interval $[t_0, t_1]$ with initial states from a given compact set is compact with respect to the $C[t_0, t_1]$ topology (see [Filippov, 1988, Thm. 2.7.3]),
- uniqueness of Filippov solutions implies continuous dependence on the initial states(see [Filippov, 1988, Thm. 2.8.2]).

In what follows, we will focus on the Filippov solutions of (1). To do so, we first summarize the results of the paper Imura and Van der Schaft [2000] which provides a detailed and meticulous treatment for the forward Carathéodory solution concept. Essentially, it gives necessary and sufficient conditions for the existence and uniqueness of such solutions. To formalize these conditions, let the nonnegative integers h_i for $i = 1, 2$ be such that the matrices

$$T_i = \begin{bmatrix} c^T \\ c^T A_i \\ c^T A_i^2 \\ \vdots \\ c^T A_i^{h_i} \end{bmatrix} \tag{4}$$

are of full row rank.

Theorem 2.2. [Imura and Van der Schaft, 2000, Thm. 4.2] *The following statements are equivalent.*

(1) The conditions
- *(a)* $h_1 = h_2$,
- *(b)* $T_1 = MT_2$ *where* $M \in \mathbb{R}^{h_1 \times h_1}$ *is a lower triangular matrix with positive diagonal entries, and*
- *(c)* $A_1 x = A_2 x$ *for all* $x \in \ker T_1$

hold.

(2) *The system* (1) *admits a unique forward Carathéodory solution for all initial states.*

A natural question is to ask whether the uniqueness holds for Filippov solution concept under the same conditions. The main contribution of this paper is to show that the answer is affirmative.

Theorem 2.3. Suppose that the conditions

(1) $h_1 = h_2$,
(2) $T_1 = MT_2$ *where* $M \in \mathbb{R}^{h_1 \times h_1}$ *is a lower triangular matrix with positive diagonal entries, and*
(3) $A_1 x = A_2 x$ *for all* $x \in \ker T_1$

hold. Then, any Filippov solution of (1) *is both forward and backward Carathéodory solution. Moreover, if* x *is a Filippov solution then* $c^T x$ *is either identically zero or may have only finitely many zeros on every finite time interval.*

To prove this theorem, we need some auxiliary lemmas that will be presented in what follows. The first lemma deals with the solutions for initial states that are unobservable. The proof is based on two immediate consequences of the conditions 1-3:

- The unobservability subspaces of the subsystems (c^T, A_i) coincide, i.e.
$$\langle \ker c^T \mid A_1 \rangle = \langle \ker c^T \mid A_2 \rangle =: \mathcal{N}. \tag{5a}$$
- The restrictions of A_1 and A_2 on the unobservability subspace $\mathcal{N}$ coincide, i.e.
$$A_1|_{\mathcal{N}} = A_2|_{\mathcal{N}}. \tag{5b}$$

Lemma 2.4. Suppose that the relations (5) *hold. Let* x *be a Filippov solution of* (1) *for an initial state* x_0. *Then, the following statements hold:*

(1) *if* $x_0 \in \mathcal{N}$ *then* x *is both a forward and backward Carathéodory solution.*
(2) *if* $x(t^*) \in \mathcal{N}$ *for some* $t^* \geqslant 0$ *then* $x(t) \in \mathcal{N}$ *for all* $t \geqslant 0$.

Proof. For the first statement, let $x_0 \in \mathcal{N}$. It follows from (5b) that
$$x(t) := \exp(A_1 t)x_0 = \exp(A_2 t)x_0 \tag{6}$$
is both a forward and backward Carathéodory solution for the initial state x_0. Let x' be any Filippov solution for the same initial state. Note that there exists a function $\lambda : \mathbb{R}_+ \to [0,1]$ such that
$$\dot{x}'(t) = [\lambda(t)A_1 + (1-\lambda(t))A_2]x'(t) \tag{7a}$$
is satisfied for almost all $t \geqslant 0$. It follows from (5b) and (6) that
$$\dot{x}(t) = [\lambda(t)A_1 + (1-\lambda(t))A_2]x(t) \tag{7b}$$
holds for the same function λ and for almost all $t \geqslant 0$. Define
$$A(t) = \lambda(t)A_1 + (1-\lambda(t))A_2. \tag{7c}$$

Then, one has
$$\frac{d}{dt}\|x'(t) - x(t)\|^2 = (x'(t)-x(t))^T(\dot{x}'(t) - \dot{x}(t)) \tag{8}$$
$$\overset{(7)}{=} (x'(t)-x(t))^T A(t)(x'(t)-x(t)) \tag{9}$$
$$= \frac{1}{2}(x'(t)-x(t))^T[A^T(t) + A(t)](x'(t)-x(t)) \tag{10}$$
$$\leqslant \alpha\|x'(t)-x(t)\|^2 \tag{11}$$
for almost all $t \geqslant 0$ where α is the maximum of the largest eigenvalues of the symmetric part of $\lambda A_1 + (1-\lambda)A_2$ over the set $\lambda \in [0,1]$. By integrating (11), one gets
$$\|x'(t)-x(t)\|^2 \leqslant \exp(\alpha t)\|x'(0)-x(0)\|^2 \tag{12}$$
for all $t \geqslant 0$. Since $x'(0) = x(0)$, one readily gets
$$x'(t) = x(t)$$
for all t.

For the second statement, note that the very same argument works for $t \geqslant t^*$. For $t \leqslant t^*$, we apply the same argument after reversing the time. To do so, we should show that the conditions in Theorem 2.3 hold for the time-reversed bimodal system:
$$\dot{x}(t) \in \begin{cases} -A_1 x(t) & \text{if } y(t) \leqslant 0 \\ -A_2 x(t) & \text{if } y(t) \geqslant 0 \end{cases} \tag{13a}$$
$$y(t) = c^T x(t). \tag{13b}$$
Let the nonnegative integers h_i^- for $i = 1,2$ be such that the matrices
$$T_i^- = \begin{bmatrix} c^T \\ -c^T A_i \\ c^T A_i^2 \\ \vdots \\ (-1)^{h_i^-} c^T A_i^{h_i^-} \end{bmatrix}. \tag{14}$$
are of full row rank. Note that

(1) $h_1 = h_2$ if, and only if, $h_1^- = h_2^-$.
(2) if $T_1 = MT_2$ then $T_1^- = M^- T_2^-$ where
$$M_{ij}^- = \begin{cases} M_{ij} & \text{if } i+j \text{ is even,} \\ -M_{ij} & \text{if } i+j \text{ is odd.} \end{cases}$$

Therefore, the conditions of Theorem 2.3 and hence the relations (5) hold for the time-reversed system. (13). ■

The second lemma states conditions under which one of the relations (2) and (3) holds on a 'forward' and 'backward' time interval. The following definitions will be used in formulating this result.

Define the sets $\mathbb{A}_k$ and $\mathbb{B}_k$ as follows
$$\mathbb{A}_0 = \mathbb{B}_0 = \{I\} \tag{15}$$
$$\mathbb{A}_k = \{A \in \mathbb{R}^{n\times n} \mid A = A_1 A' \text{ or } A = A_2 A' \text{ for some } A' \in \mathbb{A}_{k-1}\} \text{ for } k \geqslant 1 \tag{16}$$
$$\mathbb{B}_k = \{A \in \mathbb{R}^{n\times n} \mid A = (\lambda A_1 + (1-\lambda)A_2)A' \text{ for some } A' \in \mathbb{B}_{k-1} \text{ and } \lambda \in [0,1]\}. \tag{17}$$

Clearly, $\mathbb{A}_k \subseteq \mathbb{B}_k$ for all $k \geqslant 0$. By induction, one can easily show that

$$\mathbb{B}_k = \text{conv}(\mathbb{A}_k). \tag{18}$$

Note that if x is a Filippov solution of the system (1) then for almost all $t \geqslant 0$ there exists $A \in \mathbb{B}_1$ such that

$$\dot{x}(t) = Ax(t). \tag{19}$$

Lemma 2.5. Let x be a Filippov solution of the system (1) *for some initial state and $t^* \geqslant 0$. Suppose that there exist nonnegative integers m, p and a positive number ϵ such that*

(1) $c^T Ax(t^*) = 0$ *for all* $A \in \mathbb{B}_k$ *and* $0 \leqslant k \leqslant m$,
(2) $(-1)^p c^T Ax(t) > 0$ *for all* $A \in \mathbb{B}_{m+1}$ *and* $t \in (t^*, t^* + \epsilon)$.

Then,

$$(-1)^{m+p+1} c^T x(t) > 0$$

for all $t \in (t^*, t^* + \epsilon)$.

Proof. Take $A \in \mathbb{B}_m$. It follows from (19) that for almost all $t \geqslant 0$ there exists $A' \in \mathbb{B}_1$ such that

$$c^T A\dot{x}(t) = c^T AA'x(t). \tag{20}$$

Note that $AA' \in \mathbb{B}_{m+1}$. Then, it follows from (20) and the second hypothesis that

$$(-1)^p c^T A\dot{x}(t) > 0 \tag{21}$$

for all $t \in (t^*, t^* + \epsilon)$. From the first hypothesis, we know $c^T Ax(t^*) = 0$. As x is continuous, we get

$$(-1)^{p+1} c^T Ax(t) > 0 \tag{22}$$

for all $t \in (t^*, t^* + \epsilon)$. Thus, we get

$$(-1)^{p+1} c^T Ax(t) > 0$$

for all $A \in \mathbb{B}_m$ and $t \in (t^*, t^* + \epsilon)$. By repeating the same argument, one gets

$$(-1)^{m+p+1} c^T x(t) > 0$$

for all $t \in (t^*, t^* + \epsilon)$. ■

A complete analogue of the last lemma holds in 'backward' sense.

Lemma 2.6. Let x be a Filippov solution of the system (1) *for some initial state and $t^* > 0$. Suppose that there exist nonnegative integers m, p and a positive number ϵ with $\epsilon \leqslant t^*$ such that*

(1) $c^T Ax(t^*) = 0$ *for all* $A \in \mathbb{B}_k$ *and* $0 \leqslant k \leqslant m$,
(2) $(-1)^p c^T Ax(t) > 0$ *for all* $A \in \mathbb{B}_{m+1}$ *and* $t \in (t^* - \epsilon, t^*)$.

Then,

$$(-1)^{m+p+1} c^T x(t) > 0$$

for all $t \in (t^* - \epsilon, t^*)$.

The final auxiliary lemma will be employed in showing that the hypotheses of the previous lemma hold. First, we define the equivalence relation

$$p \sim q \quad :\Leftrightarrow \quad p = \alpha q \text{ for some } \alpha > 0. \tag{23}$$

Lemma 2.7. Suppose that $h_1 = h_2$ and $T_1 = MT_2$ for some lower triangular matrix $M \in \mathbb{R}^{h_1 \times h_1}$ with positive diagonal entries. Let $\xi \in \mathbb{R}^n$ and nonnegative integer m with $m < h_1$ be such that

$$c^T A\xi = 0 \textit{ for all } A \in \mathbb{B}_k \textit{ and } 0 \leqslant k \leqslant m. \tag{24}$$

Then, the sets

$$\{c^T A\xi \mid A \in \mathbb{A}_{m+1}\}$$

and

$$\{c^T A\xi \mid A \in \mathbb{B}_{m+1}\}$$

are both equivalence classes.

Proof. Take an integer ℓ with $0 \leqslant \ell \leqslant m+1$. Let $A' \in \mathbb{A}_{m-\ell+1}$. Multiplying the $(\ell+1)$st row of $T_1 = MT_2$ by $A'\xi$ from the left and using (24) result in

$$c^T A_1^\ell A'\xi \sim c^T A_2^\ell A'\xi.$$

Now, take $A \in \mathbb{A}_{m+1}$. One can find nonnegative integers $j, k_1, k_2, \dots, k_j$ such that

$$A = A_1^{k_1} A_2^{k_2} \cdots A_1^{k_{j-1}} A_2^{k_j}.$$

By repeatedly using (2), we get $c^T A\xi \sim c^T A_1^{m+1}\xi$. Consequently, the set $\{c^T A\xi \mid A \in \mathbb{A}_{m+1}\}$ is an equivalence class. The rest follows from (18). ■

2.1 Proof of Theorem 2.3

Let x be a Filippov solution of (1) for some initial state. Let $t^* \geqslant 0$. If $x(t^*) \in \mathcal{N}$ then the claim follows from Lemma 2.4. Suppose that $x(t^*) \notin \mathcal{N}$. We want to show that there exists $\epsilon > 0$ such that at least one of relations (2) and (3) holds for all $t \in (t^*, t^*+\epsilon)$. Note that continuity of x readily implies the claim if $c^T x(t^*) \neq 0$. Suppose that $c^T x(t^*) = 0$. Since $x(t^*) \notin \mathcal{N}$, there exist nonnegative integers q and p with $0 \leqslant q < h_1$ such that

$$c^T A_1^\ell x(t^*) = 0 \text{ for all } \ell = 0, 1, \dots, q \tag{25}$$

$$(-1)^p c^T A_1^{q+1} x(t^*) > 0. \tag{26}$$

By applying Lemma 2.7 for $m = 0, 1, \dots, q$, we get that

$$c^T Ax(t^*) = 0 \text{ for all } A \in \mathbb{B}_k \text{ and } 0 \leqslant k \leqslant q \tag{27}$$

and

$$(-1)^p c^T Ax(t^*) > 0 \tag{28}$$

for all $A \in \mathbb{B}_{q+1}$. Since x is continuous, for each $A \in \mathbb{B}_{q+1}$ there exists a positive number ϵ_A such that

$$(-1)^p c^T Ax(t) > 0 \tag{29}$$

for all $t \in (t^*, t^* + \epsilon_A)$. Since $\mathbb{A}_{q+1}$ is a finite set, we can define

$$\epsilon = \max_{A \in \mathbb{A}_{q+1}} \epsilon_A.$$

As the set $\mathbb{B}_{q+1}$ is the convex hull of the set $\mathbb{A}_{q+1}$, one can conclude that

$$(-1)^p c^T Ax(t) > 0 \tag{30}$$

holds for all $A \in \mathbb{B}_{q+1}$ and for all $t \in (t^*, t^* + \epsilon)$. Together with (27), this means that the hypotheses of Lemma 2.5 hold for $m = q$. Then, we get

$$(-1)^{p+q+1} c^T x(t) > 0 \tag{31}$$

for all $t \in (t^*, t^* + \epsilon)$. Consequently, exactly one of the relations (2) and (3) hold with strict inequality on the same interval. This proves that x is a forward Carathéodory solution. The very similar arguments, together with Lemma 2.6, prove that it is also a backward Carathéodory solution. The rest follows from Lemma 2.4 and the strict inequalities of (31). ■

3. MODE TRANSITIONS AND ZENO BEHAVIOR

In this section, we investigate mode transitions for bimodal systems. We say that

- a time instant $t^* > 0$ is a *non-switching time* for a Filippov solution x if there exist an interval $(t^* - \varepsilon, t^* + \varepsilon)$ and an index i with $i \in \{1, 2\}$ such that $\dot{x}(t) = A_i x(t)$ for all $t \in (t^* - \varepsilon, t^* + \varepsilon)$.
- a time instant $t^* \geqslant 0$ is a *switching time* for a Filippov solution x if it is not a non-switching time for the same solution.

The distribution of the switching times along a solution is an important issue for various reasons. For instance, the so-called event-driven simulation methods (see e.g. Van der Schaft and Schumacher [2000]) may fail if the switching times accumulate around a point. This type of phenomenon is known as Zeno behavior in hybrid systems literature. We say that

- a time instant $t^* \geqslant 0$ is a *left Zeno time* for a Filippov solution x if for each $\varepsilon > 0$ the interval $(t^*, t^* + \varepsilon)$ contains a switching time for the same solution.
- a time instant $t^* > 0$ is a *right Zeno time* for a Filippov solution x if for each $t^* > \varepsilon > 0$ the interval $(t^* - \varepsilon, t^*)$ contains a switching time for the same solution.

The system (1) is called *Zeno-free* if it has no solutions for which there are left or right Zeno times. The following theorem is a direct consequence of Theorem 2.3.

Theorem 3.1. Suppose that the conditions

(1) $h_1 = h_2$,
(2) $T_1 = MT_2$ *where* $M \in \mathbb{R}^{h_1 \times h_1}$ *is a lower triangular matrix with positive diagonal entries, and*
(3) $A_1 x = A_2 x$ *for all* $x \in \ker T_1$

hold. Then, the system (1) *is Zeno-free.*

Proof. Let x be a Filippov solution of (1) and t^* be a nonnegative time instant. If $c^T(t^*) \neq 0$ then t^* is a non-switching time due to the continuity of x. In other words, switching times are necessarily zeros of the function $t \mapsto c^T x(t)$. As a result of this observation, the claim follows from Theorem 2.3. ■

4. CONCLUSIONS

We studied bimodal piecewise linear systems that are described by possibly discontinuous vector fields. Based on the well-posedness conditions that are stated in the work of Imura and Van der Schaft [2000], we showed that Filippov solutions of these systems coincide with both the so-called forward and backward Carathéodory solutions. As such, we concluded that these systems do not exhibit Zeno behavior provided that they are well-posed, i.e. there exists a unique solution starting from each initial state.

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