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Enhancing Complex Network Synchronization Based on the Node Betweenness ⋆

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Abstract: In this paper, we present a weighted method based upon the node betweenness, and discover that synchronizability of the complex networks can be enhanced by the weighted method. And we demonstrate the validity of this method by applying this weighted method to two classes of networks with high homogeneous and heterogeneous degree distribution. The optimal synchronable condition corresponds to (Motter *et al.*, 2005a) and can also be obtained by a tunable parameter α. We hope the research can be useful for comprehensively understanding the synchronization behavior of networks and design more effective networks.

1. INTRODUCTION

Over the past few years, the analysis of complex systems from the viewpoint of networks has become an important interdisciplinary issue (Albert *et al.*, 2002). Complex networks have been intensively studied in many fields, such as social, biological, mathematical, and engineering sciences. Generally, a complex network is made up of interconnected nodes in which a node is a basic unit with detailed contents. These interactions between nodes determine many basic properties of a network. To well understand the complex dynamical behaviors of many natural systems, we need to study the topological structures of the underlying networks. In fact, the properties of a complex network are mainly determined by its topological structures, *i.e.* the connections between nodes. At one extreme are the regular networks, such as chains, lattices, grids, and fully connected graphs, while at the other extreme are the random networks where the nodes are connected randomly. Small-world networks are objects in between regular and random networks characterized by a small average distance between any two nodes, while keeping a relatively highly clustered structure (Watts *et al*, 1998). Scale-free networks are characterized by highly heterogeneous distribution of degrees (number of links per node) and display a power-law distribution $p(k) \sim k^{-\gamma}$ in the node connectivity k (degree) (Barabási *et al.*, 1999).

Synchronization is a basic phenomenon in a wide range of real systems, such as neural networks, physiological process, biology, and so on (Lu *et al.*, 2004). It has been demonstrated that many real-world complex networks display various synchronization phenomena. Network synchronization is strongly influenced by the structure of the underlying network. Previous work on synchronization has focused mainly on the influence of the topology of connections by assuming that the coupling strength is uniform. However, synchronization is influenced not only by the topology, but also by the strength of the connections (Motter *et al.*, 2005a). Most complex networks where synchronization is relevant are indeed weighted. Examples include brain networks (Scannell *et al.*, 1999), networks of coupled populations in the synchronization of epidemic outbreaks (Grenfell *et al.*, 2001), and technological networks whose functioning relies on the synchronization of interacting units (Korniss *et al.*, 2003). The distribution of connection weights in real networks is often highly heterogeneous (Barrat, 2004). The study of synchronization in weighted networks is thus of substantial interest. A basic assumption of previous work is that the oscillators are coupled symmetrically and with the same coupling strength. But to get a better synchronizability the couplings are not necessarily symmetrical. Moreover, many realistic networks are actually directed and weighted. Since the networks considered in the original study (Nishikawa *et al.*, 2003) are unweighted and undirected, recent efforts have been focused on searching for network configurations incorporating weights and directionality, to achieve more efficient synchronization in scale-free networks (Motter et al., 2005a 2005b; Hwang *et al.*, 2005; Nishikawa & Motter, 2006). A first attempt at assessing enhancement of synchronization due to weighted connections was proposed in Ref. (Motter *et al.*, 2005a), where the coupling strength was taken to be $\frac{\sigma}{k_i^{\beta}} L_{ij}$ (L_{ij} being the Laplacian matrix). This asymmetric wiring provides a spectrum of real eigenvalues, and an optimal condition $\beta = 1$ for synchronization was found (Motter *et al.*, 2005a). Chavez *et al* (2005) show enhancement in synchronization is achieved by scaling the coupling strength to the load of each link that the information contained in the overall topology. Wang *et al.*

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(2007) have proposed a coupling scheme that enhancing synchronization based on complex gradient networks.

In this paper, we propose a scheme to address the synchronizability of asymmetrical and weighted complex networks. By exploiting the global structure of shortest paths among nodes, the weight of a link will be related to its node betweenness (The definition will be introduce in section 3). We show that a weighted method based upon the betweenness of node enhance the synchronizability of complex networks. And, we apply this weighted method to both high homogeneous and heterogeneous degree distribution, which demonstrate that this weighted method significantly improves the synchronization of these complex networks.

The organization of the paper is as follows. In Section 2, Some approaches to generate complex networks are reviewed and the issue of synchronization in a network is discussed. In Section 3, we gives a weighted method to enhance the synchronizability based on the nodes betweenness and demonstrate the validity of this method by applying this weighted method to two classes of networks with high homogeneous and heterogeneous degree distribution. We end this paper with some conclusions, in section 5.

2. PRELIMINARIES: COMPLEX DYNAMICAL NETWORKS

2.1 Some approaches to generate a complex topology

A brief summary of the most popular ways to generate a complex network is presented in this section. The descriptions only try to give a flavor of the type of techniques one can use to stochastically generate a complex network.

1) *E-R Random Network Model*

Not limited to small graphs with a high degree of regularity as did in the classical graph theory, Erdös and Rényi cast more complexity in network topology with statistical algorithms. The basic E-R random network model is defined as a random graph of N labeled nodes connected by n edges, which are chosen randomly from all the $N(N-1)/2$ possible edges. The network evolution is uniform: Start with N nodes, and every pair of nodes are connected with the same probability p

The main goal of the random graph theory is to determine in what connection probability p a particular property of a graph will likely arise. For a large N, the E-R model generates a homogenous random network, whose connectivity approximately follows a Poisson distribution

$$P(k) \approx e^{-<k>} \frac{<k>^k}{k!}$$

where $<k>$, the so-called average degree of the network, is the average of k_i over all nodes i in the network. With this connectivity distribution, nodes in the network are quite uniformly spread out, which is known as a homogenous feature of the distribution.

2) *Small-World Modes*

Watts and Strogatz (1998) introduced a single parameter small-world network model that bridges the gap between a regular network and a random graph. The original WS model is described as follows.

(1) *Start with order*: Start with a nearest neighbor coupled ring lattice with N nodes, in which each node is connected to its k neighboring nodes $i \pm 1$, $i \pm 2$, $i \pm K/2$, where k is an even integer.

(2) *Randomize*: Randomly rewire each link of the network with probability p such that self-connections and duplicated links are excluded. Rewiring in this sense means transferring one end of the connection to a randomly chosen node. This process introduces $pNK/2$ long-range links, which connect some nodes that otherwise would not have direct connections. One can thus closely monitor the transition between order ($p = 0$) and ($p = 1$) randomness by adjusting p.

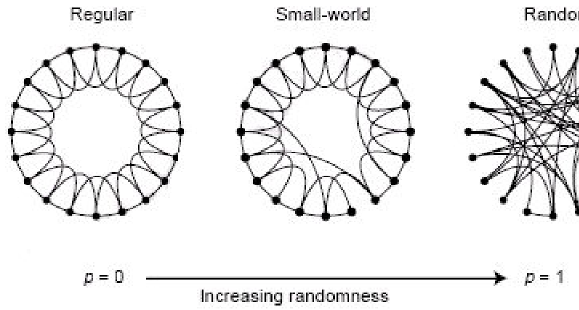

Fig. 1. Formation of a small-world network

Figure 1 shows that a small-world network lies along a continuum of network models between the two extreme networks: regular and random ones. Note, however, that there is a possibility for the SW model to be broken into unconnected clusters. This problem can be circumvented by a slight modification of the SW model, suggested by Newman & Watts (1999), which is referred to as the NW model hereafter. In the NW model, we do not break any connection between any two nearest neighbors. We add with probability p a connection between each other pair of nodes. Likewise, we do not allow a node to be coupled to another node with itself. The degree distribution of NW model is also homogenous. For $p = 0$, it reduces to the originally nearest-neighbor coupled system; for $p = 1$, it becomes a globally coupled system.

3) *BA Scale-Free Network Model*

The algorithm of the BA scale-free model is generated as follows.

(1) *Growth*: Starting with a small number (m_0) of nodes, at every time step, add a new node with $m(m \leq m_0)$ edges that link the new node to m different nodes already presented in the network.

(2) *Preferential attachment*: When choosing the nodes to which the new node connects, assume that the probability $\Pi(k_i)$ that a new node will be connected to node i depends on the degree k_i of node i, in such way that

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}.$$

After t time steps, we get a network having $N = t + m_0$ nodes and mt edges. This network evolves into a scale-invariant state with the probability that a node has edges following a power-law distribution $P(k) \sim 2m^2 k^{-\gamma_{BA}}$ with an exponent $\gamma_{BA} = 3$, where the scaling exponent is independent of m, i.e., γ_{BA} is scale-invariant, and in this sense the network is said to be scale-free. However, the degree distribution of BA model is heterogeneous.

It has been suggested that the BA scale-free network model has captured the basic mechanisms, growth, and preferential attachment, responsible for the scale-free feature and "rich gets richer" phenomenon in many real-life complex networks.

2.2 Network synchronization

Consider a dynamical network of N linearly coupled identical oscillators, with each oscillator being an n-dimensional dynamical system. Let each oscillator of the network be assigned a dynamical variable x_i $(i = 1, 2, \cdots, N)$. The evolution of the dynamical variables is written in the form (Motter *et al.*, 2005b)

$$\dot{x_i} = f(x_i) + \sigma \sum_{j=1}^{n} G_{ij} h(x_j) \tag{1}$$

where f describes the dynamics of each individual oscillator, h is the inner coupling function, σ is the coupling strength, and $G = (G_{ij})_{N\times N}$ is the coupling matrix.

Suppose that all eigenvalues of the matrix G satisfy

$$0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \tag{2}$$

The variational equation governing the linear stability of the synchronous state $\{\dot{x_i}(t) = s(t), \forall i\}$ of the network can be diagonalized into blocks of the form

$$\dot{\eta} = [Df(s) + \alpha Dh(s)]\eta \tag{3}$$

where $\alpha = \sigma\lambda_i$, and λ_i are the eigenvalues of the coupling matrix G. The largest Lyapunov exponent $\Lambda(\alpha)$ linked to $\alpha = \sigma\lambda_i$, the so-called master stability function (MSF), which determines the linear stability of the synchronized state (Pecora, *et al.*, 1998). The synchronized state is stable if $\Lambda(\sigma\lambda_i) < 0$ for $i = 2, \cdots N$. For many widely studied oscillatory systems (Barahona *et al.*, 2002; Fink *et al.*, 2000; Pecora *et al.*, 1998), the master stability function $\Lambda(\alpha)$ is negative in a finite interval (α_1, α_2). Therefore, the network is synchronizable for some σ when the eigenratio $R = \lambda_N/\lambda_2$ satisfies $R < \alpha_2/\alpha_1$. The ratio α_2/α_1 depends only on the dynamics (f, h, and s), while the eigenratio R depends only on the coupling matrix G. The problem of synchronization is the reduced to the analysis of eigenvalues of the coupling matrix (Barahona *et al.*, 2002). The synchronizability of the network can be characterized by the eigenratio R of the coupling matrix. Smaller R lead to better synchronizability.

3. ENHANCING SYNCHRONIZATION BY WEIGHTED METHOD

The first attempt at assessing enhancement of synchronization by weighted connections in Ref. (Motter *et al.*, 2005a) only retains information on the local features of the network. We show that enhancement in synchronization can be achieved by exploiting the information contained in the overall topology. We propose to scale the coupling strength by the betweenness of nodes. The node betweenness B_n is the number of shortest paths between two (other) nodes that pass through node n, therefore it reflects the global information. Precisely, the betweenness of node n is defined to be

$$B_n = \sum_{(i,j)} g_{inj} \tag{4}$$

where g_{inj} is the number of shortest paths between i and j passing through node n. The summation is to be performed over all pairs of nodes (i, j), $i \neq j$.

We consider

$$G_{ij} = l_{ij} \frac{B_i^\alpha}{\sum_{j\in K_i} B_j^\alpha} \tag{5}$$

where B_i is the betweenness of node i; α is a real tunable parameter, and K_i is the set of neighbors to the ith node. The underlying network associated with the coupling matrix L (The diagonal entries are $l_{ii} = -k_i$. The off-diagonal entries are $l_{ij} = 1$, if nodes i and j are connected, and 0 otherwise.) is undirected and unweighted, but with the introduction of the weights in equation (5), the network of couplings becomes not only weighted but also directed because the resulting matrix G is in general asymmetric.

In matrix notation, equation (5) can be written as

$$G = DL \tag{6}$$

where $D = \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$, $d_i = \frac{B_i^\alpha}{\sum_{j\in K_i} B_j^\alpha}$. From the identity

$$\begin{aligned} &\det(G - \lambda I) = \det(DL - \lambda I) \\ &= \det(D^{\frac{1}{2}} L D^{\frac{1}{2}} - \lambda I) \end{aligned} \tag{7}$$

valid for any λ, we have that the spectrum of eigenvalues of matrix G is equal to the spectrum of a symmetric matrix denoted as

$$H = D^{\frac{1}{2}} L D^{\frac{1}{2}} \tag{8}$$

As a result, all the eigenvalues of matrix G are real, because H is a real symmetric matrix. Moreover, because H is negative semidefinite, all the eigenvalues are negative or zero and, because the rows of G have zero sum, the largest eigenvalue λ_1 is always zero, as assumed above. If the network is connected, then $\lambda_i < 0$ $(i = 2, 3, \cdots, N)$ for any finite α. Another important point to be stressed concerns the various limits the coupling term can assume when changing α. The limit $\alpha = 0$ corresponds to the best synchronizability condition of (Motter *et al.*, 2005a).

By varying α in the Eq. (5), and by monitoring the ratio $R = \lambda_N/\lambda_2$ of the coupling matrix G, we can now study the propensity for synchronization of the complex networks with different degree distributions. For the scale free networks, the degree distribution is heterogeneous, for example the BA model. The used scale-free networks is a generalization of the preferential attachment growing procedure introduced in (Barabási *et al.*, 1999). Namely, starting from $m + 1$ to all connected nodes, at each time step a new node is added with m links. These m links point to old nodes with probability $\frac{k_i}{\sum_j k_j}$, where k_i is the degree of the node i. Figure 2 shows the logarithm of $R = \lambda_N/\lambda_2$ as a function of α for BA scale-free networks of different value m. Synchronizability (characterized by the eigenratio R) of weighted scale-free networks is denoted by solid line and unweighted case is denoted by dashed. We can see from Figure 2, the eigenratio R of the coupling matrix is diminished, when round $-1 < \alpha < 1$, in comparison with the unweighted case. This indicate that synchronizability is enhanced by introduce the weighed in coupling matrix, when round $-1 < \alpha < 1$. It is notice that the curve of $R = \lambda_N/\lambda_2$ has a pronounced minimum at $\alpha = 0$ for all values of m. Because here $\alpha = 0$, we can get the $\sum_{j \in K_i} B_j^{\alpha} = k_i$. Then the condition $d_i = 1/k_i$ recovers the optimal condition when the information on node degrees is used (the condition $\beta = 1$ in (Motter *et al.*, 2005a)), this indicates that our weighting procedure based on the nodes betweenness greatly enhances the scale free network propensity for synchronization. In all our results, each curve is the result of an average of 20 realizations for $N = 300$. And N has been varied form 300 to 1000 without significant qualitative differences.

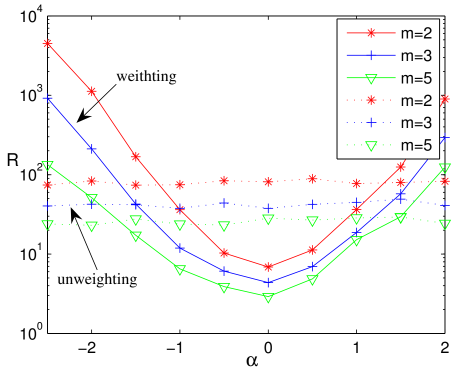

Fig. 2. Eigenratio R as a function of α, the BA scale-free network: $m = 2(*)$ $m = 3(+)$ $m = 5(\triangle)$ for nodes $N = 300$. unweighting (dashed); weighting (solid line). Each curve is the result of an average 20 realizations.

For comparison, we apply our coupling scheme to the other class of networks with high homogeneous degree distribution. The WS small-world model, obtained as in (Watts *et al.*, 1998) is one of complex networks with homogeneous degree distribution. Roughly speaking, start with a nearest-neighbor coupled ring lattice with N nodes, in which each node is connected to its K neighboring nodes $i \pm 1, i \pm 2, \cdots i \pm K_i/2$, where K is an even integer. Randomly rewire each link of the network with probability p such that self-connections and duplicated links are excluded. The synchronizability of random networks (the WS small-world model, when $p = 1$) is illustrated by the $lg(R)$ in Figure 3. And the figure indicates that here our weighted procedure based on node betweenness also enhances the synchronizability of random network, when round $-1 < \alpha < 1$. And the minimum of the curve is always positioned at $\alpha = 0$. Namely the best synchronizability also is obtained, when $\alpha = 0$. The reason is similar to the case of heterogeneous degree distribution. There is a possibility for the WS model to be broken into unconnected clusters. This problem can be circumvented by a slight modification of the WS model, suggested by Newman and Watts (1999), which is referred to as the NW model hereafter. In the NW model, any connection between any two nearest neighbors not be broken. The NW network model also is another complex networks with homogeneous degree distribution. We also apply our coupling scheme to NW small-world networks model. Figure 4 shows $\lg(\frac{\lambda_N}{\lambda_2})$ vs α of the NW small-world network model for different the parameter p, and indicates that our weighting procedure also enhanced the network synchronizability, when round $-1 < \alpha < 1$. And the minimum of the curve is also positioned at $\alpha = 0$. The reason is the condition $\alpha = 0$ recover the optimal condition when the information on node degrees is used (the condition $\beta = 1$ in (Motter *et al.*, 2005a)). Observably, the weighted scheme can greatly enhance the synchronizability of complex networks, for both homogeneous and heterogeneous networks. And, the optimal condition of synchronizability can be obtained by the global information B_i. So, we can design more efficient networks for synchrony by this weighted method.

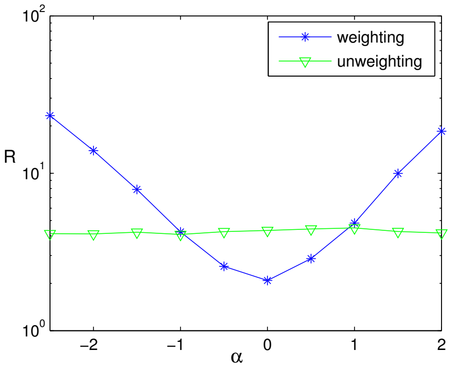

Fig. 3. Eigenratio R as a function of α, random network for nodes $N = 300$. unweighting (dashed); weighting (solid line). Each curve is the result of an average 20 realizations.

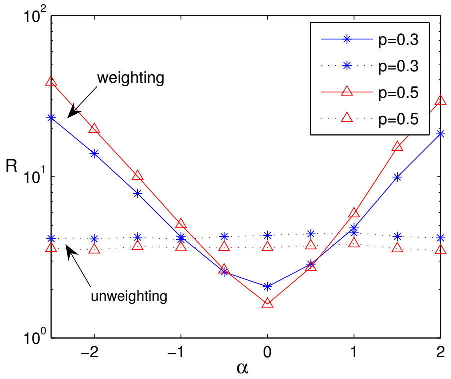

Fig. 4. Eigenratio R as a function of α, the NW small-world network: nodes N=300 $p = 0.3(*)$ $p = 0.5(\triangle)$ for $k = 20$. unweighting (dashed); weighting (solid line). Each curve is the result of an average 20 realizations.

4. CONCLUSION

We have shown that a weighted method based upon the betweenness of node enhances the synchronizability of complex networks. Within this method, we have shown that suitably weighted networks display significantly improve synchronizability for both homogeneous and heterogeneous networks. And the optimal condition $\alpha = 0$ for synchronization was found. Moreover, the condition corresponds to (Motter *et al.*, 2005a) ($\beta = 1$). The research may be useful for comprehensively understanding the synchronization behavior of networks and design more effective networks for synchrony. However, there is a long way to fully uncovering the truth of synchronization in complex networks.

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