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Stabilization of Nonlinear Systems using Weak-Control-Lyapunov Functions ⋆

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Abstract: This paper proposes a recursive method of constructing weak-control-Lyapunov functions for nonlinear systems. Lyapunov function is one of effective tools to study stability and stabilization in nonlinear system control design. However, a general way of finding Lyapunov functions has not been known yet. Our method is introduced by an explicit topological-geometric assumption for a state space manifold, called a Morse-Smale. The assumption indicates that there exists a sequence of inclusions of the manifold and its singular structures, called a weak-Lyapunov filtration. From this structure, we can construct a finite number of iterations to define weak-control-Lyapunov functions. As a result, the existence of the weak-control-Lyapunov functions can be specified by the investigation of property of manifolds.

Keywords: Nonlinear systems, Iterative modeling and control design

1. INTRODUCTION

Lyapunov functions play a critical role in stability analysis and stabilization in nonlinear system control design. However, it is difficult to find a (strictly negative) Lyapunov function in nonlinear system in general. Therefore, we use a weak- (non-positive) Lyapunov function to moderate the requirement and simultaneously introduce LaSalle's theorem and Barbalat's lemma for technical reasons for stability discrimination Karil [2002]. On the other hand, control-Lyapunov functions, which are forced to be negative by using control inputs, are frequently applied to this problem Freeman and Kokotović [1996], Sontag [1998]. A unified approach to the construction of control-Lyapunov functions is still being developed, especially for global manifolds that include multi critical points.

The purpose of this study is to specify the condition for the existence of weak-control-Lyapunov functions by introducing a topological geometric assumption for a state space manifold. This paper shows a recursive method of constructing weak-control-Lyapunov functions based on a Morse-Smale flow Smale [1960], Meyer [1968], Franks [1979], Robinson [1999] for nonlinear systems. The fundamental idea of the method is as follows. First, let us consider a closed (compact and without boundaries) manifold M as a state space. A system $\dot{x} = f(x, u)$ is considered as a vector field on the tangent space TM, where $x \in M$, and the control input $u = k(x) \in U$ is constructed by using a feedback law $k\colon M \to U$. We assume that a smooth global weak-control-Lyapunov function $V_0\colon M \to \mathbb{R}$ on M has been found, the system has already been stabilized by pre-feedback based on V_0 (e.g., universal formula Sontag [1998]), and all of the invariant sets on M are compact. At this point, the main problem is the behavior of the system state that stays on an invariant set $\Delta_1 = \{x \in M \mid \dot{V}_0 = 0\}$. Now we suppose that there exists one compact invariant set Δ_1 on M to simplify this illustration. (1) If a system state on $\sigma_1 \subset \Delta_1$ is 'escapable' to an adjacent set $L_0^{\bowtie}$ of Δ_1 by an appropriate input, a global asymptotically stability holds by finding a new weak-control-Lyapunov function $V_1\colon \Delta_1 \to \mathbb{R}$ converging on a point $x_1 \subset \sigma_1$ for $x \in \Delta_1$, where l_0 is the level-set defined by $V_0^{-1}\colon \mathbb{R}_{[0,m]} \to M$ for

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an integer $m > 0$ and $L_0^{\bowtie}$ is the reachable set of $l_0 \setminus \Delta_1$ toward the direction of a positive time evolution. (2) Since V_1 is a weak-control-Lyapunov function, there may exist an invariant set $\Delta_2 = \{x \in \Delta_1 \setminus x_1 \,|\, \dot{V}_1 = 0\}$. In this case, it is not global asymptotically stable. Then, we try to find another weak-control-Lyapunov function $V_2 \colon \Delta_2 \to \mathbb{R}$ converging on a set $\Delta_1 \setminus \Delta_2$ for $x \in \Delta_1$. (3) Next, we have to consider an invariant set $\Delta_3 = \{x \in \Delta_2 \setminus x_2 \,|\, \dot{V}_2 = 0\}$. (4) The above procedure called weak-Lyapunov filtration is repeated over and over again until we obtain a Lyapunov function.

In the filtration, each dimension of invariant sets Δ_i $(1 \leq i \leq m)$ decreases 1-dimension every iteration and this situation corresponds with the most strict condition for the Lyapunov functions. Please note that, actually, there is the possibility of existence of another set of Lyapunov functions, which is less than the above filtration. However, the purpose in this paper is to state a topological-geometric condition of manifolds explicitly for the construction of weak-control-Lyapunov functions. As a result, we obtain the fact that weak-Lyapunov filtration can be finished in a finite number of iterations.

2. MATHEMATICAL PRELIMINARY

In this section, we quote existing results. Let M be a closed smooth manifold of dimension m with a distance function d inherited from some Riemannian metric.

2.1 Invariant sets

Let us consider a continuous dynamical system $\{\varphi^t\}_{t\in\mathbb{R}}$, where $\{\varphi^t \colon M \to M \,|\, t > 0\}$ is a 1-parameter family of continuous maps. If X is a smooth vector field on M, then φ^t is the 1-parameter group of diffeomorphisms generated by X. The state of initial condition x after time t is $x(t) = \varphi^t(x)$. In this case, the *positive semi-orbit* passing through the point x is defined by $\mathcal{O}_+(x) = \{\varphi^t(x) \,|\, t \geq 0, t \in \mathbb{R}\}$. The set of limit points of $\mathcal{O}_+(x)$, that is $\omega(x) = \cap_{\tau\geq 0} \cup_{t\geq\tau} \varphi^t(x)$ is called an *ω-limit set*. A *negative semi-orbit* $\mathcal{O}_-(x)$ and an *α-limit set* such that $\alpha(x) = \cap_{\tau\leq 0} \cup_{t\leq\tau} \varphi^t(x)$ are defined by the inverse time limit $t \to -\infty$ in the same manner.

2.2 Lyapunov functions

A closed invariant set I is called *stable in the sense of Lyapunov* if there exists a neighborhood T involved in any small neighborhood Y such that $^\forall x \in T$ and $\mathcal{O}_+(x) \subset Y$. The C^1 function $V \colon M \to \mathbb{R}$ is called a *weak Lyapunov function* of flow φ^t if $V \circ \varphi^t(x) \leq V(x)$ for $^\forall x \in M$ and $^\forall t \geq 0$. In other words, $\dot{V}(x) \leq 0$ for $^\forall x \in M$ if and only if V is a weak Lyapunov function, where $\dot{V}(x) \equiv \frac{d}{dt} V\big(\varphi^t(x)\big)|_{t=0}$. Moreover, if $V \circ \varphi^t(x) < V(x)$, that is $\dot{V}(x) < 0$ for $^\forall x \notin C(\varphi^t)$ and $^\forall t \geq 0$, then V is called a *Lyapunov function*.

We consider a dynamical system $\dot{x} = f(x, u)$, where $x \in M$, $u \in U$, and U is an appropriate manifold. If a proper smooth positive function $V \colon M \to \mathbb{R}$ satisfies

$$\inf_{u\in U} \operatorname{grad} V \cdot f(x, u) < 0 \tag{1}$$

(or ≤ 0) for $^\forall x \in M \setminus \{0\}$, then V on M is called a *control-Lyapunov function* (or a *weak-control-Lyapunov function*, respectively), where the function $V \colon M \to \mathbb{R}$ is called *proper* if a set $\{x \in M \,|\, V(x) \leq a\}$ is compact for any $a > 0$.

2.3 Morse theory

The basic concept of the Morse theory is to extract topological invariant properties of manifolds from the behavior of critical points of an arbitrary function Milnor [1963], Matsumoto [2002].

Let $f \colon M \to \mathbb{R}$ be a smooth function. If the differential $Df(p) \colon T_pM \to \mathbb{R}$ is a zero map, then p is a *critical point* of f. The f is called a *Morse function* if every critical point p is a non-degenerate $\det Hf(p) \neq 0$, where $Hf(p) = \partial^2 f(p)/(\partial x_i \partial x_j)$ is a Hessian. The number of negative eigenvalues of $Hf(p)$ is called the *Morse index* of p. Morse's lemma, which is one of the most important results in Morse theory, says that we can take a suitable local coordinate $(x_1, \cdots, x_m)$ in the neighborhood of p of index λ so that the function f has a standard form given by

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2 \,. \tag{2}$$

Let us consider a gradient flow $\dot{x} = -\nabla f(x)$ on M. Let ϕ^t be the generated invertible 1-parameter family of $-\nabla f$. Now we interpret the flow as a differentiable manifold itself. For all $p \in M$ of f, we define

$$W^s(p) = \big\{x \in M; \lim_{t\to+\infty} \phi^t(x) = p\big\} \,, \tag{3}$$

$$W^u(p) = \big\{x \in M; \lim_{t\to-\infty} \phi^t(x) = p\big\} \tag{4}$$

as a *stable manifold* and an *unstable manifold*, respectively, where $W^s(p)$ is an $(n - \lambda)$-dimensional submanifold of M and the $W^u(p)$ is a λ-dimensional submanifold of M. All points on M except for critical points are on one integral curve. Every integral curve starting from p of index λ arrives at critical points of index $(\lambda - 1)$ or less.

2.4 Morse-Smale systems

In contrast with (2), from Thom's splitting lemma Gilmore [1993], Thom [1989], the local structure around degenerate critical points is given by the differentiable function germ:

$$f(x) = f(p) \pm x_1^2 \pm \cdots \pm x_\eta^2 + h(x_{\eta+1}, \cdots, x_m) \,, \tag{5}$$

where h is a function germ having order over three called the *residual singularity* of f and $Hh(p) = 0$, and the germ, which is defined by an equivalent class containing f itself, expresses the behavior of f in the neighborhood of p.

Morse-Smale systems (or flows) are defined by a class of vector fields on a manifold like gradient fields generated by Morse functions Smale [1960]. Morse-Smale systems consist of a finite number of closed orbits and singular points such as α and ω-limit sets of every trajectory Smale [1961], Meyer [1968]. Note that the nonsingular quadratic form $(x_1, \cdots, x_\eta)$ in (5) corresponds to the regular part of the coordinate: $\mathbb{B}^{m-1}$ in the following *Definition 3* iii).

Definition 1. A smooth vector field X is called a *Morse-Smale system* provided

i) X has a finite number of singular points: $\beta_1, \cdots \beta_k$, and closed orbits: $\beta_{k+1}, \cdots \beta_n$.
ii) For any $x \in M$, $\alpha(x) = \beta_i$ and $\omega(x) = \beta_j$ for some i and j.
iii) For a closed orbit β_i, there is no $x \in M \setminus \beta_i$ such that $\alpha(x) = \beta_i$ and $\omega(x) = \beta_i$.
iv) The stable and unstable manifolds associated with β_i have transversal intersection.

The set $\beta_1, \cdots \beta_n$ is called the *singular elements* of the field X.

Lemma 2. Let X be a Morse-Smale system on M. Let $\beta_i \succ \beta_j$ mean that there is a trajectory from β_i to β_j whose α-limit set is β_i and whose ω-limit set is β_j. Then $\succ$ satisfies:

i) $\beta_i \not\succ \beta_i$.
ii) If $\beta_i \succ \beta_j$ and $\beta_j \succ \beta_l$, then $\beta_i \succ \beta_l$.
iii) If $\beta_i \succ \beta_j$, then $\dim W_i^u \geq \dim W_j^u$ and equality can occur only if β_j is a closed orbit, where W_i^u is the unstable manifold associated with β_i.

A Morse-Smale system without closed orbits is called *gradient-like*. From the previous subsection, there exists a Lyapunov function (i.e., Morse function) that is decreasing along trajectories for every gradient-like system. It is known that for every Morse-Smale system, there exists a Lyapunov-Morse function, called a ξ-function, that is decreasing along the trajectories of the system Meyer [1968].

Let f be a smooth function from M into $\mathbb{R}$ and let Δ denote the set of critical points of f. Let us define a *nullity* r with a corank r of f such that $r = m - \operatorname{rank} Hf(p)$ for a degenerate singular point ($\det Hf(p) = 0$). Let Δ^r denote the set of i points δ_i^r with a nullity r in Δ.

Definition 3. A smooth function $f \colon M \to \mathbb{R}$ is called a ξ-function for M provided

i) $\Delta = \Delta^0 \cup \Delta^1$.
ii) Δ^1 is the disjoint union $\sqcup_{i=k+1}^n \delta_i^1$ of a finite number of circles such that the index of f is constant on each circle.
iii) For $i = k+1, \cdots, n$, there exists an orientable neighborhood N_i of δ_i^1 and a diffeomorphism x_i such that x_i maps N_i into $\mathbb{B}^{m-1} \times \mathbb{S}^1$ with the local coordinate consisting of a nonsingular quadratic form in $x_1, \cdots, x_{m-1}$ (the coordinates in $\mathbb{B}^{m-1}$) and is periodic with period 1 in x_m, the coordinate in $\mathbb{S}^1$, where $\mathbb{B}^i$ is the open unit ball in $\mathbb{R}^i$ and $\mathbb{S}^i$ is the unit sphere in $\mathbb{R}^{i+1}$. Moreover, for each point in $\mathbb{S}^1$, the quadratic form has an index equal to the index of f of δ_i^1.

The ξ-function decreasing along trajectories is closely related to the field.

Definition 4. Let X be a smooth vector field on M. Then a ξ-function f for M is called a ξ-function for X provided that

i) $Xf < 0$ for all $p \in M \setminus \Delta$, i.e., f is decreasing along the trajectories of X or the trajectories of X are transversal to the level lines of f.
ii) If p is a singular point of X, then $p \notin \Delta^1$.
iii) There exists a constant $\kappa > 0$ such that $-Xf(p) \geq \kappa d(p, \delta_i)^2$ for $p \in N_i$ on each N_i.

Theorem 5. If X is a Morse-Smale system, then there exists a ξ-function for X.

3. MAIN RESULTS

In this section, we introduce a recursive method of constructing weak-control-Lyapunov functions based on a Morse-Smale flow Smale [1960], Meyer [1968], Franks [1979], Robinson [1999] for nonlinear systems. For this purpose, some basic concepts in global stability based on Morse-Smale flows are defined first. Next, we clarify the requirements of control inputs for global stabilization. Finally, the precise procedure for constructing a finite set of weak-control-Lyapunov functions is given.

3.1 Problem statement

Let us consider a closed smooth manifold M as a state space. Let

$$\dot{x} = f(x, u) \tag{6}$$

be a dynamical system for $x \in M$ which can be considered as a vector field on the tangent space TM. The control input $u = k(x) \in U$ is constructed by using a feedback law $k \colon M \to U$. We assume the following:

i) A smooth global weak-control-Lyapunov function V_0 on M has been found and the system has already been stabilized by pre-feedback based on V_0 (e.g., universal formula Sontag [1998]).
ii) All of the invariant sets Δ of V_0 consist of Δ^r, where $r = 0, 1$.
iii) Then, the singular point $\{0\} \subset \Delta^0$ with index 0 is equal to the unique global asymptotically stable point of M.
iv) The system (6) always has a local Carathéodory solution Bacciotti and Rosier [2005].

Though the above conditions seem a strong limitation at first glance, actually, these mean a quite wide situation in comparison with the problem of the conventional nonlinear system $\dot{x} = f(x) + g(x)u$ discussing a local system around one critical point of index 0.

Note that the autonomous vector field (without control inputs) of the original system (6) might be non-smooth. However, a weak-Lyapunov function shaped by pre-feedback control inputs is smooth. Such a situation frequently appears in practical control problems, e.g., oscillating systems, redundant freedom systems, constrained systems, non-holonomic systems, and homogeneous systems.

3.2 Stability

First of all, the main problem on stabilization is the behavior of the system state that stays on an invariant set $\Delta_1^r = \{x \in M \mid \dot{V}_0(x) = 0\}$ except for $\{0\}$, where $r = 0, 1$.

Definition 6. Consider $\Delta' = \Delta \backslash \{0\}$. Let $S|_\epsilon = \cup_i W^s(\delta_i)|_\epsilon$ be a union of stable manifolds restricted on a closed neighborhood within a radius of ϵ around $^\forall \delta_i \in \Delta'$. $S|_\epsilon$ is called a *locally invariant singular structure*. $S|_\infty$ (we denote by S simply) is called an *invariant singular structure* if $\epsilon \to \infty$. In other words, any $x \in S$ is asymptotically stable to Δ'.

Theorem 7. Consider a Morse-Smale system X on M. Let R be a submanifold of M such that $R = \mathrm{cl}(M \backslash S)$, where S is an invariant singular structure for Δ' of M. A restricted flow $X|_R$ on R is semi-global asymptotically stable with respect to $\{0\}$ for $^\forall x \in R$.

Proof. The solutions on S arrive at some point $x \in \Delta'$ along a positive time evolution. Then, such a solution remains in Δ' and never converges into $\{0\}$. On the other hand, there exists a ξ-function that is decreasing along the trajectories on the submanifold $R = \mathrm{cl}(M \setminus S)$ containing only regular points on M according to *Theorem 5*. Thus, all of the solutions on R converge to $\{0\}$. □

Theorem 8. Consider a Morse-Smale system X on M. There exists a weak-Lyapunov function V_0 for $\{0\}$.

Proof. M consists of a union of the submanifold R of regular points and the invariant singular structure S for Δ'. V_0 on R and $S \backslash \Delta'$ are decreasing along the trajectories. Thus, the singular points Δ' in S correspond to a set of $\dot{V}_0 = 0$. □

Remark 9. Since M is compact, there always exist a maximum and a minimum on M from maximum value theorem. In this case, the maximum corresponds to singular points with index m and the minimum corresponds to singular points with index 0. Then, the image of V_0 exists in the interval $[0, m]$. The critical points p_i and p_j can be arranged in such a way that $V_0(p_j) \succ V_0(p_i)$ implies that $\mathrm{ind}(p_i) \geq \mathrm{ind}(p_j)$ while keeping the topological property of M Matsumoto [2002], where $\mathrm{ind}(p)$ is an index of p. That is, suppose that $V_0(p) = \mathrm{ind}(p)$ for all critical points p. We call V_0 *self-indexed* if $V_0 = \mathrm{ind}(p)$ whenever p is a critical point or a closed orbit. Thus, the ξ-function can be considered as a self-indexed Lyapunov-Morse function.

3.3 Controllability

Definition 10. Let N be the closed neighborhood within a radius of ϵ around a critical point $\delta_i \in \Delta$ for any i. Let $\mathcal{R}(\delta_i, T)$ be the reachability set of the local system around δ_i for some T. Consider the submanifold $R_i = \mathrm{cl}\big(N \setminus W^s(\delta_i)|_\epsilon\big)$. If $\mathcal{R}(\delta_i, T) \cap R_i \neq \emptyset$, we call δ_i *locally escapable*.

The above condition can be stated using the limited version of the locally accessible sufficient condition Isidori [1995]. That is, now we only have to find 1 degree of freedom, at least for unstabilization of critical points δ_i.

Proposition 11. Consider a Morse-Smale system X on M. Let N be the closed neighborhood within a radius of ϵ around a critical point $\delta_i \subset \Delta$ for any i, where $x \in N \subset M$. Let $\mathcal{C}(\delta_i)$ be the distribution of the local system $\dot{x} = g_{i,0}(x) + \sum_j g_{i,j}(x) u_{i,j}$ on N, which corresponds to the regular part of the local coordinates which is expressed as a quadratic form in (2) and (5). Consider the restricted distribution $\mathcal{C}(\delta_i)|_{R_i}$ on $R_i = \mathrm{cl}\big(N \setminus W^s(\delta_i)|_\epsilon\big)$. If $\dim \mathcal{C}(\delta_i)|_{R_i} \neq 0$, then δ_i is locally escapable at x.

Proof. There exist canonical local coordinates (2) and (5) around Δ. Then, we can define the local system. If there exists a control that can drive the system to R_i, the state will never return to δ_i because δ_i is isolated; that is there exists a gradient-like flow decreasing along trajectories in the neighborhood of δ_i from *Theorem 5*. □

Corollary 12. The singular points $\delta_i \subset \Delta^0$ of V_0 with the index 0 are non-escapable.

Proof. Since $N \cong W^s(\delta_i)|_\epsilon$, $R_i = \emptyset$. □

In the same way as for δ_i, we have to consider the attracting orbits $S \setminus \delta_i$ to δ_i.

Proposition 13. Consider a Morse-Smale system X on M. Let N be the closed neighborhood within a radius of ϵ around a regular point x_0 on a stable manifold $W^s(\delta_i)$ of $\delta_i \subset \Delta$ for any i, where $N \subset M$. Let $X_{x_0} \in T_{x_0} W^s(\delta_i)$ be the vector field tangent to $W^s(\delta_i)$ at x_0. Let $\mathcal{C}(x_0)$ be the distribution of X_{x_0}. Consider the restricted distribution $\mathcal{C}(x_0)|_{R_i}$ on $R_i = \mathrm{cl}\big(N \setminus W^s(\delta_i)\big)$. If $\dim \mathcal{C}(x_0)|_{R_i} \neq 0$, then $W^s(\delta_i)$ is locally escapable at x_0.

Proof. The proof is given in the same manner as for *Proposition 11*. □

Theorem 14. Consider a Morse-Smale system X on M. If S is escapable, then the system is global asymptotically stable to $\{0\}$.

Proof. Since S is escapable, all the solution on S can be converged into R by using a control input. On the other hand, R is semi-global asymptotically stable to $\{0\}$. Since $M = R \cup S$, the global stable point of the system is equal to $\{0\}$. □

As a result, if we can find a smooth global weak-control-Lyapunov function V_0 on M satisfying *Definition 3*, the controlled system by pre-feedback can behave as a Morse-Smale system. Moreover, if the invariant singular structure S is escapable, the system is global asymptotically stable.

3.4 Stabilization

In the case of degenerate critical points, the escapability condition in the previous section is quite strict for a practical situation because the control is required to be available on all of the attracting orbits $S \setminus \delta_i$ to δ_i or all of the closed orbits δ^1. In the first case, usually $S|_\epsilon$ is considered as an escapable region. This section is devoted to the last case, that is the relaxation of *Theorem 14* in a constructive way.

Definition 15. Consider a Morse-Smale system X on M. Let us define a level-set $l_0 := \{p \in M \mid V_0(p) = V_0(x), x \in \Delta'\}$, where we denote level-sets for each δ_i by $l_{0,i}$.

In other words, l_0 is equivalent to the level-set defined by the inverse of self-indexed Lyapunov-Morse function $V_0^{-1}(y)\colon \mathbb{R}_{[0,m]} \to M$ for an integer $y \in \mathbb{R}_{[0,m]}$.

Definition 16. Let us define a finite disjoint union $L_0 := \sqcup_i^{n'} l_{0,i}$ of level-sets $l_{0,i}$, where $n' < n$, because we removed singular points of index 0.

Lemma 17. Each level-set l_0 is compact.

Proof. By the implicit function theorem, V_0^{-1} is a submanifold of M. □

Lemma 18. Let $\bar{l}_0$ be a closure $\mathrm{cl}(l_0 \setminus \Delta)$ of regular subset of l_0. There exists a collar neighborhood $l_0^{\bowtie}$ of $\bar{l}_0$ with a diffeomorphism $h\colon \bar{l}_0 \times [0,1) \to l_0^{\bowtie}$ on M, where $[0,1)$ is a half-open interval toward a positive time direction and $h(\bar{l}_0, 0) = \bar{l}_0$.

Proof. Since $l_0 = V_0^{-1}(x)$ is compact from *Lemma 17*, $l_0 \setminus \Delta$ is an open set. Then, $\bar{l}_0 = \mathrm{cl}(l_0 \setminus \Delta)$ is compact. $V_0'\colon M \setminus \Delta \to \mathbb{R}$ has no critical values in $[0,\epsilon)$ for a small enough positive number ϵ. The restriction of V_0' to $T_{[0,\epsilon)} = {V_0'}^{-1} \times [0,\epsilon)$ can be considered as a ξ-function on $T_{[0,\epsilon)}$. Let X be a gradient-like vector field for V_0' on $T_{[0,\epsilon)}$. If we define $Y = (1/X \cdot V_0')X$, then the integral curve $c_x(t)$ of Y starting at $x \in \bar{l}_0$ flows down with constant speed 1 with respect to the height defined by V_0' because

$$\frac{d}{dt} V_0'\big(c_x(t)\big) = \frac{dc_x}{dt}(t) \cdot V_0' = Y \cdot V_0' = 1\,. \tag{7}$$

Define a map $h\colon \bar{l}_0 \times [0,\epsilon) \to T_{[0,\epsilon)}$ by using $h(x,t) = c_x(t)$. For the collar neighborhood $l_0^{\bowtie}$, we can take $T_{[0,\epsilon)}$, $h|_{T_{[0,\epsilon)}} = h|_{{V_0'}^{-1} \times [0,\epsilon)}$, and we have

$$\bar{l}_0 \times [0,1) \xrightarrow{\cong} T_{[0,\epsilon)} \xrightarrow{h|_{T_{[0,\epsilon)}}} l_0^{\bowtie} \tag{8}$$

for the diffeomorphism. □

Definition 19. Consider $\bar{L}_0 := \sqcup_i^{n'} \bar{l}_{0,i}$ for $n' < n$. Then, the collar neighborhood of all level-sets l_0 on M for the positive time direction such as $\bar{L}_0 \times [0,1)$ is defined by $L_0^{\bowtie}$.

From the above preparations, at last, we can define a set of weak-Lyapunov functions on $L_0^{\bowtie}$ and its submanifolds.

Definition 20. Let $V_0\colon M \to \mathbb{R}$ be a weak-control-Lyapunov function for a global asymptotically stable point $\{0\}$ for any $x \in R$. Consider $\Delta_1 := \{x \in M \mid \dot{V}_0 = 0\}$. Then, let $V_1\colon \Delta_1 \to \mathbb{R}$ be a weak-control-Lyapunov function for any point $x_1 \in \sigma_1$ in an escapable region $\sigma_1 \subset \Delta_1$ to $L_0^{\bowtie}$ for any $x \in \Delta_1$. Next, $\Delta_i := \{x \in \Delta_{i-1} \setminus \{x_{i-1}\} \mid \dot{V}_{i-1} = 0\}$, where $2 \le i \le m$. In the same manner, let $V_i\colon \Delta_i \to \mathbb{R}$ be a weak-control-Lyapunov function for any point $x_i \in \sigma_i$ in an escapable region $\sigma_i \subset \Delta_i$ to a point in $\Delta_{i-1} \setminus \Delta_i$.

The following is the procedure of stabilization by using the sequence of weak-control-Lyapunov functions V_i ($0 \le i \le m$). From here on, we assume that the system $R = M \setminus S$ has been already stabilized to a global asymptotical stable point $\{0\}$ by pre-feedback based on V_0. At this point, we concentrate on the behavior of the system state that stays on an invariant set $\Delta_1 = \{x \in M \mid \dot{V}_0 = 0\}$. Then, we attempt to find a new weak-control-Lyapunov function V_1 converging on an escapable point $x_1 \in \sigma_1 \subset \Delta_1$ to the set $L_0^{\bowtie}$ for $x \in \Delta_1$. If the state moves to $L_0^{\bowtie}$ once, then it flows along the monotone decreasing direction of V_0, because $L_0^{\bowtie}$ consists of regular points. However, there may exist an invariant set $\Delta_2 = \{x \in \Delta_1 \setminus x_1 \mid \dot{V}_1 = 0\}$ in Δ_1, because of non-positiveness of V_1. Next, we try to find another weak-control-Lyapunov function V_2 converging on an escapable point to $\Delta_1 \setminus \Delta_2$ for $x \in \Delta_2$. In the same manner, we have to consider on an invariant set $\Delta_3 = \{x \in \Delta_2 \setminus x_2 \mid \dot{V}_2 = 0\}$. Finally, if we obtain V_i for all $x \in R$, the global asymptotical stability holds. A sequence $M \supset \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_m \supset \emptyset$ of inclusions regarding the manifold and its singular structures that is defined by the above procedure repeated over and over again until we obtain the set of weak-control-Lyapunov functions is called weak-Lyapunov filtration.

Theorem 21. Consider a sequence of filtered invariant sets $M \supset \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_m \supset \emptyset$. The filtration is a finite degree.

Proof. The intersection between the level-set generated by V_0 and an m-dimensional closed manifold is an $(m-1)$-dimensional closed surface. Thus, the intersection between an $(m-1)$-dimensional closed manifold and the level-set generated by V_1 is an $(m-2)$-dimensional surface in the same manner. Finally, we obtain a 0-dimensional intersection for V_{m-1}. □

Corollary 22. Consider a Morse-Smale system X on M. The weak Lyapunov filtration is 2 degrees: $\Delta_0 \supset \Delta_1 \supset \Delta_2$.

Proof. The closed orbits δ^1 created by V_0 have two critical points $\delta_{2,0}$ and $\delta_{2,1}$ whose indexes 0 and 1, respectively for a new negative gradient flow of a Morse function that can be considered as a V_1. Thus, for $\delta_{2,0}$ and $\delta_{2,1}$, V_2 should be constructed to escape themselves at least. Here, the facts: V_2 is defined on the critical point $\delta_{2,1}$ and the existence of V_2 indicate that there exists inputs to be escapable form the critical point to $\Delta_1 \setminus \Delta_2$. □

4. EXAMPLE

Let us consider the following system on (x,y)-plane:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \{1 - (x^2 + y^2)\} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} u_1 + \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} u_2 + a(\theta) \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} u_3\,, \tag{9}$$

where θ is an angle to the positive direction of x-axis at the origin

$a(\theta)$ is the function defined as follows:

$$a(\theta) = \begin{cases} \sin\theta - \dfrac{\sqrt{3}}{2} & \left(\sin\theta > \dfrac{\sqrt{3}}{2}\right) \\ 0 & \left(\sin\theta \leq \dfrac{\sqrt{3}}{2}\right) \end{cases} . \quad (10)$$

We define $V_0 = x^2 + y^2$ as a weak-control-Lyapunov function for the global asymptotical point 0. Δ_1 is a circle with a radius 1 at the origin. If we select u_1 based on V_0, after a enough long time, any states starting from the outside of Δ_1 converge to Δ_1 and any states starting from the inside of Δ_1 converge to 0.

Now, to simplify a problem, we consider only the situation that the system state exists on Δ_1 in the first case. In the case, a controllable region on Δ_1 toward radial directions to the origin is in the range $\sin\theta > \sqrt{3}/2$ in which u_3 is effective. Furthermore, we can use u_2 to drive the state into the region. Thus, we should find a new weak-control-Lyapunov function V_1 for any $x_1 \in \sigma_1 = \sin\theta > \sqrt{3}/2$. For example, we set $V_1 = x^1 + (y-1)^2$. V_1 generate $\Delta_2 = (0, -1)$, however Δ_2 is escapable to Δ_1.

Form the above, the design of weak-control-Lyapunov functions has been completed.

5. CONCLUSION AND FUTURE WORK

In this paper, a recursive method of constructing weak-control-Lyapunov functions based on a Morse-Smale flow for nonlinear systems was presented by limiting the topological situation to weak-control-Lyapunov functions.

The presented recursive procedure still holds in the general case of the nullity $r \leq n$, leaving aside the development of definite calculations. In this study, we took notice of the condition of the nullity $r \leq 1$. As a result, we found that the procedure could be finished in a finite number of steps.

The stability and the controllability were defined on the assumption that the state-space manifold is closed. The assumption can be relaxed on the boundary of M. That is, in the case that M has a boundary, we can carry out the same discussion by considering the flow on the boundary.

We consider detailed discussion regarding the following advanced topic to be a future work. On the residual singularity of the local structure around degenerate critical points in Thom's splitting lemma, for example, the case of $r = 1$, the singular point p of f is called A_k-type if $f'(p) = \cdots = f^{(k)}(p) = 0$ and $f^{(k+1)}(p) \neq 0$. For such an A_k-type singular point, there exists a formal local coordinate such that $f(x) = f(p) \pm x^{k+1}$ for p. The classification of residual singularity has been moved ahead by Arnol'd Arnol'd [Ed.]. The simple singular points, which do not have moduli, are classified by the series of simple Lie algebras: $A_k, (k > 1)$, $D_k, (k > 4)$ and E_6, E_7, E_8 through a Dynkin diagram. Such a classification has the capability of dealing with a more unified definition of stability of degenerate critical points. On the other hand, it is known from Hironaka's resolution of the singularity theorem that there exists a resolution $\varphi\colon \widetilde{X} \to X$ of singularity for any algebraic variety X. Then, φ is obtained by making several blowing-ups on the submanifold Hironaka [1984]. This method may be used for changing the degenerate cases into Morse-type regular problems.

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