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Robust Output Feedback MPC for Linear Systems via Interpolation Technique

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Abstract: This paper provides a simple approach to the problem of robust output feedback model predictive control (MPC) for linear discrete-time systems with state and input constraints, subject to bounded state disturbances and output measurement errors. The problem of estimating the state is addressed by using a fixed linear observer. The state estimation error converges and stays in some set of the error dynamics, which is taken into account in the design of MPC controllers. In the MPC optimization where the nominal system is considered, the constraints are tightened in a monotonic sequence such that the satisfaction of input and state constraints for the original system is guaranteed. Robust stability of an invariant set for the closed-loop original system is ensured. Furthermore, in order to reduce the inherent computational complexity of the MPC controller design, interpolation techniques are introduced in the proposed approach, where the resulting controller interpolates among several MPC controllers. This procedure leads to a relatively large domain of attraction even by employing short prediction horizons. Therefore, with short horizons, a low computational complexity is expected.

1. INTRODUCTION

Model predictive control (MPC) is a feedback scheme in which an optimal control problem is solved at each time step and only the first step of the control sequence is applied, see Maciejowski (2002). Since MPC has the ability to handle hard constraints, it has received great attention in the literature. However, most traditional MPC are based on mathematical models which may has mismatch with the physical systems. Therefore, robust MPC to address both model uncertainty and disturbances is an important issue and many researchers have contributed in this area, see Bemporad et al. (2003); Kerrigan and Maciejowski (2004); Mayne et al. (2005); Manthanwar et al. (2005); Wan et al. (2006). In most MPC formulations, the state feedback is assumed, which requires full knowledge of the state (Findeisen et al. (2003); Mayne et al. (2000)). In practice, the measurements contain noise, and often internal states are not measurable. Ignoring measurement errors may result in degradation in performance or even cause instability.

This paper considers the problem of robust output feedback MPC for linear systems with state and input constraints, subject to bounded state disturbances and output measurement errors. The motivation of this paper is to provide an approach for computing output feedback MPC controllers that respect state and input constraints and ensure the robust stability of the closed-loop system.

For robust output feedback MPC, a common approach is to combine an observer with a standard predictive scheme, where the state estimate substitutes for the true system state. The design of output feedback MPC can be tackled by two approaches. One approach is to pursue the ”certainty equivalence” principle and try to separate the estimation error from the state feedback by time scale separation and therefore make the observer dynamics sufficiently faster than controller dynamics. This may be achieved using high-gain (Imsland et al. (2003)) or deadbeat observers (Pannocchia and Kerrigan (2005)). However, such approaches are not expected to be useful in the presence of noise, and therefore of little practical value in low-level control. Another approach is based on accounting for the errors in the state estimate by robust MPC controller design. With such an approach, a state estimator that provides estimation error bounds is typically required, see Chisci and Zappa (2002); Bemporad and Garulli (2000); Mhaskar et al. (2004); Goulart and Kerrigan (2007). In Mayne et al. (2006), state estimates with bounded error within an invariant set are provided by a simple Luenberger observer, and a tube-based robust predictive controller design is used, while the control paradigm is shifted from control of true process states to control of nominal estimator states.

In this paper, the proposed output feedback controller consists of a Luenberger observer and a robustly stabilizing, tube-based, MPC controller. The estimation error dynamics is stable and errors converge to the minimal disturbance invariant set, $\mathbb{E}$. The errors are taken into account by introducing the set $\mathbb{E}$ in the controller design. Like the approach proposed in Mayne et al. (2006), the controller uses a tube, the center of which is obtained by solving a nominal MPC problem and within which the estimated state is guaranteed to remain. The control problem is addressed by steering the tube to the origin. Due to considering the nominal system, the constraints in the optimization are tightened such that satisfaction of the input and state constraints for the original system is guaranteed. Unlike the work in Mayne et al. (2006), in our approach the constraints are tightened in a monotonic sequence and relaxed. Robust stability of an invariant set for the closed-loop original system is guaranteed. The computational complexity of the resulting controller is similar to that of the standard, nominal MPC controller. Moreover, in order to further relax the computational

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load, an interpolation technique is introduced in the proposed approach, inspired by the advanced work in Bacić et al. (2003); Rossiter et al. (2004). The resulting controller interpolates among several MPC controllers based on the current estimated state decomposition. This procedure leads to a relatively large domain of attraction, which is a convex hull of all domains of attraction of the constituent MPC controllers. By employing short horizons, low computational complexity is expected.

The paper is organized as follows. Section 2 discusses the class of systems considered, states several assumptions and reviews some definitions. Section 3 introduces the Luenberger observer. In Section 4, the framework of the robust output feedback MPC is introduced and its properties are stated. Section 5 applies the interpolation technique into the proposed approach, leading to low complexity. The effectiveness of the proposed output feedback controller is illustrated in Section 6. Followed by some conclusions made in Section 7.

Notation and Basic Definitions: Positive definite (semi-definite) square matrix A is denoted by $A \succ 0$ ($A \succeq 0$) and $A \succ (\succeq) B$ means $A - B \succ (\succeq) 0$. $\|x\|_A^2 = x^T A x$ with $A \succ 0$. $\|\cdot\|$ is the Euclidean norm. Let $\rho(A)$ denote spectral radius of a square matrix A. A set $X \subset \mathbb{R}^n$ is a C set if it is a compact, convex set that contains the origin in its (non-empty) interior. Suppose $X, Y \subset \mathbb{R}^n$, the interior of X is $int(X)$; $|X|$ is its cardinality; the P-difference of X and Y is $X \ominus Y = \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$ and the Minkowski sum is $X \oplus Y = \{z \in \mathbb{R}^n : z = x + y, x \in X, y \in Y\}$. Suppose $X_1, \cdots, X_k \subset \mathbb{R}^n$, their convex hull $Co(X_1, \cdots, X_k)$ = $\{\lambda_1 X_1 \oplus \cdots \oplus \lambda_k X_k : 0 \leqslant \lambda_i \leqslant 1, \sum_{i=1}^k \lambda_i = 1\}$. A polyhedron is the (convex) intersection of a finite number of open and/or closed half-spaces and a polytope is the closed and bounded polyhedron.

2. BACKGROUND

The following discrete-time, linear time-invariant system is considered,

$$x(t+1) = Ax(t) + Bu(t) + Dw(t), \ \forall t \geq 0 \quad (1a)$$

$$y(t) = Cx(t) + Ev(t), \ \forall t \geq 0, \quad (1b)$$

where t is the discrete time index, $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ are the state, input and measured output respectively and $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$. $w \in \mathbb{R}^{n_w}$ is the unknown state disturbance, $v \in \mathbb{R}^{n_v}$ is the measurement noise and disturbances w, v are known only to the extent that they lie, respectively, in the C sets $\mathbb{W}$ and $\mathbb{V}$. System (1) is subject to the following sets of hard state and input constraints:

$$x(t) \in \mathbb{X}, \ u(t) \in \mathbb{U}, \ \forall t \geq 0. \quad (2)$$

It is assumed in this paper that:

(A1) the couple (A, B) is controllable and (A, C) is observable;
(A2) $\mathbb{X}$ and $\mathbb{U}$ are polyheral and polytopic sets respectively, and both contain the origin as an interior point.

To make the results in the subsequent sections explicit, a few definitions are reviewed below.

Definition 1. (d-invariant set) A set $T \subset \mathbb{R}^{n_x}$ is disturbance invariant (d-invariant) for the system $x(t+1) = Ax(t) + Dw(t)$ and the constraint set $(\mathbb{X}, \mathbb{W})$ if $T \subseteq \mathbb{X}$ and $x(t+1) \in T$ for all $w(t) \in \mathbb{W}$ and $x(t) \in T$.

Definition 2. (Minimal d-invariant set, see Kolmanovsky and Gilbert (1998)) The minimal d-invariant set of the system $x(t+1) = Ax(t) + Dw(t)$ is d-invariant that is contained in every closed, d-invariant set of the system $x(t+1) = Ax(t) + Dw(t)$.

Definition 3. (Maximal d-invariant set, see Kolmanovsky and Gilbert (1998)) The maximal d-invariant set of the system $x(t+1) = Ax(t) + Dw(t)$ is d-invariant that contains every closed, d-invariant set of the system $x(t+1) = Ax(t) + Dw(t)$.

3. LUENBERGER OBSERVER

In most control problems, state feedback is assumed. In practice, the measurements contain noise, and perfect knowledge of the state is not realistic. A common approach is therefore to employ an observer and substitute the resulting state estimate for the true system state in the controller design. When the system dynamics is linear, a Luenberger observer is often employed, see Chisci and Zappa (2002); Mayne et al. (2006); Wan and Kothare (2002). A Luenberger observer estimates the state, *i.e.*

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)), \ \forall t \geq 0 \quad (3a)$$

$$\hat{y}(t) = C\hat{x}(t), \ \forall t \geq 0, \quad (3b)$$

where $\hat{x} \in \mathbb{R}^{n_x}$ is the current observer state, u is the current control action, $\hat{y} \in \mathbb{R}^{n_y}$ is the current observer output and the observer matrix is defined by $L \in \mathbb{R}^{n_x \times n_y}$. Let the state estimation error be

$$e(t) = x(t) - \hat{x}(t), \ \forall t \geq 0. \quad (4)$$

Then the estimated state $\hat{x}$ satisfies the following uncertain dynamics

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + LCe(t) + LEv(t), \quad (5)$$

while the state estimation error satisfies

$$e(t+1) = A_L e(t) + (Dw(t) - LEv(t)), \ \forall t \geq 0, \quad (6)$$

where L satisfies $\rho(A_L) < 1$ ($A_L := A - LC$). Let $\alpha(t) = Dw(t) - LEv(t)$. Thus $\alpha(t)$ lies in a C set $\mathbb{Q}$ defined by

$$\mathbb{Q} = D\mathbb{W} \oplus (-LE\mathbb{V}). \quad (7)$$

Equation (6) can be rewritten, *i.e.*

$$e(t+1) = A_L e(t) + \alpha(t). \quad (8)$$

Due to $\rho(A_L) < 1$, there exists a C set $\mathbb{E}$ such that it is d-invariant for system (8). It follows

$$A_L \mathbb{E} \oplus \mathbb{Q} \subseteq \mathbb{E}, \quad (9)$$

which implies that if $e(0) \in \mathbb{E}$, $e(t) \in \mathbb{E}$, $\forall t \geq 0$. Since the set $\mathbb{E}$ is the upper set of the error, it is desired to be as small as possible. In this paper the set $\mathbb{E}$ is chosen as the outer bound of the minimal d-invariant set of system (8). Efforts to compute such a set for linear systems have appeared in the literature, see for example Ong and Gilbert (2006); Raković et al. (2005). If $e(t) \in \mathbb{E}, \forall t \geq 0$, the dynamics (5) also follows that

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \beta(t), \quad (10)$$

where the uncertainty $\beta(t) := LCe(t) + LEv(t) \in \mathbb{T}$ and $\mathbb{T}$ is a C set such that

$$\mathbb{T} := LC\mathbb{E} \oplus LE\mathbb{V} \quad (11)$$

which is bounded. Therefore, the estimated state dynamics (5) can be regard as a nominal system of (1a) with an additional, unknown but bounded uncertainty, as stated in Mayne et al. (2006).

Proposition 1. (Mayne et al. (2006)) If the initial system and observer states, $x(0)$ and $\hat{x}(0)$, respectively, satisfy $e(0) \in \mathbb{E}$, then $x(t) \in \hat{x}(t) \oplus \mathbb{E}$ for all $t \geq 0$ and all admissible disturbances $w(t), v(t)$, $\forall t \geq 0$.

4. OUTPUT FEEDBACK MPC

4.1 Problem formulation

The output feedback MPC controller $u(\cdot)$ takes the form following Lee and Kouvaritakis (2000); Chisci et al. (2001), which is parameterized by $c(\cdot) \in \mathbb{R}^{n_u}$ as

$$u(t) = K\hat{x}(t) + c(t) \tag{12}$$

for some given $K \in \mathbb{R}^{n_u \times n_x}$ such that $\Phi := A + BK$ is asymptotically stable ($\rho(\Phi) < 1$). The motivation of the proposed approach is to find $c(t)$ such that robust constraint satisfaction can be guaranteed for all $t \geq 0$ and robust closed-loop stability can be ensured. To achieve it, the state estimation error should be taken into account by introducing its associated estimation error set $\mathbb{E}$. Given an initial state estimate and bounded on the estimation error such that $e(0) \in \mathbb{E}$, the estimated state satisfies

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + \beta(t), \tag{13}$$

where $\beta(t) \in \mathbb{T}$. To design the robust output feedback MPC controller, the constraint tightening approach is used, which is introduced by Gossner et al. (1997) and extended by Chisci et al. (2001). Recently, some modifications are developed based on it, see Mayne et al. (2005); Sui and Ong (2007); Richards and How (2006). This approach avoids huge complexity by using only a nominal prediction model and modifying the constraints to achieve robustness. The key idea is that the effect of the persistent disturbance β is taken into account by using the strengthened input/output constraints. Moreover, like the approach proposed in Mayne et al. (2006), the controller uses a tube, within which the estimated state is guaranteed to remain.

The disturbance-free system is

$$\widetilde{x}(t+1) = A\widetilde{x}(t) + B\widetilde{u}(t), \ \forall t \geq 0. \tag{14}$$

At time t with the given $\hat{x}(t)$, the finite horizon MPC problem over $\mathbf{c}(t) = [c^T(0|t), c^T(1|t), \cdots, c^T(N-1|t)]^T$ is:

$$\min_{\mathbf{c}(t),\widetilde{x}(0|t)} J(\mathbf{c}(t), \widetilde{x}(0|t); \hat{x}(t)) = \sum_{k=0}^{N-1} \|c(k|t)\|^2_{\Psi} \tag{15a}$$

$$s.t. \quad \hat{x}(t) \in \widetilde{x}(0|t) \oplus \mathbb{T}, \tag{15b}$$

$$\widetilde{x}(k+1|t) = A\widetilde{x}(k|t) + B\widetilde{u}(k|t), \forall k \geq 0, \tag{15c}$$

$$\widetilde{u}(k|t) = K\widetilde{x}(k|t) + c(k|t), k = 0, 1, \ldots, N-1, \tag{15d}$$

$$\widetilde{u}(k|t) = K\widetilde{x}(k|t), \forall k \geq N, \tag{15e}$$

$$\widetilde{x}(k|t) \in \mathbb{X}_k, \ \widetilde{u}(k|t) \in \mathbb{U}_k, k = 0, 1, \ldots, N-1, \tag{15f}$$

$$\widetilde{x}(N|t) \in \widetilde{\mathbb{X}}_f, \tag{15g}$$

where $\Psi \succ 0$; N is the prediction horizon; the notations $\widetilde{x}(k|t)$ and $\widetilde{u}(k|t)$ denote the state and input at time $t+k$ derived by using (15c)-(15e) based on the estimated state $\hat{x}(t)$. The sets $\mathbb{X}_k$, $\mathbb{U}_k$ and $\mathbb{X}_f$ are appropriately strengthened, given by

$$\mathbb{X}_k = \mathbb{X}_t \ominus F_k, \ \ \mathbb{U}_k = \mathbb{U} \ominus KF_k, \ \ \widetilde{\mathbb{X}}_f = \mathbb{X}_f \ominus F_N \tag{16}$$

where

$$F_k := \mathbb{T} \oplus \Phi\mathbb{T} \oplus \cdots \oplus \Phi^k\mathbb{T} \tag{17}$$

and $\mathbb{X}_t = \mathbb{X} \ominus \mathbb{E}$. The terminal set $\mathbb{X}_f$ is chosen to be the maximal d-invariant set of system

$$x(t+1) = \Phi x(t) + \beta(t), \tag{18a}$$

$$s.t. \ \ x(t) \in \mathbb{X}_t, \ Kx(t) \in \mathbb{U}, \ \beta(t) \in \mathbb{T}, \tag{18b}$$

in the sense that $\Phi\mathbb{X}_f \oplus \mathbb{T} \subseteq \mathbb{X}_f$. In problem (15), the center of the tube $\mathbb{T}$ at the initial time is treated as a decision variable. The MPC controller applied to system (1) at time t is

$$u^*(t) := K\hat{x}(t) + c^*(0|t) \tag{19}$$

where $c^*(0|t)$ is the first control of the optimal solution $\mathbf{c}^*(t)$ of problem (15). Let

$X_N := \{\hat{x}(t) : \exists \mathbf{c}(t), \widetilde{x}(0|t) \text{ such that } (15b) - (15g) \text{ are feasible}\}$ be the domain of attraction of system (13) under (19).

4.2 Feasibility and stability

To show the feasibility and robust stability of the proposed output feedback MPC, we first define the following set sequence

$$\mathbf{c}(t+1) = [c^{*T}(1|t), \cdots, c^{*T}(N-1|t), 0]^T \tag{20}$$

which is obtained by the concatenation of the optimal "tail" at time t, with a terminal zero element.

Lemma 2. Suppose Assumptions (A1)-(A2) hold and $e(t) \in \mathbb{E}, \forall t \geq 0$. For system (1) under the output feedback MPC controller (19), if there exists a feasible solution of problem (15) for $\hat{x}(t)$, then there also exists a feasible solution for $\hat{x}(t+1)$.

Proof At time $t+1$, the estimated state $\hat{x}(t+1)$ is

$$\hat{x}(t+1) = \Phi\hat{x}(t) + Bc^*(0|t) + \beta(t). \tag{21}$$

Since $\hat{x}(t) \in \widetilde{x}^*(0|t) \oplus \mathbb{T}$ and $\beta(t) \in \mathbb{T}$, we have

$$\hat{x}(t+1) \in \Phi\widetilde{x}^*(0|t) + Bc^*(0|t) \oplus \Phi\mathbb{T} \oplus \mathbb{T}, \tag{22}$$

or $\hat{x}(t+1) \in \widetilde{x}(1|t) \oplus \Phi\mathbb{T} \oplus \mathbb{T}$. Hence,

$$\hat{x}(t+1) \in \widetilde{x}(0|t+1) \oplus \mathbb{T}, \tag{23}$$

where $\widetilde{x}(0|t+1) \in \widetilde{x}(1|t) \oplus \Phi\mathbb{T}$. Employing the control sequence $\mathbf{c}(t+1)$, we have

$$\widetilde{x}(k|t+1) \in \widetilde{x}(k+1|t) \oplus \Phi^{k+1}\mathbb{T}, \ \ k = 0, 1, \ldots, N,$$
$$\widetilde{u}(k|t+1) \in \widetilde{u}(k+1|t) \oplus K\Phi^{k+1}\mathbb{T}, \ k = 0, 1, \ldots, N.$$

Due to the fact that $\widetilde{x}(N|t) \in \widetilde{\mathbb{X}}_f$, we have $\widetilde{x}(N|t) + y \in \mathbb{X}_f$ for all $y \in F_N$. Thus $K\widetilde{x}(N|t) + Ky \in \mathbb{U}$, which implies $\widetilde{u}(N|t) = K\widetilde{x}(N|t) \in \mathbb{U} \ominus KF_N = \mathbb{U}_N$. The fact $\mathbb{X}_f \subseteq \mathbb{X}_t$ means $\widetilde{\mathbb{X}}_f \subseteq \mathbb{X}_N$. It implies $\widetilde{x}(N|t) \in \mathbb{X}_N$. From equation (15f) and the above discussion, we know that $\widetilde{x}(k|t) \in \mathbb{X}_k$, $\widetilde{u}(k|t) \in \mathbb{U}_k$, $k = 0, 1, \ldots, N$, which implies that

$$\widetilde{x}(k|t+1) \in \mathbb{X}_{k+1} \oplus \Phi^{k+1}\mathbb{T}, \ \ k = 0, 1, \ldots, N-1,$$
$$\widetilde{u}(k|t+1) \in \mathbb{U}_{k+1} \oplus K\Phi^{k+1}\mathbb{T}, \ k = 0, 1, \ldots, N-1.$$

This, together with (17), show that

$$\widetilde{x}(k|t+1) \in \mathbb{X}_k, \ \widetilde{u}(k+1|t) \in \mathbb{U}_k, \ k = 0, 1, \ldots, N-1.$$

Since $\mathbb{X}_f$ is a d-invariant set, $\widetilde{x}(N|t) + y \in \mathbb{X}_f, \ \forall y \in F_N \Rightarrow \Phi\widetilde{x}(N|t) + \Phi y \oplus \mathbb{T} \in \mathbb{X}_f \Rightarrow \widetilde{x}(N+1|t) \in \mathbb{X}_f \ominus \mathbb{T} \ominus \Phi F_N = \mathbb{X}_f \ominus F_{N+1}$. Hence,

$$\widetilde{x}(N|t+1) \in \widetilde{x}(N+1|t) \oplus \Phi^{N+1}\mathbb{T} \subseteq \mathbb{X}_f \ominus F_{N+1} \oplus \Phi^{N+1}\mathbb{T}$$
$$\subseteq \mathbb{X}_f \ominus F_N = \widetilde{\mathbb{X}}_f.$$

From the above, the set sequence $[\mathbf{c}(t+1), \widetilde{x}(0|t+1)]$ is feasible for $x(t+1)$.

We can now establish our first main result:

Theorem 3. Suppose Assumptions (A1)-(A2) hold, system (1) with the proposed output feedback MPC controller (19) has the following properties, for any $\hat{x}(0) \in X_N$ and $e(0) \in \mathbb{E}$: (i) $x(t) \in X$ and $u^*(t) \in U$ for all $t \geqslant 0$; (ii) $\lim_{t\to\infty} c(t) = 0$, where $c(t) = c^*(0|t)$; (iii) $\hat{x}(t) \to F_\infty$, as $t \to \infty$, where $F_\infty = \lim_{k\to\infty} F_k$; (iv) $x(t) \to F_\infty \oplus \mathbb{E}$, as $t \to \infty$.

Proof Since, by assumption, $\hat{x}(0)$ is feasible, $\hat{x}(t)$ is feasible for all $t \geqslant 0$ following Lemma 1. Due to the fact that $\tilde{x}(0|t) \in \mathbb{X}_t \ominus \mathbb{T}$, $\tilde{u}(0|t) \in \mathbb{U} \ominus K\mathbb{T}$ and $\hat{x}(t) \in \tilde{x}(0|t) \oplus \mathbb{T}$, we have $\hat{x}(t) \in \mathbb{X}_t$, $u^*(t) \in \mathbb{U}$, $\forall t \geq 0$. Following proposition 1, $x(t) \in \hat{x}(t) \oplus \mathbb{E}$ for all $t \geq 0$ and all admissible disturbances $w(t), v(t)$, which implies $x(t) \in \mathbb{X}$, $\forall t \geq 0$. Thus property (i) holds. Suppose the optimal cost is defined by $J^*(t) = \sum_{k=0}^{N-1} \|c^*(k|t)\|_\Psi^2$. At time $t+1$, there exists a feasible cost $J^f(t+1) = \sum_{k=1}^{N-1} \|c^*(k|t)\|_\Psi^2$. Hence,

$$J^*(t+1) - J^*(t) \leqslant -\|c^*(0|t)\|_\Psi^2. \tag{24}$$

It is easy to see that $\{J^*(t)\}$ is non-increasing and bounded by 0. As $t \to \infty$, it converges to $J^*(\infty) < +\infty$. Summing (24), we have $+\infty > J^*(0) - J^*(\infty) \geqslant \sum_{t=0}^{\infty} \|c^*(0|t)\|_\Psi^2 \geqslant 0 \Rightarrow \lim_{t\to\infty} c(t) = 0$. Therefore, property (ii) is proven. Thanks to Assumptions (A1)-(A2) and $\rho(\Phi) < 1$,

$$\begin{aligned}\lim_{t\to\infty} \hat{x}(t) &= \lim_{t\to\infty} \left[\Phi^t \hat{x}(0) + \sum_{k=1}^{t} \Phi^{k-1} Bc(t-k) + \sum_{k=1}^{t} \Phi^{k-1} \beta(t-k)\right] \\ &= \lim_{t\to\infty} \left[\sum_{k=1}^{t} \Phi^{k-1} \beta(t-k)\right],\end{aligned}$$

which, in turn, proves (iii). Property (iv) of the theorem follows from the fact that $x(t) \in \hat{x}(t) \oplus \mathbb{E}$ for all $t \geq 0$.

5. INTERPOLATED OUTPUT FEEDBACK MPC

The size of X_N depends on the size of the terminal set X_f and the length of the horizon N. By increasing the length of N, X_N is enlarged but at the expense of a higher computational complexity. Employing a large set X_f can also enlarge a set X_N for a fixed N and hence reduce the computational complexity. However, obtaining a large X_f generally (although not necessarily) implies that the corresponding controller Kx is de-tuned and thus jeopardizes the local optimal performance, see Wan and Kothare (2003). In this section, we aim to enlarge X_N while using a relatively small value of N by introducing interpolation techniques. Instead of employing (12), the output feedback MPC controller interpolates among several MPC controllers, *i.e.*

$$u(t) = \sum_{p=0}^{\nu} \lambda_p(t)(K_p \hat{z}_p(t) + c_p(t)) \tag{25}$$

with $\sum_{p=0}^{\nu} \lambda_p(t) = 1,\ 0 \leq \lambda_p(t) \leq 1$, where $\hat{z}_p(t)$ is determined by $\hat{x}(t) = \sum_{p=0}^{\nu} \lambda_p(t)\hat{z}_p(t)$ and K_p, $\forall p \in \mathscr{P} := \{0, 1, \ldots, \nu\}$ are given such that $\rho(\Phi_p) < 1$ ($\Phi_p := A + BK_p$). For convenience, a subscript is added to various quantities (F_k, F_∞, Φ, $\mathbb{X}_k$, $\mathbb{U}_k$, $\widetilde{\mathbb{X}}_f$, N etc.) to denote a particular p among the $\nu+1$ systems.

At time t, with the given $\hat{x}(t)$, the interpolated MPC problem is formulated, *i.e.*

$$\min_{\mathbf{z}(t)} J(\mathbf{z}(t); \hat{x}(t)) = \sum_{p=0}^{\nu} \sum_{k=0}^{N_p-1} \|\lambda_p(t) c_p(k|t)\|_\Psi^2 \tag{26a}$$

$$\text{s.t.}\quad \hat{x}(t) = \sum_{p=0}^{\nu} \lambda_p(t) \hat{z}_p(t), \tag{26b}$$

$$\hat{z}_p(t) \in \tilde{z}_p(0|t) \oplus \mathbb{T},\ p \in \mathscr{P}, \tag{26c}$$

$$\tilde{z}_p(k+1|t) = A\tilde{z}_p(k|t) + B\tilde{u}_p(k|t), \forall k \geq 0, p \in \mathscr{P}, \tag{26d}$$

$$\tilde{u}_p(k|t) = K_p \tilde{z}_p(k|t) + c_p(k|t),\ k = 0, \ldots, N_p - 1, p \in \mathscr{P} \tag{26e}$$

$$\tilde{u}_p(k|t) = K_p \tilde{z}_p(k|t), \forall k \geqslant N_p,\ p \in \mathscr{P}, \tag{26f}$$

$$\tilde{z}_p(k|t) \in \mathbb{X}_{k,p},\ \tilde{u}_p(k|t) \in \mathbb{U}_{k,p}, k = 0, \ldots, N_p - 1, p \in \mathscr{P} \tag{26g}$$

$$\tilde{z}_p(N_p|t) \in \widetilde{\mathbb{X}}_{f,p},\ \forall p \in \mathscr{P}, \tag{26h}$$

$$\sum_{p=0}^{\nu} \lambda_p(t) = 1,\ 0 \leq \lambda_p(t) \leq 1,\ \forall p \in \mathscr{P}, \tag{26i}$$

where $\mathbf{z}(t) = [\lambda(t), \tilde{\mathbf{z}}(t), \mathbf{c}(t)]^T$ with $\lambda(t) = [\lambda_0(t), \cdots, \lambda_\nu(t)]$, $\tilde{\mathbf{z}}(t) = [\tilde{z}_0^T(0|t), \cdots, \tilde{z}_\nu^T(0|t), \hat{z}_0^T(t), \cdots, \hat{z}_\nu^T(t)]$, $\mathbf{c}(t) = [\mathbf{c}_0(t), \cdots, \mathbf{c}_\nu(t)]$ and $\mathbf{c}_p(t) = [c_p^T(0|t), \cdots, c_p^T(N_p-1|t)]$. Let $\mathbf{z}^*(t)$ be the optimal solution of problem (26). The interpolated output feedback MPC controller applied to system (1) at time t is

$$u^*(t) := \sum_{p=0}^{\nu} \lambda_p{}^*(t) \big(K_p \hat{z}_p^*(t) + c_p^*(0|t)\big). \tag{27}$$

Define the domain of attraction for a p system to be

$$X_{N_p,p} = \{\hat{z}_p(t) : \exists \mathbf{c}_p(t), \tilde{z}_p(0|t) \text{ such that } (26c) - (26h) \text{ are feasible}\}.$$

It is shown in the following proposition that the domain of attraction of system (13) under (27) is the convex hull of the sets $X_{N_p,p}$, for all $\nu+1$ systems, *i.e.*

$$\bar{X} := Co(X_{N_0,0}, \cdots, X_{N_\nu,\nu}). \tag{28}$$

Proposition 4. Suppose Assumptions (A1)-(A2) hold. If and only if $\hat{x}(t) \in X$, there exists a feasible solution of problem (26).

Proof Since $\hat{x}(t) \in \bar{X}$, from the definition of convex hull of sets, it implies that there exist a vector $\lambda(t)$ and $\hat{z}_p(t)$ such that $\hat{x}(t) = \sum_{p=0}^{\nu} \lambda_p(t)\hat{z}_p(t)$ and $\hat{z}_p(t) \in X_{N_p,p}$, $\sum_{p=0}^{\nu} \lambda_p(t) = 1$, $0 \leq \lambda_p(t) \leq 1, \forall p \in \mathscr{P}$. This, together with the definition of the sets $X_{N_p,p}$, show that problem (26) has a feasible solution for $x(t) \in X$ and vice versa.

Following Proposition 4, the domain of attraction of system (13) is enlarged by interpolating among several MPC controllers. For each MPC controller, the pre-defined value of K_p can be chosen according to the special requirements and with the following principles

1. For all $p \in \mathscr{P}$, it is required that $\rho(\Phi_p) < 1$ to ensure the stability.
2. For all $p \in \mathscr{P}$, $\mathbb{X}_{f,p}$ always exists. K_ν should be chosen carefully. Since it is used to produce a large terminal set $\mathbb{X}_{f,\nu}$.

It is reasonable (although not necessary) that a low value of K implies a large set $\mathbb{X}_f$, see Rossiter et al. (2001); Kolmanovsky and Gilbert (1996). Therefore, unconstrained LQ methodology is one of the most straightforward methods to determine K_ν, by increasing weight matrix R or decreasing weight matrix Q in the cost function. Other methodologies are mentioned in Sui and Ong (2007). Due to the use of a large set of $\mathbb{X}_{f,\nu}$, a large domain of attraction $X_{N_\nu,\nu}$ is preserved even employing a short horizon N_ν. Following (28), a large $X_{N_\nu,\nu}$ implies a large set $\bar{X}$, which in turn means that other horizons N_p $(p \neq \nu)$ can also be chosen small. Hence, the reasonable computational complexity of the interpolated output feedback MPC is expected because of the use of the short horizons N_p. It is also necessary to note that the number of the individual MPC controllers, $\nu+1$, affects the computational complexity. The less is the number, the less is the computational work. Thereby, it is desired that the number $\nu+1$ is small. Generally ν can be chosen as 1, and when $\nu = 0$, the Interpolated MPC problem becomes the standard one.

The feasibility and the closed-loop stability of the interpolated MPC can be easily obtained based on Lemma 2 and Theorem 3.

Lemma 5. Suppose Assumptions (A1)-(A2) hold and $e(t) \in \mathbb{E}$. For system (1) under the interpolated output feedback MPC controller (27), if there exists a feasible solution of problem (26) for $\hat{x}(t)$, then there also exists a feasible solution for $\hat{x}(t+1)$.

Proof At time $t+1$, the estimated state is $\hat{x}(t+1) = A\hat{x}(t) + B\sum_{p=0}^{\nu}\lambda_p{}^*(t)\big(K_p\hat{z}_p^*(t) + c_p^*(0|t)\big) + \beta(t)$. Hence

$$\hat{x}(t+1) = \sum_{p=0}^{\nu}\lambda_p{}^*(t)\big(\Phi_p\hat{z}_p^*(t) + Bc_p^*(0|t) + \beta(t)\big). \qquad (29)$$

Choose

$$\hat{z}_p(t+1) = \Phi_p\hat{z}_p^*(t) + Bc_p^*(0|t) + \beta(t),\ \forall p \in \mathscr{P}$$

and $\lambda_p(t+1) = \lambda_p(t)$. Due to the fact that $\hat{z}_p^*(t) \in \tilde{z}_p^*(0|t) \oplus \mathbb{T}$, $\forall p \in \mathscr{P}$ and $\beta(t) \in \mathbb{T}$,

$$\hat{z}_p(t+1) \in \Phi_p\tilde{z}_p^*(0|t) + Bc_p^*(0|t) \oplus \Phi_p\mathbb{T} \oplus \mathbb{T},\ \forall p \in \mathscr{P} \qquad (30)$$

or $\hat{z}_p(t+1) \in \tilde{z}_p(1|t) \oplus \Phi_p\mathbb{T} \oplus \mathbb{T}$. Hence

$$\hat{z}_p(t+1) \in \tilde{z}_p(0|t+1) \oplus \mathbb{T},\ \ \forall p \in \mathscr{P}, \qquad (31)$$

where $\tilde{z}_p(0|t+1) \in \tilde{z}_p(1|t) \oplus \Phi_p\mathbb{T}$. The rest of proof is similar as the proof of Lemma 2, thus it is omitted due to the lack of space.

Now we establish the second main result:

Theorem 6. Suppose Assumptions (A1)-(A2) hold, system (1) with the proposed interpolated output feedback MPC controller has the following properties for any $\hat{x}(0) \in \bar{X}$ and $e(0) \in \mathbb{E}$: (i) $x(t) \in X$ and $u^*(t) \in U$ for all $t \geqslant 0$; (ii) $\lim_{t\to\infty}\lambda_p^*(t)c_p(t) = 0$, $\forall p \in \mathscr{P}$, where $c_p(t) = c_p^*(0|t)$; (iii) $\hat{x}(t) \to \sum_{p=0}^{\nu}\lambda_p^*F_{\infty,p}$ as $t \to \infty$; (iv) $x(t) \to \sum_{p=0}^{\nu}\lambda_p^*F_{\infty,p} \oplus \mathbb{E}$, as $t \to \infty$.

Proof Since, by assumption, $\hat{x}(0)$ is feasible, $\hat{x}(t)$ is feasible for all $t \geqslant 0$ following Lemma 5. Due to the fact that $\tilde{z}_p(0|t) \in \mathbb{X}_t \ominus \mathbb{T}$, $\tilde{u}_p(0|t) \in \mathbb{U} \ominus K_p\mathbb{T}$ and $\hat{z}_p(t) \in \tilde{z}_p(0|t) \oplus \mathbb{T}$, we have $\hat{z}_p(t) \in \mathbb{X}_t$, $K_p\hat{z}_p(t) + c_p(0|t) \in \mathbb{U}$, $\forall t \geq 0$, $p \in \mathscr{P}$. Thus, $\hat{x}(t) \in \mathbb{X}_t$ and $u^*(t) \in \mathbb{U}$. Similar as the proof of property (i) in Theorem 3, the rest proof of property (i) is omitted. The proof of property (ii) is similar as the proof in Theorem 3, thus is omitted. It is easy to show that $\lim_{t\to\infty}\lambda_p^*(t) \to \lambda_p^*$ and $\lim_{t\to\infty}\hat{z}_p(t) \to F_{\infty,p}$ (see property (iii) of Theorem 3). Thus property (iii) holds. Property (iv) is omitted, see the proof of property (iv) in Theorem 3.

For the interpolated output feedback MPC problem (26), let $\tilde{y}_p(0|t) = \lambda_p(t)\tilde{z}_p(0|t)$, $\hat{y}_p(t) = \lambda_p(t)\hat{z}_p(t)$ and $e_p(k|t) = \lambda_p(t)c_p(k|t)$ for all $k = 0,\ldots,N_p-1$ and $p \in \mathscr{P}$. Constraints (26b)-(26i) can be expressed collectively as a matrix inequality $G\mathbf{q}(t) \leq V + W\hat{x}(t)$, where matrixes G, V, W can be easily obtained and $\mathbf{q}(t) = [\lambda(t), \mathbf{y}(t), \mathbf{e}(t)]^T$ with $\mathbf{y}(t) = [\tilde{y}_0^T(0|t), \cdots, \tilde{y}_\nu^T(0|t), \hat{y}_0^T(t), \cdots, \hat{y}_\nu^T(t)]$, $\mathbf{e}(t) = [\mathbf{e}_0(t), \cdots, \mathbf{e}_\nu(t)]$ and $\mathbf{e}_p(t) = [e_p^T(0|t), \cdots, e_p^T(N_p-1|t)]$. Hence, optimization problem (26) is a quadratic programming problem.

6. EXAMPLE

The example is taken from Mayne et al. (2006). The system is a double integrator:

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t), \qquad (32)$$

$$y(t) = [1\ 1]x(t) + v(t) \qquad (33)$$

with additive disturbances $(w,v) \in \mathbb{W} \times \mathbb{V}$ where $\mathbb{W} = \{w \in \mathbb{R}^2 : \| w \|_\infty \leqslant 0.1\}$ and $\mathbb{V} = \{v \in \mathbb{R} : \| v \| \leqslant 0.05\}$. The state and control constraints are $(x,u) \in \mathbb{X} \times \mathbb{U}$ where $\mathbb{X} = \{x \in \mathbb{R}^2 : x_1 \in [-50,\ 3],\ x_2 \in [-50,\ 3]\}$ and $\mathbb{U} = \{u \in \mathbb{R} : \| u \| \leqslant 3\}$ (x_i is the ith coordinate of a vector of x). $K = [-1\ \ -1]$ and $L = [1\ 1]^T$. The d-invariant sets $\mathbb{E}$ is obtained using results in Ong and Gilbert (2006). The horizon is $N = 13$. Figure 1 shows the responses of the proposed controller starting from the initial state $\hat{x}(0) = [-3, -8]$. The domain of attraction X_{13} is shown as dash line. The domain of attraction for the true system is $X_{13} \oplus \mathbb{E}$, shown as dash-dot line. From Figure 1, it is shown that the estimate state $\hat{x}(t)$ finally converges to the set F_∞.

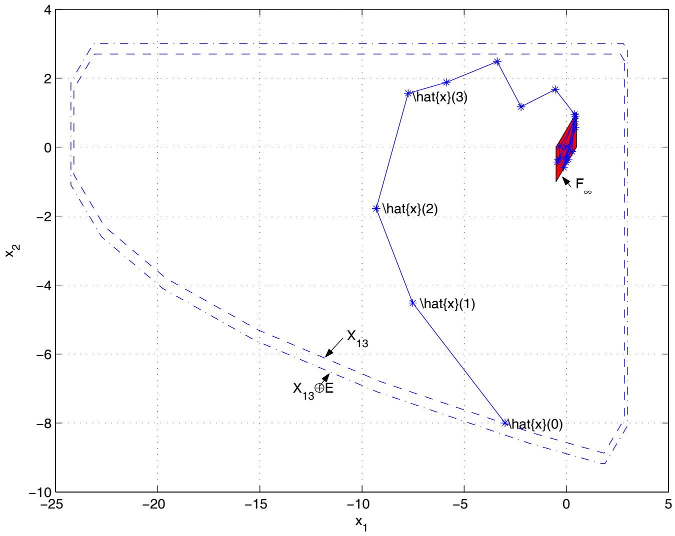

Fig. 1. Closed-loop responses of robust output-feedback MPC.

For the proposed interpolated output feedback MPC approach, it chooses 2 ($\nu = 1$) controllers with feedback gains $K_0 = K$ and $K_1 = [-0.2054\ \ -0.5781]$ respectively. Due to the use of the low gain control law $K_1 x$, for the same initial state $\hat{x}(0) = [-3, -8]$, N_1 just needs to be chosen as 7 such that MPC problem has a feasible solution. And N_0 can be chosen as 1. Therefore, for this example, with the use of relatively short horizons, low on-line computational complexity is expected.

7. CONCLUSION

The main contribution of this paper is to provide a simple approach to the problem of robust output feedback MPC for linear systems, subject to bounded state disturbances and output measurement errors, which employs a combination of a fixed linear observer with a tube-based robust MPC controller. The satisfaction of state and input constraints are guaranteed and the closed-loop stability is ensured. To reduce the computational complexity, the interpolation technique is used. As a result, the overall domain of attraction is the convex hull of all domains of attraction of the constituent controllers, leading to a large feasible set.

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