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Robust Explicit Time Optimal Controllers for Linear Systems via Decomposition Principle

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Abstract: One of the key problems in time optimal control (TOC) is the inherent computational complexity, which restricts its application to low dimensional systems. Considering a constrained linear system with bounded disturbances, this paper proposes a novel approach to reduce the computational complexity of TOC, where the terminal controller is nonlinear. It comprises several predetermined local linear feedback laws, resulting in a large terminal set. Starting from this relatively large terminal set, a large domain of attraction of the proposed TOC controller can be obtained by using a short horizon, and consequently leads to a low on-line computational effort. Furthermore, by formulating a suitable cost function, as time evolves, the TOC controller reaches the desired controller to obtain a good asymptotical behavior. The performance of the proposed approach is assessed via a numerical example.

1. INTRODUCTION

Consider the following constrained discrete-time linear system with bounded disturbances:

$$x(t+1) = Ax(t) + Bu(t) + Dw(t) \tag{1}$$
$$u(t) \in U, \qquad \forall t \geq 0, \tag{2}$$
$$x(t) \in X, \qquad \forall t \geq 0, \tag{3}$$
$$w(t) \in W, \qquad \forall t \geq 0, \tag{4}$$

where t is the discrete time index, $x(\cdot)$, $u(\cdot)$ and $w(\cdot)$ are the state, control and disturbance variables respectively, and $X \subset \mathbb{R}^{n_x}, U \subset \mathbb{R}^{n_u}, W \subset \mathbb{R}^{n_w}$ are the corresponding constraints and disturbance set. The disturbance $w(t)$ has no special structure other than being a random vector.

Recently, there emerged several interesting papers on this topic, namely, time optimal control (TOC), see *e.g.* Grieder et al. (2005); Kerrigan and Mayne (2002); Mayne and Schroeder (1997). In these papers, constrained linear systems as well as piecewise affine (PWA) systems are addressed. The typical design procedure starts with computing the set in which all states can be driven into a specific terminal (target) set X_f in one step. Using this set as the new target, the process is repeated, building up a family of sets that can be brought into the target X_f in at most N steps. Due to the input constraints, the size of these sets mainly depends on the size of the terminal set X_f and the number of steps. Generally, a larger X_f leads to a larger domain of attraction than a smaller X_f does with the same (even shorter horizon) N. Therefore, the on-line computational effort can be reduced by employing a large X_f (choosing a short value of N). However, such a large terminal set also implies that the corresponding terminal controller Kx is de-tuned and thus jeopardizes the local optimal performance, see discussions in Wan and Kothare (2003).

The disturbance plays another important role in TOC. As stated in Kolmanovsky and Gilbert (1998), in the presence of the disturbance $w(t)$ the closed-loop system (1)-(4) with some stabilizing feedback law $u = Kx$ does not converge to the origin but to some set, namely F_∞, the minimal disturbance invariant set. The size of F_∞ determines the asymptotic behavior of such a system and is usually desired to be as small as possible. Kolmanovsky and Gilbert (1998) also emphasizes that the size of F_∞ depends on the choice of the feedback gain K. In general, a small F_∞ often implies a small X_f. Therefore, a large N should be chosen to obtain a large domain of attraction but at the expense of a high computational complexity.

Up to now, we can see that a linear terminal controller Kx is a trade-off between two conflicting issues: large X_f and F_∞ (low computational effort but poor asymptotic behavior) versus small X_f and F_∞ (heavy computational effort but good asymptotic behavior). In view of this, recently a multi-mode controller, see Sui and Ong (2006), for system (1)-(4) was proposed under some connecting conditions to address this issue. However, these connecting conditions become complicated when the number of the constituting control laws is large. Also, an interpolation based MPC approach, *e.g.* Bacić et al. (2003); Pluymers et al. (2005); Rossiter et al. (2004) can be utilized. In these papers, the control strategy is interpolated among more than one MPC controllers. The terminal region is the convex hull of the constituent terminal sets, leading to a relatively large domain of attraction. However, the presence of $w(t)$ is not taken into account.

In this paper, a novel TOC approach is proposed to control constrained linear systems with bounded disturbances, where the terminal controller is nonlinear and comprises several predetermined local linear feedback laws via decomposing the state. The idea in Rossiter et al. (2005a,b) is extended here to address systems with disturbances. It aims at obtaining a

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low on-line computational effort and a good asymptotic behavior simultaneously. The resulting terminal set is enlarged by employing a nonlinear terminal controller. As time evolves, the TOC controller eventually reaches the desired controller to achieve a good asymptotic behavior.

In addition, the proposed TOC problem can be solved off-line by using multi-parametric programming. The on-line work is simplified to the identification of the region the current state belongs to and then the computation of the corresponding control law. Multi-parametric programming significantly decreases the cost of applying TOC to industrial systems. However, its complexity is the major limitation in practical applications, for example, the number of partitioned regions grows exponentially with the increase of N. Hence, reducing the partition complexity of explicit TOC is another key aspect. Thanks to the use of the shorter N, the proposed approach has a fewer number of partitions than the standard TOC approach does.

The paper is organized as follows. In Section 2, a standard TOC approach is reviewed. Section 3 discusses the framework of the proposed TOC controllers based on decomposition principle (DTOC) and states their properties. The effectiveness of the proposed controllers is illustrated via one example in Section 4. Conclusions are given in Section 5.

Notation and Basic Definitions: Positive definite (semi-definite) square matrix A is denoted by $A \succ (\succeq) 0$ and $A \succ (\succeq) B$ means $A - B \succ (\succeq) 0$. $\|x\|_\ell$ refers to the ℓ-norm of vector $x \in \mathbb{R}^n$ while the norm induced by $A \succ 0$ is $\|x\|_A^2 = x^T Ax$. Suppose $X, Y \subset \mathbb{R}^n$, the interior of X is denoted as $int(X)$, the P-difference of X and Y is $X \ominus Y = \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$ and the Minkowski sum is $X \oplus Y = \{z \in \mathbb{R}^n : z = x + y, x \in X, y \in Y\}$. Let $Y \subseteq X$, then $X \setminus Y = \{z : z \in X, z \notin Y\}$. Let $X_1, \cdots, X_k \subset \mathbb{R}^n$, their convex hull $Co(X_1, \cdots, X_k) = \{\lambda_1 X_1 \oplus \cdots \oplus \lambda_k X_k : 0 \le \lambda_i \le 1, \sum_{i=1}^k \lambda_i = 1\}$. $|I|$ is the cardinality of $I \subset N^+$ where N^+ is the set of non-negative integers.

It is assumed hereafter that system (1)-(4) satisfies the following assumptions: (A1) (A, B) is stabilizable; (A2) The sets W, X and U are non-empty convex polytopes that contain the origin in their respective interiors.

2. BACKGROUND

2.1 Standard TOC Formulation

The proposed controller uses, as its basis, a standard TOC scheme, see Mayne and Schroeder (1997). The work of Mayne and Schroeder (1997) is briefly reviewed below. Consider the closed-loop system of (1)-(4) under the given linear feedback control law $u = Kx$,

$$x(t+1) = \Phi x(t) + Dw(t), \quad \forall\, t \ge 0, \tag{5}$$

where $\Phi := A + BK$ is assumed to be asymptotically stable (spectral radius of Φ, $\rho(\Phi) < 1$). Let X_k, $k = 1, \ldots, N$, be the maximal set of states x that can be driven to X_{k-1} in one step, satisfying the state and input constraints in the presence of all allowable disturbance sequences. Each X_k is characterized as follows:

$$X_k := \{x : \exists u \in U,\ Ax + Bu + Dw \in X_{k-1},\ x \in X,\ w \in W\}, \tag{6}$$

with $X_0 := X_f$ being the maximal disturbance invariant set of system (5). For $x \in X_k \setminus X_{k-1}$, $u(x)$ is defined such that it can bring x into X_{k-1} in one step without violating any of the input constraints. One choice of $u(x)$ is based on the solution of the following optimization problem:

$$\min_{u(x)} J(u(x); x) = \|\Psi u(x)\|_\ell \tag{7}$$
$$\text{s.t. } u(x) \in U,$$
$$Ax + Bu(x) \in X_{k-1} \ominus DW,$$

for $\Psi \succ \mathbf{0}$. The TOC control law u^* at time t is

$$u^*(t) := \begin{cases} u(x(t)), & \text{if } x(t) \in X_k \setminus X_{k-1} \\ Kx(t), & \text{if } x(t) \in X_f \end{cases} . \tag{8}$$

Robust convergence of the TOC is shown in Mayne and Schroeder (1997). Due to the presence of $w(t)$ for all $t \ge 0$, $x(t)$ of (1)-(4) under (8) converges to the minimal disturbance invariant set, F_∞, of system (5), *i.e.* $F_\infty = \lim_{k \to \infty} F_k$ with $F_k := DW \oplus \Phi DW \oplus \cdots \oplus \Phi^{k-1} DW$.

Remark 1. Due to the convexity of X_f, problem (7) actually falls into a class of multi-parametric programs, see Bemporad et al. (2002); Tøndel et al. (2003), where the controller $u(x)$ can be solved off-line by multi-parametric programming. Hence, the on-line work is simplified. The characterization of X_k is a key factor which impacts on the application of TOC, whereas, becomes difficult and leads to a high computational complexity with the increase of k, especially when n_x is large. By using multi-parametric programming, the set X_k and explicit expression of $u(x)$ can be obtained simultaneously, which improves the computational efficiency of TOC.

2.2 Discussion of Standard TOC

The set X_k depends on the size of the terminal set X_f and the value of k. A large X_f implies a large X_k for a fixed k. It is reasonable (although not necessary) to expect that a large X_f is achieved by using a low gain K, see Tan and Gilbert (1992). However, such a Kx is generally de-tuned and thus jeopardizes the local optimal performance, see the detailed discussions in Wan and Kothare (2003). On the other hand, F_∞ determines the asymptotic behavior of the closed-loop system and should be small in size, an effect achieved by having a high gain K (although not necessarily). Hence, the choice of K is generally a trade-off between two conflicting requirements: a small F_∞ and a large X_f. One common approach is to choose a high gain K for a small F_∞ (and a small X_f) and increase k, to have a large X_k. This, of course, requires higher computational costs. In summary, with a linear terminal controller Kx, a good asymptotic behavior, a low computational cost and a large domain of attraction, are often not met satisfactorily.

3. TIME OPTIMAL CONTROL VIA DECOMPOSITION

3.1 Choice of the Terminal Controller

Instead of employing a linear terminal control law Kx, a good choice of the terminal controller follows the paper (Rossiter et al. (2005b)), which comprises several predetermined linear terminal controllers via decomposing the state x, *i.e.*

$$u = \sum_{p=0}^{\nu} K_p \hat{x}_p \tag{9}$$

where $x = \sum_{p=0}^{\nu} \hat{x}_p$. K_p are given such that $\rho(\Phi_p) < 1$ where $\Phi_p := A + BK_p, \forall p \in \mathscr{P} := \{0, 1, \ldots, \nu\}$. For the convenience, a subscript is added to various quantities (X_f, u, F_∞, etc.) to denote a particular p among the $\nu+1$ systems.

The closed-loop system under controller (9) becomes $x(t+1) = \sum_{p=0}^{\nu} \Phi_p \hat{x}_p(t) + Dw(t)$. Since $\hat{x}_0(t) = x(t) - \sum_{p=1}^{\nu} \hat{x}_p(t)$, we have

$$x(t+1) = \Phi_0 x(t) + \sum_{p=1}^{\nu} (\Phi_p - \Phi_0)\hat{x}_p(t) + Dw(t). \quad (10)$$

Consider the following auxiliary systems:

$$\hat{x}_p(t+1) = \Phi_p \hat{x}_p(t), \;\; p = 1, \ldots, \nu. \quad (11)$$

By combining the dynamics (10) and the auxiliary systems (11), the following augmented system is obtained:

$$\begin{bmatrix} x(t+1) \\ \hat{x}_1(t+1) \\ \vdots \\ \hat{x}_\nu(t+1) \end{bmatrix} = \begin{bmatrix} \Phi_0 & \Phi_1 - \Phi_0 & \cdots & \Phi_\nu - \Phi_0 \\ \mathbf{0} & \Phi_1 & \cdots & \mathbf{0} \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & \Phi_\nu \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}_1(t) \\ \vdots \\ \hat{x}_\nu(t) \end{bmatrix} + \begin{bmatrix} D \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} w(t) \quad (12)$$

with

$$x(t) \in X, \;\; K_0 x(t) + \sum_{p=1}^{\nu} (K_p - K_0)\hat{x}_p(t) \in U, \;\; w(t) \in W. \quad (13)$$

Define system (12) as $x^E(t+1) = \Phi^E x^E(t) + D^E w(t)$, where $x^E = [x^T, \hat{x}_1^T, \cdots, \hat{x}_\nu^T]^T$. Since $\rho(\Phi_p) < 1, \; \forall p \in \mathscr{P}$, we have $\rho(\Phi^E) < 1$. Let O_∞^E be the maximal disturbance invariant set of system (12)-(13) and $\hat{X}_f$ be the projection of O_∞^E onto x space, *i.e.*

$$\hat{X}_f = \{x : \exists [\hat{x}_1^T, \cdots, \hat{x}_\nu^T]^T \text{ such that } [x^T, \, \hat{x}_1^T, \cdots, \, \hat{x}_\nu^T]^T \in O_\infty^E\}. \quad (14)$$

Proposition 1. Suppose K_p is given such that Φ_p is asymptotically stable for all $p \in \mathscr{P}$ and Assumptions (A1)-(A2) hold. Then $\hat{X}_f$ is a constraint admissible, disturbance invariant set of system (1)-(4) under a feedback control law (9).

Proof Since O_∞^E is the maximal disturbance invariant set of system (12)-(13), the state inside $\hat{X}_f$ satisfies the constraints (13). Hence, $\hat{X}_f$ is constraint admissible. When $x(t) \in \hat{X}_f$, from (14), there must exist a sequence $[\hat{x}_1^T(t), \; \cdots, \; \hat{x}_\nu^T(t)]^T$ such that $[x^T(t), \, \hat{x}_1^T(t), \; \cdots, \; \hat{x}_\nu^T(t)]^T \in O_\infty^E$. Since O_∞^E is disturbance invariant of system (12), $[x^T(t+1), \; \hat{x}_1^T(t+1), \; \cdots, \; \hat{x}_\nu^T(t+1)]^T \in O_\infty^E$ where $x(t+1) = Ax(t) + B\sum_{p=0}^{\nu} K_p \hat{x}_p(t) + Dw(t)$ with $x(t) = \sum_{p=0}^{\nu} \hat{x}_p(t)$. Therefore, $x(t+1) \in \hat{X}_f$. The set $\hat{X}_f$ is a constraint admissible, disturbance invariant set of system (1)-(4) under a feedback control law $u = \sum_{p=0}^{\nu} K_p \hat{x}_p$.

Remark 2. The set $\hat{X}_f$ always exists if $X_{f,p}$ exists. For example, let $\hat{x}_1 = \cdots = \hat{x}_\nu = \mathbf{0}$, then $x = \hat{x}_0$. Hence, in that case, $\hat{X}_f$ is the same as $X_{f,0}$.

From Proposition 1, the terminal set corresponding to the terminal controller (9) is chosen as $\hat{X}_f$. For the nominal case ($W = \emptyset$), it is easy to see that $X_{f,p} \subseteq \hat{X}_f$ for all $p \in \mathscr{P}$. Therefore, $\bar{X}_f := Co(X_{f,0}, \cdots, X_{f,\nu}) \subseteq \hat{X}_f$. When $W \neq \emptyset$, it cannot be guaranteed that $\bar{X}_f \subseteq \hat{X}_f$. Due to the fact that $\hat{X}_f$ is the projection of the maximal disturbance set O_∞^E of system (12), it is correct to say that $\hat{X}_f$ is relatively large. Figure 1 and Figure 2 show the sets $\bar{X}_f$ and $\hat{X}_f$ when $W = \emptyset$ and $W \neq \emptyset$ relatively, where $p \in \mathscr{P} = \{0, 1\}$ of the system described in Section 4. Hence, by choosing a nonlinear terminal controller (9), the terminal set is enlarged.

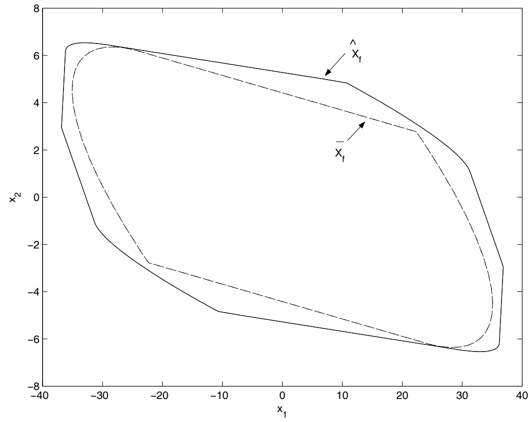

Fig. 1. $\bar{X}_f$ and $\hat{X}_f$ ($W = \emptyset$).

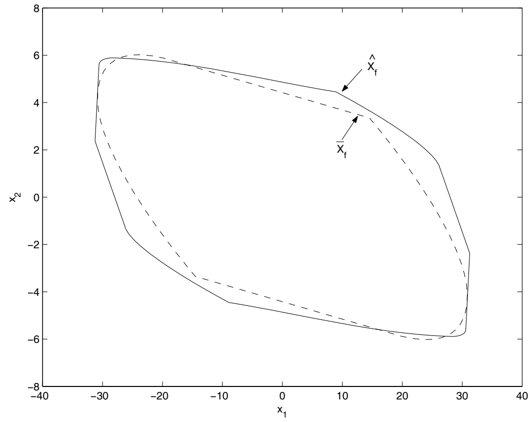

Fig. 2. $\bar{X}_f$ and $\hat{X}_f$ ($W \neq \emptyset$).

3.2 Choices of K_p

In the proposed approach, K_p with $p \in \mathscr{P}$ are chosen and ordered as follows.

(1) For all $p \in \mathscr{P}$, it is required that $\rho(\Phi_p) < 1$ to ensure the stability.
(2) For all $p \in \mathscr{P}$, $X_{f,p}$ always exists.
(3) For K_p with p in between 0 and ν, there are no other specific requirements than stability. They can be determined problem dependently. For the stability, we require that

$$\Phi_p^T P_p \Phi_p - P_p \preceq -Q_p - K_p^T R_p K_p, \;\; p = 1, \ldots, \nu, \quad (15)$$

where $P_p, Q_p, R_p \succ \mathbf{0}$.
(4) K_0 is the desired controller that leads to the best performance and the relatively small F_∞ set. As will be proven later, the DTOC controller eventually reaches $K_0 x$ as time evolves.
(5) K_ν should be chosen carefully. Since it is used in this paper to produce a large terminal set, with which a short value N can be used to reduce the computational complexity.

Several design methodologies can be used to compute K_p, including the popular LQ, H_2 and H_∞ design. For example, similar as the work of Alvarez-Ramirez and Suarez (1996), it is logical to determine the gain of K_p, that solves the standard LQ optimization problem for a range of control matrix R from the desired value R_0 to a very high value R_ν. It is reasonable (although not necessary) to expect that often $R_p < R_{p+1} \Rightarrow$

$X_{f,p} \subset X_{f,p+1}$. Given that the desired choice for R is R_0, then all other choices lead to larger terminal sets and hence extend the feasibility of control (more discussions in Rossiter et al. (2001)). Other methodologies to compute K_p can be referred to Sui and Ong (2007).

3.3 DTOC Controllers

When the state is outside of the terminal set, the computation of the DTOC controllers is standard, for example, referred to problem (7). In the sequel, the computation of the controllers when $x \in \hat{X}_f$ is provided.

At time t, given $x(t) \in \hat{X}_f$, the corresponding DTOC is based on the solution of the following optimization problem over $\hat{\mathbf{x}}(t) = [\hat{x}_0^T(t), \cdots, \hat{x}_\nu^T(t)]^T$:

$$\min_{\hat{\mathbf{x}}(t)} J(\hat{\mathbf{x}}(t)) = \sum_{p=1}^{\nu} ||\hat{x}_p(t)||_{P_p}^2 \text{ s.t. } x(t) = \sum_{p=0}^{\nu} \hat{x}_p(t), x^E(t) \in O_\infty^E, \quad (16)$$

where matrix P_p is defined by (15). Due to the convexity of O_∞^E, problem (16) is actually a quadratic programming (QP) problem. The optimal solution of (16) is denoted by $\hat{\mathbf{x}}^*(t)$. Hence, the corresponding controller is

$$u^*(t) := \begin{cases} u(x(t)), & \text{if } x(t) \in X_k \setminus X_{k-1} \\ \sum_{p=0}^{\nu} K_p \hat{x}_p^*(t) & \text{if } x(t) \in \hat{X}_f \end{cases} . \quad (17)$$

Theorem 2. Suppose K_p is given such that Φ_p is asymptotically stable for all $p \in \mathscr{P}$ and that Assumptions (A1)-(A2) hold. System (1)-(4) with $u^*(t)$ given by (17) has the following properties for any $x(0) \in X_k \subseteq X_N$: (i) the state $x(0)$ enters into $\hat{X}_f$ in no more than k steps and stays in it thereafter; (ii) $x(t) \in X$ and $u^*(t) \in U$ for all $t \geq 0$; (iii) $x(t) \to F_{\infty,0}$, as $t \to \infty$.

Proof Properties (i)-(ii) directly follow from Theorem 5 of Mayne and Schroeder (1997) and Proposition 1. (iii) At time $t+1$, the state $x(t+1)$ of system (1) with $u^*(t)$ is $x(t+1) = \hat{x}_0^f(t+1) + \sum_{p=1}^{\nu} \hat{x}_p^f(t+1)$, where $\hat{x}_0^f(t+1) = \Phi_0 \hat{x}_0^*(t) + Dw(t)$ and $\hat{x}_p^f(t+1) = \Phi_p \hat{x}_p^*(t)$, $p \neq 0$. Hence, it is easy to see that $\hat{\mathbf{x}}^f(t+1) = [(\hat{x}_0^f(t+1))^T, \cdots, (\hat{x}_\nu^f(t+1))^T]^T$ is a feasible solution of (16) at time $t+1$. Suppose the optimal cost is defined by $J^*(t) = \sum_{p=1}^{\nu} ||\hat{x}_p^*(t)||_{P_p}^2$. At time $t+1$, the feasible cost is $J^f(t+1) = \sum_{p=1}^{\nu} ||\hat{x}_p^f(t+1)||_{P_p}^2$. Due to the fact that $\Phi_p^T P_p \Phi_p - P_p \preceq -Q_p - K_p^T R_p K_p$ with $P_p, Q_p, R_p \succ \mathbf{0}$, see (15), we have $||\hat{x}_p^f(t+1)||_{P_p}^2 - ||\hat{x}_p^*(t)||_{P_p}^2 \leq -||\hat{x}_p^*(t)||_{Q_p + K_p^T R_p K_p}^2$. Hence,

$$J^*(t+1) - J^*(t) \leq \sum_{p=1}^{\nu} -||\hat{x}_p^*(t)||_{Q_p + K_p^T R_p K_p}^2. \quad (18)$$

It is easy to see that $\{J^*(t)\}$ is non-increasing and bounded by 0. As $t \to \infty$, it converges to $J^*(\infty) < +\infty$. Summing (18), we have $+\infty > J^*(t) - J^*(\infty) \geq \sum_{t=0}^{\infty} \sum_{p=1}^{\nu} ||\hat{x}_p^*(t)||_{Q_p + K_p^T R_p K_p}^2 \geq 0$ $\Rightarrow \lim_{t \to \infty} \hat{x}_p^*(t) = 0$ for all $p \neq 0$. It implies that $x(t) \to \hat{x}_0(t)$ as $t \to \infty$. Therefore, property (iii) is proven.

3.4 Multi-parametric Programming in DTOC

As stated in Remark 1, problem (7) falls into a class of multi-parametric programmes. Using the algorithm described in Kvasnica et al. (2005), one can compute the explicit solution of problem (7) off-line for all $x \in X_k \setminus X_{k-1}$, *i.e.*

$$u(x(t)) = L_i^k x(t) + g_i^k, \text{ if } x(t) \in Z_i^k, \forall i \in \mathscr{I}^k, \quad (19)$$

where $L_i^k \in \mathbb{R}^{n_u \times n_x}$, $g_i^k \in \mathbb{R}^{n_u}$ and Z_i^k is a convex polytope in $\mathbb{R}^{n_x}$ that forms a partition of $X_k \setminus X_{k-1}$ in the sense that $X_k \setminus X_{k-1} = \cup_{i \in \mathscr{I}^k} Z_i^k$ and $int(Z_i^k) \cap int(Z_j^k) = \emptyset$ for all $i \neq j, i, j \in \mathscr{I}^k$.

Remark 3. Clearly, the availability of (19) means that the $(L_i^k, g_i^k), i \in \mathscr{I}^k$ can be computed off-line leaving the on-line computational effort to the identification of Z_i^k when $x \in X_k \setminus X_{k-1}$ and the evaluation of u for $x(t)$. This implies that the on-line computational effort is further relaxed and is proportional to $\sum_{k=1}^{N} |\mathscr{I}^k|$, the number of partitions, which depends on the length of N. The larger the value of N is, the higher the computational effort will be needed. Fortunately, due to the enlargement of the terminal set, the length of N can be chosen short so that the partition complexity can be reduced.

Similarly, optimization problem (16) can be solved off-line using multi-parametric programming too, although solving it on-line is time-cheap.

4. NUMERICAL EXAMPLE

The example is taken from Chisci et al. (2001). The system is given by

$$x(t+1) = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t) \quad (20)$$

with $U = \{u \in \mathbb{R} : \| u \|_\infty \leq 2\}, W = \{w \in \mathbb{R}^2 : \| w \|_\infty \leq 0.09\}$. With $l = 1$ and $\Psi = I$, the proposed approach chooses 2 ($\nu = 1$) controllers with feedback gains $K_0 = [-0.7925, -1.1081]$ and $K_1 = [-0.0333, -0.4527]$ respectively. Let $N = 1$. Figure 3 shows the responses of the DTOC controller starting from initial state $x(0) = [-26, 5.5]$.

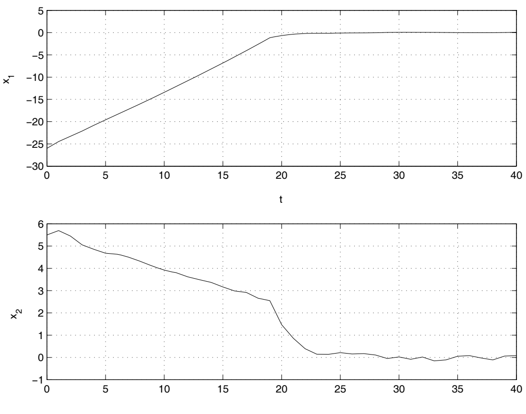

Fig. 3. States history of DTOC Controllers.

Two standard TOC controllers (see Mayne and Schroeder (1997)) described in Section 2 are used for the comparison, denoted respectively by superscripts A and B. Controller A uses the feedback gain $K^A = K_0$ while Controller B uses $K^B = K_1$.

The choice of horizon N^A is 15 such that the ratio of Area(X_1^{DTOC})/Area (X_{15}^A) $\approx$ 1.1181. As $K^A = K_0$, $F_\infty^A = F_{\infty,0}$. However, since the large value of N^A is chosen, the on-line computational effort for Controller A is higher. The number of partitions is $\sum_{k=1}^{15} |\mathscr{I}^k| = 506$ for Controller A, while $\sum_{k=1}^{1} |\mathscr{I}^k| = 24$ for DTOC Controller.

For Controller B, $N^B = 1$. The ratio of Area(X_1^{DTOC})/Area $(X_1^B) \approx 1.076$. However, the set F_∞^B is much larger than $F_{\infty,0}$ with a ratio of 821. For most linear systems with additive disturbances, the asymptotic behavior is likely to be an important consideration. Therefore, the asymptotic performance of Controller B is expected to be much worse than the proposed DTOC controllers.

5. CONCLUSION

This paper describes a DTOC approach for constrained linear systems with bounded disturbances. The DTOC terminal controller comprises several terminal controllers. It has the advantage of combining the merits of the underlying standard TOC controllers resulting in a system with a large domain of attraction and a good asymptotic performance while avoiding the associated problem of having a high on-line computational effort.

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