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On the Geometric Structure of the $\mathcal{H}_\infty$ Central Controller

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Abstract: This paper shows that the controllable and unobservable subspaces of the $\mathcal{H}_\infty$ central controller for a linear continuous-time system can be characterized by the image and kernel spaces of two matrices $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$, where $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$ are positive semidefinite solutions of two pertinent Lyapunov equations whose coefficients involve $\boldsymbol{X}_\infty$ and $\boldsymbol{Z}_\infty$, the stabilizing solutions of two celebrated algebraic Riccati equations used in solving the $\mathcal{H}_\infty$ control problem. Furthermore, under this characterization, it is shown that the unobservable subspace of the central controller contains the intersection of $\mathrm{Ker}\boldsymbol{X}_\infty$ and the unobservable subspace of the plant. In addition, it is also shown that the central controller's controllable subspace is a subspace of the sum of $\mathrm{Im}\boldsymbol{Z}_\infty$ and the plant's controllable subspace. A numerical example is also given for illustration. In terms of geometric language, all the results and proofs given are clear and simple.

1. INTRODUCTION

The very first step in control system analysis and design is to construct a mathematical model of the plant to be controlled. To make the control problem mathematically tractable, the model of the true plant is usually oversimplified, thus incurring inaccuracy in the process of modelling. Much of modern control theory addresses problems involving uncertainty. Although the celebrated $\mathcal{H}_2$ optimal control theory or its stochastic counterpart *linear quadratic Gaussian* (LQG) optimal control theory provides a powerful tool for optimizing performance [Anderson and Moore, 1971, Kawkernaak and Sivan, 1972] there is no guaranteed robustness for LQG-controllers [Doyle, 1978]. In contrast, $\mathcal{H}_\infty$ (sub)optimal control theory is intended for explicitly taking robustness issue into account; this is because the $\mathcal{H}_\infty$-norm specifies a level of disturbance attenuation[Francis, 1987] from the fact that the $\mathcal{H}_\infty$-norm is the induced norm from $\mathcal{RH}_2$ to $\mathcal{RH}_2$, and implies a prespecified level of stability robustness provided by the small gain theorem [Zames, 1966a,b].

One of the most important breakthroughs in $\mathcal{H}_\infty$ control theory was the derivation of state-space solutions, in terms of the solutions to two algebraic Riccati equations (AREs), to the standard linear $\mathcal{H}_\infty$ output feedback control problem [Doyle et al., 1989]. A parametrization of all $\mathcal{H}_\infty$ (sub)optimal output feedback controllers was also given in Doyle et al. [1989]. The full-order controllers thus obtained in Doyle et al. [1989] have a state dimension not less than that of the generalized plant.

Geometric control theory arose in the late 1960's. A central role in this theory was played by the geometric properties of the coefficient matrices appearing in system equations. In particular, the notions of controllable and observable subspaces have played an important role. These notions also turned out to be essential in understanding and classifying the fine structure of the system under consideration. Many problems were studied in a geometric framework [Silverman, 1976, Willems, 1981, Wonham and Morse, 1970]. It was proved that the kernel of any symmetric solution of an $\mathcal{H}_2$ ARE is an $\boldsymbol{A}$-invariant subspace contained in $\mathrm{Ker}\boldsymbol{C}$, hence it is contained in the unobservable subspace of $(\boldsymbol{C}, \boldsymbol{A})$, see, for example, Saberi et al [1995]. The connection between the solution set of the ARE and the set of n-dimensional invariant subspaces of the corresponding Hamiltonian matrix was also investigated in Saberi et al [1995]. In Weiland and Willems [1989], the authors solved, in terms of the geometric concepts of linear system theory, the almost disturbance decoupling problem with internal stability. The solution gave necessary and sufficient conditions for the existence of a dynamic output feedback controller such that in the closed-loop system the disturbances were quenched, say in the $\mathcal{H}_\infty$-sense, up to any degree of accuracy while maintaining a stable system matrix.

Most recently, Marro et. al. [2002] have given a solution to the cheap and singular linear quadratic (LQ) problem. Their approach does not require the solution of any ARE, or linear matrix inequality(LMI), but rather it use the basic tools of the geometric theory. Marro and Zattoni also introduced a new characterization of the invariant subspaces of the Hamiltonian systems, aimed to derive a non-recursive solution to the finite-horizon LQ control problem for stabilizable continuous-time systems [Marro and Zattoni, 2005, Zattoni, 2004].

In Yung [2000], the author constructed a reduced-order $\mathcal{H}_\infty$ controller via a basis of the image space of $\boldsymbol{W}_\infty$, where $\boldsymbol{W}_\infty$ is the stabilizing solution to an ARE in $\boldsymbol{W}$ developed in Petersen et. al. [1991]. In fact, as we will show later, this reduced-order controller is exactly the observable

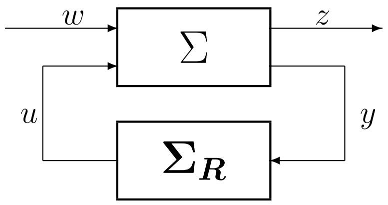

Fig. 1. Standard Block Diagram

realization of the $\mathcal{H}_\infty$ central controller. Motivated by this result, we will show in this paper that the controllable and unobservable subspaces of the $\mathcal{H}_\infty$ central controller can be characterized in terms of the image and kernel spaces of two matrices $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$, where $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$ are positive semidefinite solutions of two certain Lyapunov equations. The coefficients of the two Lyapunov equations involve $\boldsymbol{X}_\infty$ and $\boldsymbol{Z}_\infty$, which are the stabilizing solutions of two celebrated AREs used to solve the $\mathcal{H}_\infty$ control problem. Furthermore, under this characterization, it is shown that the unobservable subspace of the $\mathcal{H}_\infty$ central controller contains the intersection of Ker$\boldsymbol{X}_\infty$ and the unobservable subspace of the plant, and the central controller's controllable subspace is a subspace of the sum of Im$\boldsymbol{Z}_L$ and the plant's controllable subspace. A numerical example is also given for illustration.

2. PRELIMINARIES

2.1 Geometric Theory

Consider a linear time-invariant system described by

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \end{cases} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^l$ is the control input, and $\boldsymbol{y} \in \mathbb{R}^q$ is the output. $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are constant matrices with compatible dimensions. Let $\mathcal{B} \triangleq \mathrm{Im}\boldsymbol{B}$ and $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) \triangleq \mathcal{B} + \boldsymbol{A}\mathcal{B} + \cdots + \boldsymbol{A}^{n-1}\mathcal{B}$. The subspace $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$ is called the controllable subspace of the pair $(\boldsymbol{A}, \boldsymbol{B})$, which contains all the states reachable from $\boldsymbol{x}(0)$ and is the minimal $\boldsymbol{A}$-invariant subspace containing $\mathcal{B}$. Dually, the unobservable subspace of the pair $(\boldsymbol{C}, \boldsymbol{A})$ is $\mathcal{N}(\boldsymbol{C}, \boldsymbol{A}) \triangleq \bigcap_{i=1}^{n} \mathrm{Ker}(\boldsymbol{C}\boldsymbol{A}^{i-1})$, which is the maximal $\boldsymbol{A}$-invariant subspace contained in Ker$\boldsymbol{C}$. System (1) is controllable if and only if $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) = \mathbb{R}^n$, and is observable if and only if $\mathcal{N}(\boldsymbol{C}, \boldsymbol{A}) = \{0\}$. From the definition, we have $(\mathcal{N}(\boldsymbol{C}, \boldsymbol{A}))^\perp = \mathcal{C}(\boldsymbol{A}^T, \boldsymbol{C}^T)$. $\sigma(\boldsymbol{A}) \triangleq \{\lambda_k | k = 1, 2, \cdots \rho\}$ stands for the spectrum of $\boldsymbol{A}$ with corresponding algebraic eigenspaces $\boldsymbol{N}_k$. See Callier and Desoer [1991], for example, for more details.

2.2 $\mathcal{H}_\infty$ Optimal Control Problem

We now turn our attention to the $\mathcal{H}_\infty$ control problem. Consider the standard feedback configuration shown in Figure 1. Let the plant $\boldsymbol{\Sigma}$ be described by the dynamic equations:

$$\boldsymbol{\Sigma} : \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{w} + \boldsymbol{B}_2\boldsymbol{u}, \\ \boldsymbol{z} = \boldsymbol{C}_1\boldsymbol{x} + \boldsymbol{D}_{12}\boldsymbol{u}, \\ \boldsymbol{y} = \boldsymbol{C}_2\boldsymbol{x} + \boldsymbol{D}_{21}\boldsymbol{w}, \end{cases} \tag{2}$$

where for each t, $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^l$ is the control input, $\boldsymbol{w} \in \mathbb{R}^m$ represents a set of exogenous inputs which includes disturbances to be rejected and/or reference commands to be tracked, $\boldsymbol{z} \in \mathbb{R}^p$ is the unknown output to be controlled, and $\boldsymbol{y} \in \mathbb{R}^q$ is the measured output. $\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{D}_{12}$, and $\boldsymbol{D}_{21}$ are constant matrices with compatible dimensions. The goal of $\mathcal{H}_\infty$ control problem is finding a proper controller $\boldsymbol{\Sigma_R}$ such that the resulting closed-loop system is internally stable and the $\mathcal{H}_\infty$ norm of $\boldsymbol{T_{zw}}$ is less than γ, where $\boldsymbol{T_{zw}}$ represents the closed transfer matrix from $\boldsymbol{w}$ to $\boldsymbol{z}$.

The following proposition follows immediately from the result of Doyle et al. [1989]. See also Zhou and Dolye [1997].

Proposition 1. Consider system (2) and assume the following hypotheses hold:

(H1) $\{\boldsymbol{A}, \boldsymbol{B}_2\}$ is stabilizable, $\boldsymbol{E}_1 \triangleq \boldsymbol{D}_{12}^T\boldsymbol{D}_{12}$ is nonsingular, and $\begin{bmatrix} \boldsymbol{A} - \mathrm{j}\omega\boldsymbol{I} & \boldsymbol{B}_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

(H2) $\{\boldsymbol{A}, \boldsymbol{C}_2\}$ is detectable, $\boldsymbol{E}_2 \triangleq \boldsymbol{D}_{21}\boldsymbol{D}_{21}^T$ is nonsingular, and $\begin{bmatrix} \boldsymbol{A} - \mathrm{j}\omega\boldsymbol{I} & \boldsymbol{B}_1 \\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

Then the following statements are equivalent:
(1) There exists an internally stabilizing controller such that $\| \boldsymbol{T_{zw}} \| < \gamma$.
(2)(a) the ARE

$$\begin{aligned} &(\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1)^T\boldsymbol{X} + \boldsymbol{X}(\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1) \\ &+\boldsymbol{X}(\frac{1}{\gamma^2}\boldsymbol{B}_1\boldsymbol{B}_1^T - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{B}_2^T)\boldsymbol{X} + \boldsymbol{C}_1^T(\boldsymbol{I} - \boldsymbol{D}_{12}\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T)\boldsymbol{C}_1 \\ &= 0 \end{aligned} \tag{3}$$

has a stabilizing solution $\boldsymbol{X}_\infty \geq 0$,
(b) the ARE

$$\begin{aligned} &(\tilde{\boldsymbol{A}} - \boldsymbol{B}_1\boldsymbol{D}_{21}^T\boldsymbol{E}_2^{-1}\tilde{\boldsymbol{C}}_2)\boldsymbol{Z} + \boldsymbol{Z}(\tilde{\boldsymbol{A}} - \boldsymbol{B}_1\boldsymbol{D}_{21}^T\boldsymbol{E}_2^{-1}\tilde{\boldsymbol{C}}_2)^T \\ &+\boldsymbol{Z}(\frac{1}{\gamma^2}\boldsymbol{F}_\infty^T\boldsymbol{E}_1\boldsymbol{F}_\infty - \tilde{\boldsymbol{C}}_2^T\boldsymbol{E}_2^{-1}\tilde{\boldsymbol{C}}_2)\boldsymbol{Z} + \tilde{\boldsymbol{B}}_1\tilde{\boldsymbol{B}}_1^T \\ &= 0 \end{aligned} \tag{4}$$

has a stabilizing solution $\boldsymbol{Z}_\infty \geq 0$, where

$$\begin{aligned} \tilde{\boldsymbol{A}} &\triangleq \boldsymbol{A} + \frac{1}{\gamma^2}\boldsymbol{B}_1\boldsymbol{B}_1^T\boldsymbol{X}_\infty, \\ \tilde{\boldsymbol{B}}_1 &\triangleq \boldsymbol{B}_1(\boldsymbol{I} - \boldsymbol{D}_{21}^T\boldsymbol{E}_2^{-1}\boldsymbol{D}_{21}), \\ \tilde{\boldsymbol{C}}_2 &\triangleq \boldsymbol{C}_2 + \frac{1}{\gamma^2}\boldsymbol{D}_{21}\boldsymbol{B}_1^T\boldsymbol{X}_\infty, \end{aligned}$$

and

$$\boldsymbol{F}_\infty \triangleq -\boldsymbol{E}_1^{-1}(\boldsymbol{B}_2^T\boldsymbol{X}_\infty + \boldsymbol{D}_{12}^T\boldsymbol{C}_1).$$

Moreover, when these conditions are satisfied, one such controller (namely the central controller) is given by

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \hat{\boldsymbol{A}}_0\hat{\boldsymbol{x}} + \hat{\boldsymbol{B}}_0\boldsymbol{y}, \\ \boldsymbol{u} &= \hat{\boldsymbol{C}}_0\hat{\boldsymbol{x}}, \end{aligned} \tag{5}$$

where

$$\hat{\boldsymbol{A}}_0 = \tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2,$$
$$\hat{\boldsymbol{B}}_0 = (\boldsymbol{Z}_\infty\tilde{\boldsymbol{C}}_2^T + \boldsymbol{B}_1\boldsymbol{D}_{21}^T)\boldsymbol{E}_2^{-1},$$
$$\hat{\boldsymbol{C}}_0 = \boldsymbol{F}_\infty.$$

3. MAIN RESULTS

3.1 Characterization for the Controllable and Unobservable Subspaces of the $\mathcal{H}_\infty$ Central Controller

Suppose that the conditions (2a) and (2b) of Proposition 1 are satisfied. Then it follows from Petersen et. al. [1991] that the ARE

$$\boldsymbol{A}_0^T\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_0 + \frac{1}{\gamma^2}\boldsymbol{W}\boldsymbol{B}_0\boldsymbol{B}_0^T\boldsymbol{W} + \boldsymbol{C}_0^T\boldsymbol{C}_0 = 0 \qquad (6)$$

has also a stabilizing solution $\boldsymbol{W}_\infty \geq 0$, where $\boldsymbol{A}_0 \triangleq \hat{\boldsymbol{A}}_0 - \boldsymbol{B}_2\boldsymbol{F}_\infty = \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2$, $\boldsymbol{B}_0 \triangleq \boldsymbol{B}_1 - \hat{\boldsymbol{B}}_0\boldsymbol{D}_{21}$ and $\boldsymbol{C}_0 \triangleq \boldsymbol{E}_1^{\frac{1}{2}}\boldsymbol{F}_\infty$. Also, the matrix $\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2$ has been proved to be stable by bounded real lemma in Petersen et. al. [1991]. The matrix $\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 = (\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1) + (\frac{1}{\gamma^2}\boldsymbol{B}_1\boldsymbol{B}_1^T - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{B}_2^T)\boldsymbol{X}_\infty$ is stable because $\boldsymbol{X}_\infty$ is a stabilizing solution to the ARE (3). Since $\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0$ and $\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2$ are stable, it follows immediately from Lyapunov stability theory that the following two Lyaounov equations

$$Lya(\boldsymbol{W}) = (\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^T\boldsymbol{W} + \boldsymbol{W}(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2) + \hat{\boldsymbol{C}}_0^T\hat{\boldsymbol{C}}_0 = 0, \qquad (7)$$
$$Lya(\boldsymbol{Z}) = (\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0)\boldsymbol{Z} + \boldsymbol{Z}(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0)^T + \hat{\boldsymbol{B}}_0\hat{\boldsymbol{B}}_0^T = 0, \qquad (8)$$

have unique positive semidefinite solutions, $\boldsymbol{W}_L$ and $\boldsymbol{Z}_L$, respectively. Then we have the following.

Theorem 2. Ker$\boldsymbol{W}_L$ is the maximal $\hat{\boldsymbol{A}}_0-$invariant subspace contained in Ker$\hat{\boldsymbol{C}}_0$; that is, Ker$\boldsymbol{W}_L = \mathcal{N}(\hat{\boldsymbol{C}}_0, \hat{\boldsymbol{A}}_0)$.

Proof. It is well-known that Ker$\boldsymbol{W}_L \subset$ Ker$\hat{\boldsymbol{C}}_0$ and $\boldsymbol{W}_L(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} = 0$ for any $\boldsymbol{v} \in$ Ker$\boldsymbol{W}_L$. Thus, $\boldsymbol{W}_L\hat{\boldsymbol{A}}_0\boldsymbol{v} = \boldsymbol{W}_L(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} = 0$. This proves that Ker$\boldsymbol{W}_L$ is an $\hat{\boldsymbol{A}}_0-$invariant subspace contained in Ker$\hat{\boldsymbol{C}}_0$. Since $\mathcal{N}(\hat{\boldsymbol{C}}_0, \hat{\boldsymbol{A}}_0)$ is the maximal $\hat{\boldsymbol{A}}_0-$invariant subspace contained in Ker$\hat{\boldsymbol{C}}_0$, we get Ker$\boldsymbol{W}_L \subset \mathcal{N}(\hat{\boldsymbol{C}}_0, \hat{\boldsymbol{A}}_0)$. On the other hand, suppose $\boldsymbol{v} \in \mathcal{N}(\hat{\boldsymbol{C}}_0, \hat{\boldsymbol{A}}_0)$. Then, by definition, we have $\hat{\boldsymbol{C}}_0\hat{\boldsymbol{A}}_0^i\boldsymbol{v} = 0$ for all nonnegative integers i. This implies $\hat{\boldsymbol{C}}_0\boldsymbol{v} = 0$ and

$$\hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^i\boldsymbol{v} = 0 \qquad (9)$$

for all integers i. Then $\hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^i v = 0$ for all nonnegative integers i. This can be shown by induction as follows. When $i = 0$, $\hat{\boldsymbol{C}}_0\boldsymbol{v} = 0$. Suppose $\hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^j\boldsymbol{v} = 0$ for some positive integer j. Then

$$\begin{aligned}
&\hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^{j+1}\boldsymbol{v} \\
&= \hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^j(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} \\
&= \hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^j(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} \\
&= \hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^{j-1}(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2) \\
&\quad (\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} \\
&= \hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^{j-1}(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2) \\
&\quad (\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} \text{ (by equation (9))} \\
&= \hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^{j-1}(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^2\boldsymbol{v}
\end{aligned}$$

Repeating the arguments finally reaches the condition that $\hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^{j+1}\boldsymbol{v} = \hat{\boldsymbol{C}}_0(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^{j+1}\boldsymbol{v} = 0$. Thus, $\boldsymbol{v} \in \mathcal{N}(\hat{\boldsymbol{C}}_0, \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)$. This shows that $\mathcal{N}(\hat{\boldsymbol{C}}_0, \tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2) \subset \mathcal{N}(\hat{\boldsymbol{C}}_0, \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)$. It is well-known that $\mathcal{N}(\hat{\boldsymbol{C}}_0, \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2) \subset$ Ker$\boldsymbol{W}_L$. This completes the proof. Q.E.D.

Corollary 3. The reduced-order $\mathcal{H}_\infty$ controller given in Yung [2000] is the observable realization of the central controller (5).

Proof. It is well-known that Ker$\boldsymbol{W}_\infty$ and Ker$\boldsymbol{W}_L$ are both the maximal $(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)-$invariant subspace in Ker$\hat{\boldsymbol{C}}_0$, we have Ker$\boldsymbol{W}_\infty =$ Ker$\boldsymbol{W}_L$. So the reduced-order controller

$$\dot{\xi} = \boldsymbol{V}^T\hat{\boldsymbol{A}}_0\boldsymbol{V}\xi + \boldsymbol{V}^T\hat{\boldsymbol{B}}_0\boldsymbol{y},$$
$$\boldsymbol{u} = \hat{\boldsymbol{C}}_0\boldsymbol{V}\xi,$$

where $\boldsymbol{V}$ is the basis matrix of Ker$\boldsymbol{W}_\infty$, given in Yung [2000] is the observable realization of $\mathcal{H}_\infty$ central controller (5). This completes the proof. Q.E.D.

With a dual argument, we also have the following lemma.

Lemma 4. Ker$\boldsymbol{Z}_L$ is the maximal $\hat{\boldsymbol{A}}_0^T$-invariant subspace contained in Ker$\hat{\boldsymbol{B}}_0^T$.

The next theory is a dual result of Theorem (2).

Theorem 5. Im$\boldsymbol{Z}_L$ is the smallest $\hat{\boldsymbol{A}}_0$-invariant subspace containing Im$\hat{\boldsymbol{B}}_0$; that is, Im$\boldsymbol{Z}_L = \mathcal{C}(\hat{\boldsymbol{A}}_0, \hat{\boldsymbol{B}}_0)$.

Proof. First, we prove Im$\boldsymbol{Z}_L$ is $\hat{\boldsymbol{A}}_0$-invariant. Let $\boldsymbol{y} \in$ Im$\boldsymbol{Z}_L$, and let $\boldsymbol{z} \in$ Ker$\boldsymbol{Z}_L$. Then $\hat{\boldsymbol{A}}_0^T\boldsymbol{z} \in$ Ker$\boldsymbol{Z}_L$. Thus,

$$(\hat{\boldsymbol{A}}_0\boldsymbol{y})^T\boldsymbol{z} = \boldsymbol{y}^T\hat{\boldsymbol{A}}_0^T\boldsymbol{z} = 0. \qquad (10)$$

This implies

$$\hat{\boldsymbol{A}}_0\boldsymbol{y} \in \text{Im}\boldsymbol{Z}_L.$$

Hence, we conclude that Im$\boldsymbol{Z}_L$ is $\hat{\boldsymbol{A}}_0$-invariant. Next, suppose that $\mathcal{S}$ is any $\hat{\boldsymbol{A}}_0$-invariant subspace containing Im$\hat{\boldsymbol{C}}_0$. Then we have $\mathcal{S}^\perp \subset$ Ker$\hat{\boldsymbol{C}}_0^T$. If $\boldsymbol{w} \in \mathcal{S}^\perp$, then $\boldsymbol{w}^T\boldsymbol{u} = 0$ for all $\boldsymbol{u} \in \mathcal{S}$. Thus we have $\boldsymbol{w}^T\hat{\boldsymbol{A}}_0\boldsymbol{u} = 0$ since $\mathcal{S}$ is $\hat{\boldsymbol{A}}_0$-invariant. This implies $(\hat{\boldsymbol{A}}_0^T\boldsymbol{w})^T\boldsymbol{u} = 0$. Hence $\mathcal{S}^\perp$ is $\hat{\boldsymbol{A}}_0^T$-invariant contained in Ker$\boldsymbol{Z}_L$, and thus contained in Ker$\hat{\boldsymbol{C}}_0^T$. By Lemma 4, we thus have $\mathcal{S}^\perp \subset$ Ker$\boldsymbol{Z}_L$, which in turn implies that Im$\boldsymbol{Z}_L$ is the minimal $\boldsymbol{A}_2$-invariant subspace containing Im$\hat{\boldsymbol{C}}_0$. Q.E.D.

In view of the results given above, an obvious way to obtain a minimal realization of $\mathcal{H}_\infty$ central controller is summarized in the following statement, which is a direct application of the celebrate Kalman decomposition. See, for example, Basile and Marro [1969].

Corollary 6. Let $[\mathbf{Q}_1]$ be a basis matrix for $\mathrm{Im}\boldsymbol{Z}_L \bigcap \mathrm{Ker}\boldsymbol{W}_L$. Suppose $[\mathbf{Q}_1\,\mathbf{Q}_2]$ is a basis matrix for $\mathrm{Im}\boldsymbol{Z}_L$, and let $[\mathbf{Q}_1\,\mathbf{Q}_3]$ be a basis matrix for $\mathrm{Ker}\boldsymbol{W}_L$. Finally we extend to a basis matrix $[\mathbf{Q}_1\,\mathbf{Q}_2\,\mathbf{Q}_3\,\mathbf{Q}_4]$ for $I\!R^n$, and let $\mathbf{Q} = [\mathbf{Q}_1\,\mathbf{Q}_2\,\mathbf{Q}_3\,\mathbf{Q}_4]$. Also, let $\mathbf{Q}^{-1} = [\mathbf{R}_1^T\,\mathbf{R}_2^T\,\mathbf{R}_3^T\,\mathbf{R}_4^T]^T$. Then the dynamic equation

$$\boldsymbol{K}_r : \begin{cases} \dot{\widehat{\boldsymbol{x}}}_r = \mathbf{R}_2\hat{\boldsymbol{A}}_0\mathbf{Q}_2\widehat{\boldsymbol{x}}_r + \mathbf{R}_2\hat{\boldsymbol{B}}_0\boldsymbol{y}, \\ \boldsymbol{u} = \hat{\boldsymbol{C}}_0\mathbf{Q}_2\widehat{\boldsymbol{x}}_r, \end{cases} \tag{11}$$

is a minimal realization of the $\mathcal{H}_\infty$ central controller given in (5).

3.2 The Geometric Connection between Plant $\boldsymbol{\Sigma}$ and $\mathcal{H}_\infty$ Central Controller $\boldsymbol{\Sigma_R}$

In the previous subsection, Theorems 2 and 5 show that the controllable and unobservable subspaces of the $\mathcal{H}_\infty$ central controller can be characterized in terms of the image and kernel spaces of two matrices $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$. Although the results mentioned above seem to be natural, as we will show later, they have very important implications. Before presenting the main results of this subsection, we give some geometric motivation. First, let $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \end{bmatrix}$, and $\boldsymbol{B} = [\,\boldsymbol{B}_1\;\boldsymbol{B}_2\,]$. From the definition, it is easy to see that $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) = \mathcal{N}(\boldsymbol{C}_1,\boldsymbol{A}) \bigcap \mathcal{N}(\boldsymbol{C}_2,\boldsymbol{A})$. And $\mathcal{N}(\boldsymbol{A}^T,\boldsymbol{B}^T) = \mathcal{N}(\boldsymbol{A}^T,\boldsymbol{B}_1^T) \bigcap \mathcal{N}(\boldsymbol{A}^T,\boldsymbol{B}_2^T)$. Suppose that $\boldsymbol{v} \in \mathrm{Ker}\boldsymbol{X}_\infty \bigcap \mathrm{Ker}\boldsymbol{C}$ is any eigenvector of $(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)$. Then it can be easily observed that $\boldsymbol{v}$ generates a one-dimensional $\boldsymbol{A}$-invariant subspace $\mathcal{S}$ of the unobservable subspace of the plant $\boldsymbol{\Sigma}$, that is , $\mathcal{S} \subset \mathcal{N}(\boldsymbol{C},\boldsymbol{A})$. It is easy to see from (7) that $\mathrm{Ker}\boldsymbol{W}_L$ is the maximal $(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)$-invariant subspace contained in $\mathrm{Ker}\boldsymbol{F}_\infty$. In view of the structure of the matrix $\boldsymbol{F}_\infty$, $\mathcal{S}$ must also be a subspace of $\mathrm{Ker}\boldsymbol{W}_L$. A natural question arises: Is $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ composed of one-dimensional subspaces like $\mathcal{S}$ contained in $\mathrm{Ker}\boldsymbol{W}_L$? The answer is affirmative. To prove our main theorem in this section, we need the following proposition taken from Callier and Desoer [1991]. See also Zhou and Dolye [1997]

Proposition 7. Consider the two AREs (3)(4) and their stabilizing solutions $\boldsymbol{X}_\infty$ and $\boldsymbol{Z}_\infty$. Then $\mathrm{Ker}\boldsymbol{X}_\infty$ is an $(\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1)$−invariant subspace contained in $\mathrm{Ker}(\boldsymbol{C}_1^T(\boldsymbol{I} - \boldsymbol{D}_{12}\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T)\boldsymbol{C}_1)^{\frac{1}{2}}$, and $\mathrm{Ker}\boldsymbol{Z}_\infty$ is an $(\tilde{\boldsymbol{A}} - \boldsymbol{B}_1\boldsymbol{D}_{21}^T\boldsymbol{E}_2^{-1}\tilde{\boldsymbol{C}}_2)^T$−invariant subspace contained in $\mathrm{Ker}\tilde{\boldsymbol{B}}_1^T$.

From Proposition 7, we have the following lemma.

Lemma 8. The space $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is an $\boldsymbol{A}$−invariant subspace contained in $\mathrm{Ker}\boldsymbol{C}$.

Proof. Obviously, $\mathcal{N}(\boldsymbol{C},\boldsymbol{A})$ is also an $(\boldsymbol{A}-\boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1)$−invariant subspace. This implies that $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is $(\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{E}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1)$−invariant. Since $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty \in \mathrm{Ker}\boldsymbol{C}_1$, $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is $\boldsymbol{A}$−invariant. Q.E.D.

We are now in the position to state our main theorem in this subsection.

Theorem 9. The intersection of $\mathrm{Ker}\boldsymbol{X}_\infty$ and the unobservable subspace of the plant $\boldsymbol{\Sigma}$ is a subspace of the unobservable subspace of the $\mathcal{H}_\infty$ central controller $\boldsymbol{\Sigma_R}$. That is, $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty \subset \mathcal{N}(\hat{\boldsymbol{C}}_0,\hat{\boldsymbol{A}}_0)$.

Proof. Let $\boldsymbol{v} \in \mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$. Then we have $\boldsymbol{v} \in \mathrm{Ker}\boldsymbol{X}_\infty \bigcap \mathcal{N}(\boldsymbol{C}_2,\boldsymbol{A})$. This implies that $\boldsymbol{v} \in \mathrm{Ker}\boldsymbol{F}_\infty$. From Lemma 8, we know that $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is $\boldsymbol{A}$−invariant. Thus, we have $\boldsymbol{F}_\infty\boldsymbol{A}^i\boldsymbol{v} = 0$ for all nonnegative integer i. Since $\boldsymbol{v} \in \mathcal{N}(\boldsymbol{C}_2,\boldsymbol{A})$, it can be proven by induction that $\boldsymbol{F}_\infty(\tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)^i\boldsymbol{v} = 0$ for all nonnegative integer i. This implies that $\boldsymbol{v} \in \mathcal{N}(\boldsymbol{F}_\infty, \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)$. Since $\mathrm{Ker}\boldsymbol{W}_L = \mathcal{N}(\boldsymbol{F}_\infty, \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2) = \mathcal{N}(\hat{\boldsymbol{C}}_0, \tilde{\boldsymbol{A}} - \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)$, this completes the proof by Theorem 2. Q.E.D.

Remark 10. In view of the proof in Theorem 9, it is obvious that $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is not equal to $\mathcal{N}(\hat{\boldsymbol{C}}_0,\hat{\boldsymbol{A}}_0)$ in general.

A dual result of Theorem 9 follows immediately.

Theorem 11. The controllable subspace of the $\mathcal{H}_\infty$ central controllers is contained in the sum of $\mathrm{Im}\boldsymbol{Z}_\infty$ and the controllable subspace of the plant. That is, $\mathcal{C}(\hat{\boldsymbol{A}}_0,\hat{\boldsymbol{B}}_0) \subset \mathcal{C}(\boldsymbol{A},\boldsymbol{B}) + \mathrm{Im}\boldsymbol{Z}_\infty$.

Proof. With similar arguments, we have $\mathcal{N}(\boldsymbol{B}^T,\boldsymbol{A}^T) \subset \mathcal{N}(\hat{\boldsymbol{B}}_0^T,\hat{\boldsymbol{A}}_0^T)$. Taking orthogonal complement yields $\mathcal{C}(\hat{\boldsymbol{A}}_0,\hat{\boldsymbol{B}}_0) \subset \mathcal{C}(\boldsymbol{A},\boldsymbol{B})$. This completes the proof. Q.E.D.

Now, regard $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}_0$ as linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $\mathcal{I}$ is an $\boldsymbol{A}$−invariant subspace, let $\boldsymbol{A}|_{\mathcal{I}}$ represent the linear map restricted on the space $\mathcal{I}$. With this notation, we have the following theorem.

Theorem 12. The spectrum of $\boldsymbol{A}$ restricted to $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is contained in the spectrum of $\hat{\boldsymbol{A}}_0$ restricted to $\mathcal{N}(\hat{\boldsymbol{C}}_0,\hat{\boldsymbol{A}}_0)$; that is, $\sigma(\boldsymbol{A}|_{\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty}) \subset \sigma(\hat{\boldsymbol{A}}_0|_{\mathcal{N}(\hat{\boldsymbol{C}}_0,\hat{\boldsymbol{A}}_0)})$

Proof. Let $\lambda \in \sigma(\boldsymbol{A}|_{\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty})$. This implies that there exists a vector $\boldsymbol{v} \in \mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ such that $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$. Since $\boldsymbol{v} \in \mathrm{Ker}\boldsymbol{X}_\infty \bigcap \mathrm{Ker}\boldsymbol{C}_2$, we have $\boldsymbol{v} \in \mathrm{Ker}\tilde{\boldsymbol{C}}_2$. By Theorem 9, we have $\boldsymbol{v} \in \mathrm{Ker}\hat{\boldsymbol{C}}_0$. Hence $(\tilde{\boldsymbol{A}} + \boldsymbol{B}_2\hat{\boldsymbol{C}}_0 + \hat{\boldsymbol{B}}_0\tilde{\boldsymbol{C}}_2)\boldsymbol{v} = \lambda\boldsymbol{v}$. This completes the proof. Q.E.D.

Remark 13. Theorem 9 and 12 indicate that part of the unobservable dynamics of the plant in $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ are completely copied in the unobservable dynamics of the $\mathcal{H}_\infty$ central controller. They also imply the fact that if the plant has $\hat{r}$ unobservable modes in space $\mathcal{N}(\boldsymbol{C},\boldsymbol{A})\bigcap \mathrm{Ker}\boldsymbol{X}_\infty$, then the $\mathcal{H}_\infty$ central controller has at least $\hat{r}$ unobservable modes. Thus, the order of the $\mathcal{H}_\infty$ central controller could be reduced to $n-\hat{r}$ at least.

With dual arguments, we have the following result.

Theorem 14. The spectrum of $\boldsymbol{A}^T$ restricted to $\mathcal{N}(\boldsymbol{B}^T,\boldsymbol{A}^T)\bigcap \mathrm{Ker}\boldsymbol{Z}_\infty$ is contained in the spectrum of $\hat{\boldsymbol{A}}_0^T$ restricted to $\mathcal{N}(\hat{\boldsymbol{B}}_0^T,\hat{\boldsymbol{A}}_0^T)$

Remark 15. If γ approaches to infinity, equation (3) and (4) become $\mathcal{H}_2$ AREs and $\mathcal{H}_\infty$ central controller (5) reduces to the optimal $\mathcal{H}_2$ controller for the plant $\boldsymbol{\Sigma}$. Thus $\mathcal{N}(\boldsymbol{C},\boldsymbol{A}) \in \mathrm{Ker}\boldsymbol{X}_\infty$ and $\mathcal{N}(\boldsymbol{B}^T,\boldsymbol{A}^T) \in \mathrm{Ker}\boldsymbol{Z}_\infty$. This implies that the unobservable subspace of the plant $\boldsymbol{\Sigma}$ is a subspace of the unobservable subspace of the controller (5) and the controllable subspace of the controller (5) is contained in the controllable subspace of the plant $\boldsymbol{\Sigma}$ when γ approaches to infinity. These coincide with the results in our previous work [Wu and Yung, 2007] on geometric characterization of the unobservable and controllable subspaces for the $\mathcal{H}_2$ optimal controller.

4. NUMERICAL EXAMPLE

In this section, we give an example for illustration. Consider the plant (2) with

$$\boldsymbol{A}=\begin{bmatrix}1&0&0&0&1.5\\-2&-1&0&0&-1.5\\-4&0&-1&0&-3\\0&0&0&-1&0\\0&0&0&0&-2\end{bmatrix},\quad \boldsymbol{B}_1=\begin{bmatrix}0&0\\1&0\\0&0\\0&0\\0&0\end{bmatrix},$$

$$\boldsymbol{B}_2=[\,1\;\;0\;\;-2\;\;0\;\;0\,]^T,\quad \boldsymbol{C}_1=\begin{bmatrix}1&1&0&0&0\\0&0&0&0&0\end{bmatrix},$$

$$\boldsymbol{C}_2=[\,1\;\;0\;\;0\;\;0\;\;0.5\,],\quad \boldsymbol{D}_{12}=\boldsymbol{D}_{21}^T=\begin{bmatrix}0\\1\end{bmatrix}.$$

Take $\gamma=1$. It is easy to check that the plant satisfies Hypotheses (H1) and (H2) in Proposition 1. By Theorem 2, we can calculate the numbers of unobservable and uncontrollable modes in the $\mathcal{H}_\infty$ central controller by computing $\boldsymbol{W}_L$ and $\boldsymbol{Z}_L$. It can be shown numerically that

$$\boldsymbol{X}_\infty=\begin{bmatrix}0.6472&0.0494&0&0&0.2989\\0.0494&0.4099&0&0&-0.1802\\0&0&0&0&0\\0&0&0&0&0\\0.2989&-0.1802&0&0&0.2395\end{bmatrix},$$

$$\boldsymbol{Z}_\infty=\begin{bmatrix}4.0771&-4.9480&-8.1542&0&0\\-4.9480&6.8515&9.8961&0&0\\-8.1542&9.8961&16.3085&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$

$$\hat{\boldsymbol{A}}_0=\begin{bmatrix}-3.7243&-0.0494&0&0&-0.8347\\2.9975&-0.5901&0&0&0.7938\\5.4486&0.0989&-1&0&1.6749\\0&0&0&-1&0\\0&0&0&0&-2\end{bmatrix},$$

$$\hat{\boldsymbol{C}}_0=[\,-0.6472\;\;-0.0494\;\;0\;\;0\;\;-0.2989\,],$$

$$\hat{\boldsymbol{B}}_0^T=[\,4.0771\;\;-4.9480\;\;-8.1542\;\;0\;\;0\,].$$

Thus the $\mathcal{H}_\infty$ controller given in (5) reads:

$$\boldsymbol{\Sigma_R}:\begin{cases}\dot{\boldsymbol{\xi}}=\hat{\boldsymbol{A}}_0\boldsymbol{\xi}+\hat{\boldsymbol{B}}_0\boldsymbol{y},\\ \boldsymbol{u}=\hat{\boldsymbol{C}}_0\boldsymbol{\xi}.\end{cases}$$

Solving (7) and (8) gets $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$ as

$$\boldsymbol{Z}_L=\begin{bmatrix}-13.5685&71.2885&27.1371&0&0\\71.2885&-214.8932&-142.5769&0&0\\27.1371&-142.5769&-54.2742&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},$$

$$\boldsymbol{W}_L=\begin{bmatrix}0.0782&0.0104&0&0&0.0339\\0.0104&0.0021&0&0&0.0042\\0&0&0&0&0\\0&0&0&0&0\\0.0339&0.0042&0&0&0.0149\end{bmatrix}.$$

Thus, $\mathrm{rank}\boldsymbol{Z}_L=2$, $\mathrm{nullity}\boldsymbol{W}_L=3$. The $\mathcal{H}_\infty$ central controller thus has 2 controllable modes and 3 unobservable modes. A minimal realization of the $\mathcal{H}_\infty$ central controller thus has order two given by

$$\dot{\hat{\boldsymbol{x}}}_r(t)=\begin{bmatrix}0.5901&1.341\\-0.1105&-3.7243\end{bmatrix}\hat{\boldsymbol{x}}_r(t)+\begin{bmatrix}4.948\\-9.117\end{bmatrix}\boldsymbol{y}(t),$$

$$\boldsymbol{u}(t)=[\,0.04943\;\;0.2894\,]\,\hat{\boldsymbol{x}}_r(t).$$

It can be shown numerically that $\mathcal{N}(\boldsymbol{C},\boldsymbol{A})\bigcap \mathrm{Ker}\boldsymbol{X}_\infty = \mathrm{span}\{\begin{bmatrix}0.4082\\-0.4082\\0\\0\\-0.8165\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\}$. A direct computation shows that $\mathcal{N}(\boldsymbol{C},\boldsymbol{A})\bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ is $\boldsymbol{A}-$invariant. It is also noted that the three unobservable modes in $\mathcal{N}(\boldsymbol{C},\boldsymbol{A})\bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ are -1, -1, -2, and the three unobservable modes in the $\mathcal{H}_\infty$ central controller are also -1, -1, -2. It is thus verified that the unobservable dynamics of the plant in $\mathcal{N}(\boldsymbol{C},\boldsymbol{A})\bigcap \mathrm{Ker}\boldsymbol{X}_\infty$ are completely copied in the unobservable dynamics of the $\mathcal{H}_\infty$ central controller.

5. CONCLUSION

In this paper, we have shown that the controllable and unobservable subspaces of the $\mathcal{H}_\infty$ central controllers can be described by the image and kernel spaces of two matrices $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$, where $\boldsymbol{Z}_L$ and $\boldsymbol{W}_L$ are positive semidefinite solutions of two Lyapunov equations. Furthermore, under this characterization, it has been shown that the unobservable subspace of the $\mathcal{H}_\infty$ central controller contains the the intersection of $\mathrm{Ker}\boldsymbol{X}_\infty$ and the unobservable subspace of the plant, and the $\mathcal{H}_\infty$ central controller's controllable subspace is a subspace of the sum of $\mathrm{Im}\boldsymbol{Z}_\infty$ and the plant's controllable subspace. A numerical example has also been given for illustration.

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