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On Mismatch between Initializations at Coder/Decoder in Quantized Control ⋆

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Abstract: This paper analyzes the stability of linear systems with quantized feedback in the presence of a mismatch between the initial conditions at the coder and decoder. Under the assumption of the prefect channel, we show that using the scheme proposed in [Liberzon, Nešić (2007)] it is possible to achieve global exponential stability of linear systems with quantized feedback when the coder and decoder are initialized at different initial conditions.

Keywords: communication channel, quantization, controller design, stabilization

1. INTRODUCTION

The subject of this paper is the analysis of robustness of linear systems with quantized feedback with respect to a mismatch between the initial conditions at the coder and decoder. Control systems with quantized feedback are increasingly used in control practice due to advances in computer, sensor and actuator technologies, as well as our desire to decrease costs, simplify installation and maintenance.

While a number of important results have been published on the topic of the quantized control systems, including Brockett, Liberzon (2000), Elia, Mitter (2001), Li, Baillieul (2004), Liberzon (2003), Nair, Evans (2004), Liberzon, Nešić (2007), to our best knowledge, none of them consider the issue of the mismatch between the initial conditions at the coder and decoder.

To simplify the presentation, we assume that the channel is perfect and concentrate on the robustness properties of the systems when the coder and decoder are initialized at different initial conditions. While the issues of the robustness with respect to the time-delays, data dropouts, bit-errors, corrupted signals and control, in general, over noisy channels were investigated Gupta *et.al.* (2006), Martins, Dahleh (2005), Matveev, Savkin (2003), Tatikonda, Mitter (2004), the robustness with respect to the computational errors at the coder and decoder did not receive much attention in the literature.

The device which at each instant of time maps the value of the plant output measurements into one of all possible symbols is called the coder (pre-processing device). The symbol generated by the coder is then transmitted through the channel to the receiver. At the reception, the decoder (after-processing device) generates an estimated value of the state from the received symbol. Note, that to deal with a finite capacity of the channel, we have to run two copies of the system on both sides of the digital channel. A common assumption made in the literature is that these two systems are initialized at the same initial condition and, hence, the issue of the discrepancy in the initialization at the coder and decoder is ignored. This may not be implementable in practice due to hardware imperfections. Even if the coder and decoder are initialized at the same value and the channel is perfect, since the coder and decoder dynamics evolve independently, the computational errors in the algorithm implemented at the coder and decoder can occur. At this point of time the mismatch in the coder and decoder takes place.

In other words, even if the channel is perfect and the internal coder/decoder factors are initialized at the same value, due to a finite precision of encoding and decoding schemes for transmitted information there might exist time such that these internal factors start to differ. We treat this time instant as the initial time when the mismatch occurs.

In this paper we investigate this phenomenon further. We explore the following question: does the system preserve stability properties when the coder and decoder are initialized at different initial conditions?

We explore the robustness with respect to the mismatch between the initial conditions at the coder and decoder of the discrete linear time-invariant systems with quantized feedback.

As a particular example of the quantizer-coder-decoder scheme, we analyze in detail the sampled-data hysteresis switching scheme proposed in Liberzon, Nešić (2007). We use this scheme as a representative example of other quantized control schemes, that have adaptive quantization as their main feature, that is: the quantizer's range and quantization error are changing adaptively depending on the quantized measurements of the plant.

This scheme is known to lead to the various stability properties of the linear time-invariant systems when the coder and decoder are initialized at the same initial conditions Kameneva, Nešić (2007), Liberzon, Nešić (2007). In the simulations we have observed, that when the mismatch

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between the coder and decoder initialization is sufficiently large, the system dynamics become unstable.

Under the assumption that the channel is perfect, using Liberzon and Nešić scheme, we analyze the robustness properties of the quantized control system with respect to the computational errors that occur due to the independent evaluation of the adaptive "scaling" factors at the coder and decoder. We give a quantitative measure on how much these adaptive "scaling" factors at the coder and decoder can differ so that the system preserves stability.

We show that if the channel is perfect and the coder and decoder are initialized at different initial conditions (but a bound on the mismatch holds), then using the time-sampled scheme introduced in Liberzon, Nešić (2007), it is possible to adjust the parameters of the quantizer so that the systems is global exponential stable (GES) (refer to Definition 3 in Section 4).

Note, that in Liberzon, Nešić (2007) the modified version of the hysteresis scheme, that we use in this paper was also introduced. That scheme was developed to handle disturbances, while here we consider systems with no disturbances. An interesting future research topic will be to use the modified version of the scheme that able to handle disturbances and analyze the robustness properties with respect to the initial coder/decoder mismatch of the nonlinear systems and systems with input disturbances.

Our Theorem 1 in Section 4 shows that GES (in the sense of Definition 3) is possible when the channel is perfect, the parameters of the scheme are adjusted appropriately and the bound on the mismatch between the initial conditions at the coder and decoder holds. In other words, we show that the scheme has some intrinsic robustness properties with respect to small mismatches in the coder/decoder initialization. We believe that these results shed a light on the robustness properties of other quantized control scheme in the literature.

The remainder of the paper is organized as follows. In Section 2 we give definitions that are used in the sequel. The closed loop system, switching rules and protocol are given in Section 3. The main results are presented in Section 4. Section 5 offers the conclusions.

2. NOTATION AND PRELIMINARIES

In this section we introduce some notation and give the definitions that will make the discussed concepts precise. In what follows, $|\cdot|$ denotes the Euclidean norm, $\|\cdot\|$ denotes the corresponding matrix induced norm. The infinity-norm of a sequence of vectors on a time-interval $[k_1, k_2]$ is denoted $\|z\|_{[k_1,k_2]} := \sup_{k\in[k_1,k_2]} |z_k|$.

A quantizer is a piecewise constant function $q : \mathbb{R}^n \to Q$, where Q is a finite subset of $\mathbb{R}^n$. We use the following assumption:

Assumption 1. There exist strictly positive numbers $M_1 \geq M > \Delta > 0$, Δ_0 such that the following holds:

1. If $|z| \leq M$ then $|q(z) - z| \leq \Delta$;

2. If $|z| > M$ then $|q(z)| > M - \Delta$;

3. For all $|z| \leq \Delta_0$ we have that $q(z) = 0$;

4. $|q(z)| \leq M_1$ for all $z \in \mathbb{R}^n$.

M is called the range of the quantizer; Δ is called the quantization error. The first condition gives a bound on the quantization error when the state is in the range of the quantizer, the second gives the possibility to detect saturation. The third condition is needed to preserve the origin as an equilibrium and, moreover, together with the forth condition it guarantees that there exists $L_q > 0$ such that $|q(z)| < L_q|z|$ $\forall z \in \mathbb{R}^n$. The last conditions guarantees that the quantized values of z are globally bounded. Note, that for a sufficiently large M_1 without loss of generality we can assume that the following holds: $M_1 = L_q M$. We will use the following definitions:

Definition 1. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded.

Definition 2. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be class $\mathcal{KL}$ if $\beta(\cdot, t)$ is continuous, strictly increasing and zero at zero and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

3. CLOSED-LOOP SYSTEM

Consider the continuous-time linear system with a control input:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are respectively the state and control. The matrix A is nonzero and non-Hurwitz. Define $t_k = kT$ for $k = 0, 1, 2, \ldots,$ where $T > 0$ is a given sampling period. We shortly denote $x(t_k) = x_k$, $u(t_k) = u_k$, $k = 0, 1, 2, \ldots$. The plant (1) induces the following discrete-time system which is more amenable to analysis:

$$x_{k+1} = \Phi x_k + \Gamma u_k, \quad x_0 \in \mathbb{R}^n, \tag{2}$$

where $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{As}B\, ds$. Note, that due to a finite capacity of the channel the state measurements are quantized into a finite subset of $\mathbb{R}^n$. We use the quantized measurements in the following form:

$$q_k^c := \mu_k^c q\left(\frac{x_k}{\mu_k^c}\right), \quad \mu_0^c > 0$$

on the coder side of the channel (see Figure 1) and

$$q_k^d := \mu_k^d \left(q\left(\frac{x_k}{\mu_k^c}\right) + v_k\right), \quad \mu_0^d > 0$$

on the decoder side of the communication channel. μ_k^c, μ_k^d are the adjustable parameters, called "zoom" variables, that are updated at discrete instants of time; μ_k^c and μ_k^d correspond to the coder and decoder dynamics respectively.

The symbol $q\left(\frac{x_k}{\mu_k^c}\right)$ is send via the communication channel. At the reception, the decoder receives the symbol $q\left(\frac{x_k}{\mu_k^c}\right) + v_k$, which is, in general, not necessarily identical to the symbol which was send by the coder. The term v_k

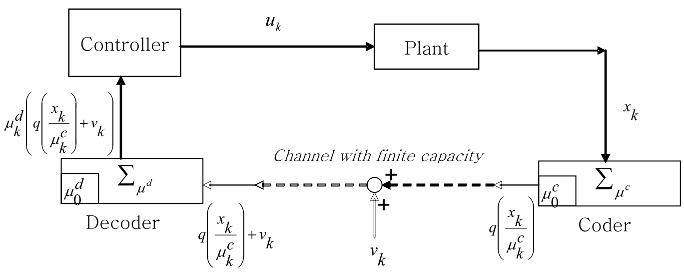

Fig. 1. A feedback system with a digital channel.

corresponds to a general noise in the channel, it can model the pure (propagation) time-delay, packets loss, bit-errors etc. To simplify the presentation, we do not combine the issues of the noisy channel with the issue of the robustness with respect to the mismatch in the coder/decoder initialization. This problem is outside the scope of this paper.

Assumption 2. Assume that

$$v_k \equiv 0 \quad \forall k \geq 0.$$

Assumption 2 guarantees that the channel is perfect: the data that the coder sends, the decoder receives without delay and without errors. The scheme of the discrete closed-loop system (2) is given in Figure 1.

To control the system (2) we use the quantized hybrid feedback that was introduced in Liberzon, Nešić (2007). We assume that (Φ, Γ) is stabilizable and let K be such that $\Phi + \Gamma K$ is Schur. Then the feedback is defined by the following equations:

$$u_k = U(\Omega_k^d, \mu_k^c, \mu_k^d, x_k), \quad t \in [t_k, t_{k+1}) \tag{3}$$

$$U(\Omega_k^d, \mu_k^c, \mu_k^d, x_k) := \begin{cases} 0 & \text{if } \Omega_k^d = \Omega_{out} \\ Kq_k^d & \text{if } \Omega_k^d = \Omega_{in}, \end{cases} \tag{4}$$

where the variable Ω_k^d determines the switching rules for the decoder. It can take only two strictly positive values Ω_{out} and Ω_{in}, that will be defined next. If $\Omega_k^d = \Omega_{out}$ we say that a zoom-out condition is triggered at the decoder at time k. If $\Omega_k^d = \Omega_{in}$ we say that a zoom-in condition is triggered at the decoder at time k. During the zoom-out stage of the decoder the system is running in an open loop: $u_k = 0$. During the zoom-in stage of the decoder the certainty equivalence feedback $u_k = Kq_k^d$ is applied. The variable Ω_k^c determines the switching rules for the coder in the same manner as the variable Ω_k^d determines the switching rules for the decoder.

The protocol dynamics is described by the following:

$$\mu_{k+1}^c = G(\Omega_k^c, \mu_k^c, x_k), \quad \mu_0^c \in \mathbb{R}_{>0} \tag{5}$$

$$G(\Omega_k^c, \mu_k^c, x_k) := \begin{cases} \Omega_{out}\mu_k^c & \text{if } \Omega_k^c = \Omega_{out} \\ \Omega_{in}\mu_k^c & \text{if } \Omega_k^c = \Omega_{in} \end{cases} \tag{6}$$

$$\mu_{k+1}^d = G(\Omega_k^d, \mu_k^d, x_k), \quad \mu_0^d \in \mathbb{R}_{>0} \tag{7}$$

$$G(\Omega_k^d, \mu_k^d, x_k) := \begin{cases} \Omega_{out}\mu_k^d & \text{if } \Omega_k^d = \Omega_{out} \\ \Omega_{in}\mu_k^d & \text{if } \Omega_k^d = \Omega_{in} \end{cases} \tag{8}$$

The adjustment policy for μ_k^c, μ_k^d can be thought of as implemented on both ends of the communication channel (at the coder and decoder) from some known initial values μ_0^c, μ_0^d.

Note, that in Liberzon, Nešić (2007) it is assumed, that the coder and decoder are initialized at the same initial condition $\mu_0^c = \mu_0^d$, therefore $\mu_k^c = \mu_k^d$ for all $k = 0, 1, 2, \ldots$. We, on the other hand, assume that there may be a mismatch between the initial conditions at the coder and decoder and in our case, generally, $\mu_k^c \neq \mu_k^d$. In particular, we assume that the ratio (the mismatch) of the initial conditions at the coder and at the decoder is $r > 0$ (and not necessarily $r = 1$, as assumed in Liberzon, Nešić (2007)):

$$r := \frac{\mu_0^c}{\mu_0^d}, \quad \mu_0^c > 0, \; \mu_0^d > 0.$$

The adjustment policies for μ_k^c and μ_k^d are composed of two stages: a zoom-out stage and a zoom-in stage. During the zoom-out stage of the coder (respectively the decoder) the value of an adjustable parameter μ^c (respectively μ^d) is increased at the rate faster than the growth of $|x_k|$ until the state can be adequately measured. During the zoom-in stage of the coder (respectively the decoder) the value of an adjustable parameter μ^c (respectively μ^d) is decreased in such way as to drive the state to the origin. The hysteresis switching is used to switch between the zoom-in and zoom-out stages. It is described by the following:

$$\Omega_k^c = H(\Omega_{k-1}^c, \mu_k^c, x_k), \quad \Omega_{-1}^c = \Omega_{out} \tag{9}$$

$$H(\Omega_{k-1}^c, \mu_k^c, x_k) := \begin{cases} \Omega_{out} & \text{if } |q_k^c| > l_{out}\mu_k^c \\ \Omega_{in} & \text{if } |q_k^c| < l_{in}\mu_k^c \\ \Omega_{k-1}^c & \text{if } |q_k^c| \in [l_{in}\mu_k^c, l_{out}\mu_k^c] \end{cases} \tag{10}$$

$$\Omega_k^d = H(\Omega_{k-1}^d, \mu_k^d, x_k), \quad \Omega_{-1}^d = \Omega_{out} \tag{11}$$

$$H(\Omega_{k-1}^d, \mu_k^d, x_k) := \begin{cases} \Omega_{out} & \text{if } |q_k^d| > l_{out}\mu_k^d \\ \Omega_{in} & \text{if } |q_k^d| < l_{in}\mu_k^d \\ \Omega_{k-1}^d & \text{if } |q_k^d| \in [l_{in}\mu_k^d, l_{out}\mu_k^d] \end{cases} \tag{12}$$

where l_{out} and l_{in} are strictly positive numbers such that $l_{out} := M - \Delta$, $l_{in} := \Delta_M - \Delta$ and $\Delta_M > \Delta$, Δ_M will be defined later.

Similarly to Liberzon, Nešić (2007), we assume that the coder and decoder are initialized at the same synchronized stage:

Assumption 3. Assume that

$$\Omega_0^c = \Omega_0^d = \Omega_{out}.$$

Remark 1. Note, that the evaluation of Ω_k^c and Ω_k^d do not require integration of any equation. The coder/decoder evaluators for Ω_k^c/Ω_k^d use the dynamic "look-up tables" (10)/(12) to set up the values of Ω_k^c/Ω_k^d based on the values of $q\left(\frac{x_k}{\mu_k^c}\right)$ and $q\left(\frac{x_k}{\mu_k^c}\right) + v_k$ respectively. Note, that due to Assumption 2, the values of both, Ω_k^c and Ω_k^d, depend on the same symbol $q\left(\frac{x_k}{\mu_k^c}\right)$. Therefore, if Ω_k^c and Ω_k^d initial stage is synchronized, then their synchronization is enforced at every time step.

We assume that the coder and decoder evaluators for Ω_k^c and Ω_k^d are reliable, that there are no mistakes in the dynamic "look-up tables" for Ω_k^c/Ω_k^d at the coder and decoder.

Assumption 4. Assume that

$$\text{if for some } k \geq 0 \; v_k = 0 \text{ and } \Omega_k^c = \Omega_k^d, \quad \text{then } \Omega_{k+1}^c = \Omega_{k+1}^d.$$

Note, that if Assumptions 2 and 4 hold, the coder and decoder switching will be synchronized. Cancelling μ_k^c, μ_k^d in (10) and (12) we can conclude, that if the channel is perfect (Assumption 2 holds) and the coder/decoder evaluators for Ω_k^c/Ω_k^d are reliable (Assumption 4 holds), then the switching depends only on the value of $q\left(\frac{x_k}{\mu_k^c}\right)$. This can be interpreted as the fact that the switching is governed by the variable $\xi_k^c := \frac{x_k}{\mu_k^c}$ (see Remark below). Therefore, the coder and decoder switching conditions are the same. That is, the coder and decoder switching will be synchronized: if the coder is zooming-in, then the decoder is zooming-in; and vice versa.

Remark 2. Consider the switching conditions for the coder. Note, that whenever $\left|\frac{x_k}{\mu_k^c}\right| < l_{in} - \Delta$ holds, $\left|\mu_k^c q\left(\frac{x_k}{\mu_k^c}\right)\right| < l_{in}\mu_k^c$ holds. Also, the zoom-out switching condition $\left|\mu_k^c q\left(\frac{x_k}{\mu_k^c}\right)\right| > l_{out}\mu_k^c$ implies that $\left|\frac{x_k}{\mu_k^c}\right| > l_{out}+\Delta$. The same observation holds for the decoder.

Due to Assumption 2, in the sequel we treat the decoder quantized measurements as $q_k^d = \mu_k^d q\left(\frac{x_k}{\mu_k^c}\right)$.

Next we present a straightforward result (Proposition 1 below), that guarantees that if

(i) the channel is perfect (Assumption 2 holds);

(ii) the coder and decoder are initialized at the same synchronized stage (Assumption 3 holds);

(iii) the coder/decoder evaluators for Ω_k^c/Ω_k^d are reliable (Assumption 4 holds);

then the coder and decoder stage will be always synchronized.

Proposition 1. Suppose Assumption 2 - 4 hold. Then

$$\Omega_k^c = \Omega_k^d \quad \forall k \geq 0.$$

The proof of Proposition 1 is by induction and not presented here due to space limitations.

Remark 3. Note, that the difference of Proposition 1 from Assumption 4 is that Assumption 4 guarantees the synchronized stage of coder and decoder only for one step ahead. Proposition 1, on the other hand, guarantees that if the coder and decoder stage is synchronized at some point of time, it will be synchronized for all future time. In other words, if Ω_0^c and Ω_0^d are synchronized at the first step, then the synchronization of Ω_k^c and Ω_k^d is enforced at each time step.

Remark 4. We will analyze only the stability properties of the discrete-time system (2) with (3) - (12) induced by the sampled-data system (1). It was shown in Nešić *et. al.* (1999) how to use the underlying discrete-time model to conclude appropriate stability properties of the sampled-data system.

We introduce some notation. Due to Assumptions 2 - 3 the coder and decoder switching will be synchronized. We have that $\Omega_k^c = \Omega_k^d$ for all $k \geq 0$. We introduce $k_j \in \mathbb{N}$ such that

$$\begin{aligned} &\Omega_k^c = \Omega_k^d = \Omega_{out} \text{ if } k \in [k_{2i}, k_{2i+1}-1], \quad i = 0,1,2,\ldots,N \\ &\Omega_k^c = \Omega_k^d = \Omega_{in} \text{ if } k \in [k_{2i+1}, k_{2i+2}-1], \end{aligned}$$

That is: k_{2i+1} is the time instant at which the coder and decoder switch from the zoom-out stage to the zoom-in stage; k_{2i+2} is the time instant at which the coder and decoder switch from the zoom-in stage to the zoom-out stage. We assume that $k_0 = 0$ and that the first interval is always the zoom-out. We will adjust the quantizer, coder, decoder and controller so that $N = 0$. In other words, the coder and decoder will zoom-out for $k \in [0, k_1 - 1]$ and zoom-in for all $k \geq k_1$.

To understand the operation of the plant (2) we need to consider two modes of the operation of the plant:

Mode 1. The coder and decoder are zooming-out;

Mode 2. The coder and decoder are zooming-in.

The plant dynamics during each mode is considered in full details in Lemmas 1 and 2 in Section 4. Lemmas 1 and 2 show that if the quantizer, coder, decoder and controller are appropriately adjusted, then the following holds:

• Mode 1 can happen only on the first zooming interval, after which the system switches to Mode 2;

• If Mode 2 happens then system stays in Mode 2 for all future time.

The dynamics of the plant during Modes 1 and 2 is described below.

Mode 1: $k \in [0, k_1 - 1]$. The coder and decoder are zooming-out. During this mode the system dynamics is described by the following equations:

$$\begin{aligned} x_{k+1} &= \Phi x_k, \quad x_0 \in \mathbb{R}^n, \qquad (13)\\ \mu_{k+1}^c &= \Omega_{out}\mu_k^c, \quad \mu_0^c > 0, \\ \mu_{k+1}^d &= \Omega_{out}\mu_k^d, \quad \mu_0^d > 0. \end{aligned}$$

The dynamics of ξ_k^c is described by the following equation:

$$\xi_{k+1}^c = \frac{1}{\Omega_{out}}\Phi\xi_k^c. \qquad (14)$$

Note that during this mode the ratio $\frac{\mu_k^c}{\mu_k^d} = \frac{\Omega_{out}^k\mu_0^c}{\Omega_{out}^k\mu_0^d} = \frac{\mu_0^c}{\mu_0^d} = r$ stays constant for all $k \in [0, k_1]$.

Mode 2: $k \geq k_1$. The coder and decoder are zooming-in. During this mode the system dynamics is described by the following equations:

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma K \mu_k^d q\left(\frac{x_k}{\mu_k^c}\right), \qquad (15)\\ \mu_{k+1}^c &= \Omega_{in}\mu_k^c, \\ \mu_{k+1}^d &= \Omega_{in}\mu_k^d. \end{aligned}$$

The dynamics of ξ_k^c is described by the following equation:

$$\xi_{k+1}^c = \frac{1}{\Omega_{in}}\Phi\xi_k^c + \frac{1}{\Omega_{in}}\Gamma K\frac{\mu_k^d}{\mu_k^c}q(\xi_k^c). \quad (16)$$

Note that during this mode the ratio $\frac{\mu_k^c}{\mu_k^d} = \frac{\Omega_{in}^{k-k_1}\mu_{k_1}^c}{\Omega_{in}^{k-k_1}\mu_{k_1}^d} = \frac{\mu_{k_1}^c}{\mu_{k_1}^d} = \frac{\mu_0^c}{\mu_0^d} = r$ stays constant for all $k \geq k_1$.

Adding and subtracting $\frac{1}{\Omega_{in}}\Gamma K\xi_k^c$, $\frac{1}{\Omega_{in}}\Gamma K q(\xi_k^c)$ terms to the equation (16), we can say, that during Mode 2 the system dynamics for ξ_k^c satisfies the following:

$$\xi_{k+1}^c = \frac{1}{\Omega_{in}}(\Phi + \Gamma K)\xi_k^c + \frac{1}{\Omega_{in}}\Gamma K\bar{\nu}_k, \quad (17)$$

where $\bar{\nu}_k = \nu_k^c + (\frac{1}{r} - 1)q(\xi_k^c)$, $\nu_k^c = q(\xi_k^c) - \xi_k^c$. Note that when the initial conditions at the coder and at the decoder are the same ($\mu_0^c = \mu_0^d$), the ξ_k^c dynamics satisfies (17) with $\bar{\nu}_k = \nu_k^c$ (since $r = \mu_0^c/\mu_0^d = 1$ in this case). Now we can state the following results, that are similar to Lemma III.2 and Corollary III.3 from Liberzon, Nešić (2007). The first result follows directly from Jiang, Wang (2001), Example 3.4. We omit the proof of Corollary 2 due to space limitations.

Corollary 1. Suppose that $\Phi + \Gamma K$ is Schur. Then, there exists an $\Omega_{in}^* \in (0,1)$ such that for all $\Omega_{in} \in [\Omega_{in}^*, 1)$, $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur. Moreover, for any such Ω_{in}, there exist strictly positive K_1, λ, γ such that the solutions of the system (17) satisfy the following:

$$|\xi_k^c| \leq K_1 \exp(-\lambda(k - k_1))|\xi_{k_1}^c| + \gamma\|\bar{\nu}\|. \quad (18)$$

In particular, let $\kappa > 0$ and $\sigma \in (0,1)$ be such that[1] $\|\frac{1}{\Omega_{in}^k}(\Phi + \Gamma K)^k\| \leq \kappa\sigma^k$ for all $k \geq 0$. Then, we can let

$$K_1 = \kappa, \quad \lambda = -\ln(\sigma), \quad \gamma = \frac{\kappa\|\Gamma K\|}{\Omega_{in}(1-\sigma)}. \quad (19)$$

Corollary 2. Suppose

$$-1 < \frac{\mu_0^d - \mu_0^c}{\mu_0^c} < \frac{1}{\gamma L_q}. \quad (20)$$

Let Ω_{in}, K_1, γ come from Corollary 1 and let strictly positive M and Δ be such that the following holds:

$$M > \frac{(2 + K_1 + \gamma)\Delta}{1 - \gamma(\frac{1}{r} - 1)L_q}. \quad (21)$$

Then there exists a $\Delta_M > 0$ with $\Delta_M - \Delta > 0$, such that whenever $|\xi_0^c| \leq \Delta_M$ and $|\nu_k| \leq \Delta$ the following two conditions hold for all $k \geq k_1$:

$$|\xi_k^c| \leq M \quad (22)$$

and

$$\left|q\left(\frac{x_k}{\mu_k^c}\right)\right| \leq M - \Delta. \quad (23)$$

4. STABILITY

The main results are presented in this section.

Definition 3. The system (2) is Globally Exponentially Stable [2] (GES) in x if for a fixed $\mu_0^c > 0, \mu_0^d > 0$ with $\mu_0^c/\mu_0^d = r$ there exists $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}$ and we have:

$$|x_k| \leq \varphi(|x_0|) \;\; \forall k \geq 0 \quad (24)$$

and $|x_k| \to 0$ as $k \to \infty$ exponentially fast.

Remark 5. Note that $\varphi(|x_0|)$ depends on μ_0^c and μ_0^d.

Definition 4. The system $x_{k+1} = Ax_k + Dw_k$, $x_0 \in \mathbb{R}^n$, where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^l$ are respectively the state and the disturbance, is said to be Input-to-State Stable (ISS) with a linear gain $\tilde{\gamma} > 0$ if for every initial condition $x_0 \in \mathbb{R}$ and every bounded disturbance w there exist positive $\hat{K}, \hat{\lambda}$ such that we have: $|x_k| \leq \hat{K}\exp(-\hat{\lambda}(k - k_0))|x_0| + \tilde{\gamma}\|w\|$ $\forall k > k_0$.

The main contribution of our work is the following theorem, which shows that the system (2) with (3) - (12) is GES in the sense of our non-standard Definition 3 if the mismatch between μ_0^c and μ_0^d is sufficiently small.

Theorem 1. Consider the system (2) with (3) - (12), when $\mu_0^c/\mu_0^d = r > 0$. Let q be a quantizer fulfilling Assumption 1. Suppose Assumptions 2 - 4 hold and for a given sampling period $T > 0$ the pair (Φ, Γ) is stabilizable. Let

(*i*) K be such that $\Phi + \Gamma K$ is Schur,

(*ii*) $\Omega_{in} \in (0,1)$ be such that $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur,

(*iii*) Ω_{out} be such that $\Omega_{out} > \|\Phi\|$,

(*iv*) $-1 < \frac{\mu_0^d - \mu_0^c}{\mu_0^c} < \frac{1}{\gamma L_q}$, where L_q comes from Assumption 1 and γ is defined in (19),

(*v*) M and Δ in Assumption 1 be such that $M > \frac{(2+K_1+\gamma)\Delta}{1-\gamma(\frac{1}{r}-1)L_q}$, where K_1, γ are defined in (19),

(*vi*) $l_{out} = M - \Delta$,

(*vii*) $l_{in} = \Delta_M - \Delta$, where Δ_M comes from Corollary 2.

Then, the system (2) is GES in x.

Remark 6. The first item of Theorem 1 requires, that the system is stabilizable with a certainty-equivalence controller; the second is a condition on how slow the μ^c, μ^d-subsystems have to be during the zoom-in stage; the third is a condition on how fast the μ^c, μ^d-subsystems have to be during the zoom-out stage; the forth is the bound on the mismatch between the initial conditions at the coder and decoder; the fifth is a condition on the data-rate of the channel; the sixth and seventh are the conditions on the switching parameters.

Remark 7. The fifth item of Theorem 1 (which is the condition (21) from Section 3) means that the range of the quantizer M has to be large enough compared to the quantization error Δ (i.e. the quantizer takes sufficiently many levels). Note that when the initial conditions at the coder and at the decoder are the same ($\mu_0^c = \mu_0^d$, i.e. $r = 1$), the condition on the data rate

$$M > (2 + K_1 + \gamma)\Delta \quad (25)$$

for the system (17) with $\bar{\nu}_k = \nu_k^c$ (which is the condition used in Liberzon, Nešić (2007)) can be recovered from (21). On the other hand, since (25) is a strict inequality, whenever it holds, there exists r sufficiently close to one, such that the forth and the fifth items of Theorem 1 hold (conditions (20) and (21) from Section 3). Hence, this implies that the scheme, proposed in Liberzon, Nešić (2007)

[1] These numbers always exist since $\frac{1}{\Omega_{in}}(\Phi + \Gamma K)$ is Schur

[2] Note that this is not widely used standard definition. Here we talk only about the stability of the state x of the plant, not μ. Also an overshoot may depend on the initial condition.

has some intrinsic robustness properties with respect to the mismatch between the initialization at the coder and decoder.

Remark 8. Note that the forth condition of Theorem 1 shows a relationship between a ratio (mismatch) of the initial conditions at the coder and at the decoder r, the robustness measure (gain) γ of the plant and the quantizer characteristics L_q. It shows that for a fixed L_q, when the gain is large, the smaller mismatch can be tolerated. Also when the gain is small, the large mismatch can be tolerated. Note that without loss of generality we can assume that $L_q = 1$, since many quantizers satisfy this property.

The Proof of Theorem 1 consists of Lemmas 1 and 2. These lemmas capture the dynamics of the system during two modes considered in the end of Section 3. The proof of Theorem 1 and Lemmas 1, 2 are omitted due to space limitations.

The first lemma considers the plant dynamics during **Mode 1**. It claims that if the initial conditions are such that the zoom-out is triggered initially at both the coder and decoder, then both of them will switch to the zoom-in stage in the same finite time.

Lemma 1. Consider the system (2) with (3) - (12). Suppose all conditions of Theorem 1 hold. Suppose the initial conditions are such that the zoom-out stage is triggered at both the coder and decoder **(Mode 1)**. Then there exists $k_1 > 0$ such that

$$\Omega^c_{k_1} = \Omega^d_{k_1} = \Omega_{in}.$$

Moreover,

$$k_1 \leq \left\lfloor \frac{1}{\ln(\Omega_{out}/\|\Phi\|)} \ln\left(\frac{|\xi^c_0|}{l_{in} - \Delta}\right)\right\rfloor.$$

The next lemma considers the plant dynamics during **Mode 2**. It claims that if the zoom-in stage is triggered at both the coder and decoder, then the coder and decoder will always stay in the zoom-in stage.

Lemma 2. Consider the system (2) with (3) - (12). Suppose all conditions of Theorem 1 hold. Then

$$\Omega^c_k = \Omega^d_k = \Omega_{in}$$

for all $k \geq k_1$, where k_1 comes from Lemma 1.

The proof of Theorem 1 is a direct consequence of the fact that the system during Mode 2 behaves as a cascade of ISS $x-$subsystem and GES μ^c-, μ^d- subsystems.

5. CONCLUSIONS

This paper is the first investigation of the problem of robustness of linear control systems with quantized control with respect to the computational errors at the coder and decoder. In this paper we analyze the stability of the quantized control systems when the data is transmitted via a perfect channel and the coder and decoder are initialized at different initial conditions. Using a trajectory-based scheme proposed in Liberzon, Nešić (2007), under the assumption that the channel is perfect, we derived the bound on the mismatch between the initial conditions at the coder and decoder that can be tolerated in order to achieve GES. We believe that similar results can be proven for other quantized control schemes published in the literature.

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