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Constrained Stabilization of Bilinear Discrete-Time Systems Using Polyhedral Lyapunov Functions

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Abstract: In this paper, the stabilization problem of discrete-time bilinear systems by linear state-feedback control is investigated. First, conditions guaranteeing the positive invariance of polyhedral sets with respect to nonlinear systems with second order polynomial nonlinearity are established. Then these results are used for the determination of linear state-feedback unconstrained and constrained control laws making a prespecified polyhedral set a domain of attraction of the resulting closed-loop system.

1. INTRODUCTION

Bilinear systems are systems linear in state, linear in control, but not jointly linear in state and control. Many processes in biology, socioeconomics, immunology, quantum mechanics, biomedical applications, and engineering can be naturally modeled by bilinear systems (Bruni et al. [1974], Mohler et al. [1980], and Mohler et al. [2000]). Also, many bilinear systems arise from the approximation of nonlinear systems which in most cases is much more accurate when bilinear terms are included in the expansion of the Taylor series.

The importance of this class of nonlinear systems has led to an extensive study in terms of analysis and developing control techniques. Most work deal with continuous time systems. In Khapalov et al. [1998], a piecewise-constant feedback control law was used to stabilize asymptotically a continuous-time bilinear system. This was done by introducing an auxiliary bilinear system with additional control in the drift term. Chen et al. [2000] present a bang-bang sliding mode control technique where the stability region strongly depends on the sliding function which is designed via a pole assignment based method. In Amato et al. [2007], quadratic Lyapunov functions are used for the development of a method for the determination of a linear control law making a polyhedral set a domain of attraction. The optimal quadratic cost control problem has also been studied for bilinear systems. In Benallou et al. [1988], a globally stabilizing nonlinear optimal control strategy for bilinear systems possessing rather strong properties was found. In a recent work (Ekman [2005]), a nonlinear suboptimal control was computed through an approximate solution of the Hamilton-Jacobi-Bellman equation.

Very few works dealing with the stabilization problem of discrete-time systems have been reported. Kim et al. [2002] using quadratic Lyapunov functions derive conditions for a globally stabilizing nonlinear control law for multi-input bilinear systems which, however, are assumed to be open-loop stable. Bacic et al. [2003] deal with SISO bilinear systems with constraints in the control input. A switching control technique between controllers produced from input output feedback linearization and bilinear controllers that render a polytopic set invariant and feasible was used. An extension of this work is in Liao et al. [2005].

It is not surprising that very few works dealing with the stabilization problem of discrete-time systems have been reported. This is due to the fact that quadratic functions which can be viewed as the "natural" Lyapunov functions for linear systems lead to very complex computational problems when they are applied to the stabilization problem of nonlinear discrete-time systems. For this reason, in this paper we propose the use of "polyhedral" Lyapunov functions, that is Lyapunov functions that provide polyhedral invariant sets and/or domains of attraction. This however requires the development of the appropriate theoretical background, namely the establishment of conditions guaranteeing that a polyhedral subset of the state space is positively invariant with respect to a nonlinear system with second order polynomial nonlinearity. This is the object of Section 3 of this paper. In Section 4, two problems are investigated: The first one is the derivation of linear state feedback control laws so that a prespecified subset of the state space is a domain of attraction of the resulting closed-loop system. Then this problem is also investigated when in addition hard linear constraints on the control input are imposed.

2. PROBLEM STATEMENT

Throughout the paper, capital letters denote real matrices and lower case letters denote column vectors or scalars. $\mathbb{R}^n$ denotes the real n-space and $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices. Given a real $n \times m$ matrix $A = (a_{ij})$, $A^+ = (a^+_{ij})$ and $A^- = (a^-_{ij})$ are $n \times m$ matrices with entries defined by the relations $a^+_{ij} = \max\{a_{ij}, 0\}$ and $a^-_{ij} = -\min\{a_{ij}, 0\}$. Thus, $A = A^+ - A^-$. Given a square matrix $D = (d_{ij})$, $D^\delta = (d^\delta_{ij})$ denotes the diagonal matrix with $d^\delta_{ii} = d_{ii}$ and $D^\mu = (d^\mu_{ij})$ denotes the square matrix

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with $d^{\mu}_{ii} = 0$ and $d^{\mu}_{ij} = d_{ij}$ for $i \neq j$. Thus $D = D^{\delta} + D^{\mu}$. For two $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \odot B = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}$ denotes their component-wise inner product, called the Frobenius inner product.

The inequality $A \leq B$ $(A < B)$ with $A, B \in \mathbb{R}^{n\times m}$ is equivalent to $a_{ij} \leq b_{ij} (a_{ij} < b_{ij})$. Similar notation holds for vectors.

Given a function $v(x), v : \mathbb{R}^n \to \mathbb{R}^p$ and a set $X \subseteq \mathbb{R}^n$, then $v(X) = \{y \in \mathbb{R}^p : (\exists x \in \mathbb{R}^n : v(x) = y)\}$.Finally, T denotes the time set $T = \{0, 1, 2, ...\}$.

We consider bilinear discrete-time systems described by difference equations of the form

$$x(t+1) = Ax(t) + Bu(t) + \begin{bmatrix} x^T(t)C_1 \\ x^T(t)C_2 \\ \vdots \\ x^T(t)C_n \end{bmatrix} u(t) \qquad (1)$$

where $x \in \mathbb{R}^{n}$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $t \in T$ is the time variable and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, n$.

For linear state-feedback control laws of the form

$$u(k) = Kx(t) \qquad (2)$$

with $K \in \mathbb{R}^{m\times n}$, the resulting closed-loop system is described by the equation

$$x(t+1) = (A+BK)x(t) + \begin{bmatrix} x^T(t)C_1Kx(t) \\ x^T(t)C_2Kx(t) \\ \vdots \\ x^T(t)C_nKx(t) \end{bmatrix}. \qquad (3)$$

This equation describes a nonlinear system with second order polynomial nonlinearity.

The unconstrained stabilization problem to be investigated is formulated as follows: Given system (1) and a bounded subset of the state space defined by the inequalities

$$-w_2 \leq Gx \leq w_1 \qquad (4)$$

with $G \in \mathbb{R}^{r\times n}$, $w_1 > 0, w_2 > 0$, determine a linear state-feedback control law (2) making this set a domain of attraction of the resulting closed-loop system (3).

In the constrained stabilization problem, control constraints of the form

$$-u_m \leq u(t) \leq u_M \qquad (5)$$

with $u_M > 0$, $u_m > 0$ are imposed. The problem is the determination of a linear state-feedback control law (2) so that all initial states belonging to the set defined by inequalities (4) are transferred asymptotically to the origin while the control constraints (5) are satisfied.

3. POLYHEDRAL POSITIVELY INVARIANT SETS

Given a dynamical system, a subset of its state space is said to be *positively invariant* if all trajectories starting from this set remain in it for all future instances. This property is very important for control problems with state constraints. Thus, if the state constraints define an admissible subset of the state space then a solution to the control problem under state constraints is a stabilizing linear control law making this admissible set positively invariant with respect to the resulting closed-loop system. Since in practical control problems the state constraints are usually expressed by linear inequalities, the admissible set is a polyhedron. Therefore, it is necessary to establish conditions guaranteeing positive invariance of polyhedral sets of the form (4) with respect to nonlinear systems of the form (3).

The following Lemma which provides necessary and sufficient conditions for a set defined by a nonlinear vector inequality of the form $v(x) \leq w$ to be positively invariant with respect to a nonlinear discrete-time system is very important for the development of the results of this paper.

Lemma 1. (Bitsoris et al. [1995],Bitsoris et al. [2006]). The set

$$R(v, w) \triangleq \{x \in \mathbb{R}^n : v(x) \leq w)\}$$

with $v(x)$, $v : \mathbb{R}^n \to \mathbb{R}^p$ and $w \in \mathbb{R}^p$ is a positively invariant set of system

$$x(t+1) = f(x(t)) \qquad (6)$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$, if and only if there exists a nondecreasing function $h(y), h : \mathbb{R}^p \to \mathbb{R}^p$ such that

$$v(f(x)) \leq h(v(x))$$

and

$$h(w) \leq w.$$

We shall use this result to establish conditions guaranteeing that a polyhedral set defined by

$$-\rho_2 \leq Sx \leq \rho_1$$

with $S \in \mathbb{R}^{r\times n}$, $\rho_1 > 0, \rho_2 > 0$, is positively invariant with respect to a nonlinear system with second order polynomial nonlinearity. Let

$$y_1 = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1r} \end{bmatrix} = Sx, y_2 = \begin{bmatrix} y_{21} \\ y_{21} \\ \vdots \\ y_{2r} \end{bmatrix} = -Sx \qquad (7)$$

and Y^M, Y^m $r \times r$ be matrices defined by the relations

$$Y^M = (y^M_{ij}) \text{ with } y^M_{ij} \triangleq \max(y_{1i}y_{1j}, y_{2i}y_{2j}) \qquad (8)$$

$$Y^m = (y^m_{ij}) \text{ with } y^m_{ij} \triangleq \max(y_{1i}y_{2j}, y_{2i}y_{1j}). \qquad (9)$$

Theorem 2. The polyhedral set

$$Q(S, \rho_1, \rho_2) \triangleq \{x \in \mathbb{R}^n : -\rho_2 \leq Sx \leq \rho_1\}$$

with $S \in \mathbb{R}^{r\times n}$, $rankS = n$, $\rho_1 \in \mathbb{R}^r, \rho_1 > 0, \rho_2 \in \mathbb{R}^r, \rho_2 > 0$ is positively invariant with respect to the nonlinear system

$$x(t+1) = Ax(t) + \begin{bmatrix} x^T(t)M_1x(t) \\ x^T(t)M_2x(t) \\ \vdots \\ x^T(t)M_nx(t) \end{bmatrix} \qquad (10)$$

if there exist matrices $H \in \mathbb{R}^{r\times r}$ and $D_j \in \mathbb{R}^{r\times r}$ $j = 1, 2, \ldots, r$ such that

$$SA = HS \qquad (11)$$

$$\sum_{i=1}^{n} s_{ji}M_i = S^TD_jS \qquad j = 1, 2, \ldots, r \qquad (12)$$

and

$$h(\rho) \leq \rho \qquad (13)$$

where

$$h(y) = \begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} D_1^{\delta^+} \odot Y^M + D_1^{\mu^+} \odot Y^M + D_1^{\mu^-} \odot Y^m \\ \vdots \\ D_r^{\delta^+} \odot Y^M + D_r^{\mu^+} \odot Y^M + D_r^{\mu^-} \odot Y^m \\ D_1^{\delta^-} \odot Y^M + D_1^{\mu^-} \odot Y^M + D_1^{\mu^+} \odot Y^m \\ \vdots \\ D_r^{\delta^-} \odot Y^M + D_r^{\mu^-} \odot Y^M + D_r^{\mu^+} \odot Y^m \end{bmatrix} \tag{14}$$

and

$$\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}.$$

Proof. Let

$$v(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} = \begin{bmatrix} Sx \\ -Sx \end{bmatrix}.$$

Then inequalities

$$-\rho_2 \le Sx \le \rho_1$$

can be equivalently written in the form

$$v(x) \le \rho.$$

Consequently,

$$Q(S, \rho_1, \rho_2) = R(v, \rho).$$

On the other hand,

$$v_i(x(t+1)) = (-1)^{i+1} SAx(t) + (-1)^{i+1} S \begin{bmatrix} x^T(t) M_1 x(t) \\ x^T(t) M_2 x(t) \\ \vdots \\ x^T(t) M_n x(t) \end{bmatrix}$$

and taking into account (11) and (12) we establish the relations

$$v_i(x(t+1)) = (-1)^{i+1} HSx(t) + (-1)^{i+1} \begin{bmatrix} x^T(t) S^T D_1 Sx(t) \\ x^T(t) S^T D_2 Sx(t) \\ \vdots \\ x^T(t) S^T D_r Sx(t) \end{bmatrix} \tag{15}$$

for $i = 1, 2$. Since $H = H^+ - H^-$,

$$HSx = H^+(Sx) + H^-(-Sx) \tag{16}$$

$$H(-Sx) = H^-(Sx) + H^+(-Sx) \tag{17}$$

Furthermore,

$$x^T S^T D_j Sx = x^T S^T D_j^\delta Sx + x^T S^T D_j^\mu Sx =$$

$$= x^T S^T D_j^\delta Sx + x^T S^T D_j^{\mu^+} Sx - x^T S^T D_j^{\mu^-} Sx \tag{18}$$

and

$$-x^T S^T D_j Sx =$$

$$= -x^T S^T D_j^\delta Sx - x^T S^T D_j^{\mu^+} Sx + x^T S^T D_j^{\mu^-} Sx \tag{19}$$

because

$$D_j = D_j^\delta + D_j^\mu$$

$$D_j^\mu = D_j^{\mu^+} - D_j^{\mu^-}.$$

Finally, using notation (7)-(9), from (16)-(19) it follows that

$$HSx = H^+ y_1 + H^- y_2 \tag{20}$$

$$-HSx = H^- y_1 + H^+ y_2 \tag{21}$$

$$x^T S^T D_j^\delta Sx \le x^T S^T D_j^{\delta^+} Sx = D_j^{\delta^+} \odot Y^M \tag{22}$$

$$x^T S^T D_j^\mu Sx = x^T S^T D_j^{\mu^+} Sx - x^T S^T D_j^{\mu^-} Sx \le$$

$$\le D_j^{\mu^+} \odot Y^M + D_j^{\mu^-} \odot Y^m \tag{23}$$

$$-x^T S^T D_j^\delta Sx \le x^T S^T D_j^{\delta^-} Sx = D_j^{\delta^-} \odot Y^M \tag{24}$$

$$-x^T S^T D_j^\mu Sx = -x^T S^T D_j^{\mu^+} Sx + x^T S^T D_j^{\mu^-} Sx \le$$

$$\le D_j^{\mu^+} \odot Y^m + D_j^{\mu^-} \odot Y^M \tag{25}$$

because matrices $D_j^{\delta^+}, D_j^{\delta^+}, D_j^{\mu^+}$ and $D_j^{\mu^-}$ have nonnegative elements and for a nonnegative matrix D

$$x^T S^T DSx =$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r} d_{ij}(Sx)_i(Sx)_j = \sum_{i=1}^{r}\sum_{j=1}^{r} d_{ij}(-Sx)_i(-Sx)_j \le$$

$$\le \sum_{i=1}^{r}\sum_{j=1}^{r} d_{ij} \max\{(Sx)_i(Sx)_j, (-Sx)_i(-Sx)_j\} = D \odot Y^M$$

and

$$-x^T S^T DSx$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} d_{ij}(Sx)_i(-Sx)_j = \sum_{i=1}^{r}\sum_{j=1}^{r} d_{ij}(-Sx)_i(Sx)_j \le$$

$$\le \sum_{i=1}^{r}\sum_{j=1}^{r} d_{ij} \max\{(Sx)_i(-Sx)_j, (-Sx)_i(Sx)_j\} = D \odot Y^m.$$

Thus, taking into account (20)-(25), and using (16)-(19) from (15) it follows that

$$y(t+1) \le h(y(t))$$

or, equivalently,

$$v(x(t+1)) \le h[v(x(t))]$$

with function $h(y)$ defined by (14). By construction, this function is nondecreasing. Therefore, by virtue of Lemma 1, from (13) it follows that the set $Q(S, \rho_1, \rho_2) = R(v, \rho)$ is positively invariant with respect to the nonlinear system (10).

This result can be used for solving a control problem that has been the object of much research work during the last years, namely to determine a control law that makes a given subset of the state space positively invariant. Thus, given a bilinear system (1) and a polyhedral subset $Q(S, \rho_1, \rho_2)$ of its state space, a linear control law $u(t) = Kx(t)$ that makes this subset positively invariant with respect to the resulting closed-loop (3), can be determined by solving the linear algebraic relations

$$S(A + BK) = HS$$

$$\sum_{i=1}^{n} s_{ji} C_i K = S^T D_j S \qquad j = 1, 2, \ldots, r$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} D_1^{\delta^+} \odot P^M + D_1^{\mu^+} \odot P^M + D_1^{\mu^-} \odot P^m \\ \vdots \\ D_r^{\delta^+} \odot P^M + D_r^{\mu^+} \odot P^M + D_r^{\mu^-} \odot P^m \\ D_1^{\delta^-} \odot P^m + D_1^{\mu^-} \odot P^M + D_1^{\mu^+} \odot P^m \\ \vdots \\ D_r^{\delta^-} \odot P^m + D_r^{\mu^-} \odot P^M + D_r^{\mu^+} \odot P^m \end{bmatrix} \le \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}.$$

where

$$P^M = (\rho_{ij}^M) \text{ with } \rho_{ij}^M = \max(\rho_{1i}\rho_{1j}, \rho_{2i}\rho_{2j}) \tag{26}$$

$$P^m = (\rho_{ij}^m) \text{ with } \rho_{ij}^m = \max(\rho_{1i}\rho_{2j}, \rho_{2i}\rho_{1j}). \tag{27}$$

4. STABILIZATION

The usual approach to derive domains of attraction of nonlinear systems is the application of Lyapunov's direct method. If $v^*(x)$ is a continuous positive definite function and its total time difference $\Delta v^*(x)_{(\Sigma)} = v^*(f(x)) - v^*(x)$ w.r.t system Σ is negative definite, then any set $R(v^*, a)$ with $a > 0$ is positively invariant w.r.t. this system. In addition, any subset of $R(v^*, a)$ is a domain of attraction of system Σ. Such a function is said to be a Lyapunov function of the system under consideration. Thus a control law $u(t) = Ku(t)$ is a solution to the unconstrained stabilization problem if there exist a continuous positive definite function $v(x)$ and a positive real number a such that $\Delta v^*(x)_{(3)}$ is negative definite in $R(v^*, a)$ and

$$Q(G, w_1, w_2) \subseteq R(v^*, a).$$

For linear systems a natural Lyapunov function is a quadratic one. However, this is not the case for nonlinear systems. In particular, for discrete-time nonlinear systems the use of quadratic Lyapunov functions leads to high order nonlinear algebraic problems. For example, the use of quadratic Lyapunov functions to the study of nonlinear systems with second order nonlinearity leads to a fourth order nonlinear algebraic problem. We shall show that these difficulties can be overcome by using "polyhedral" Lyapunov functions as it has been proposed for studying constrained control problems for linear systems. To this end we can use the following result:

Lemma 3. (Bitsoris [1984], Bitsoris et al. [1995]). If for a vector valued function $v(x)$, $v : \mathbb{R}^n \to \mathbb{R}^p$, the scalar function

$$v^*(x) = \max_{i=1,2,\dots,p}\{v_i(x)\}$$

is positive definite and there exist a nondecreasing function $h(y)$, $h : \mathbb{R}^p \to \mathbb{R}^p$ and a vector $\rho \in \mathbb{R}^p$, $\rho > 0$ such that

$$v(f(x)) \le h(v(x)) \tag{28}$$

and

$$h(r\rho) < r\rho \quad r \in (0, 1] \tag{29}$$

then the equilibrium $x = 0$ of system $x(t+1) = f(x(t))$ $f(0) = 0$ is asymptotically stable, $v^*(x)$ is a Lyapunov function, and $R(v, w)$ is a domain of attraction.

We shall use this result by choosing a polyhedral function $v(x)$ of the form

$$v(x) = \begin{bmatrix} Sx \\ -Sx \end{bmatrix}. \tag{30}$$

Theorem 4. The control law $u(t) = Kx(t)$ is a solution to the unconstrained stabilization problem if there exist matrices $H \in \mathbb{R}^{r\times r}$ and $D \in \mathbb{R}^{r\times r}$, $j = 1, 2, \dots, r$, $L \in \mathbb{R}^{2r\times 2r}$ with $L \ge 0$, $S \in \mathbb{R}^{r\times n}$ with $rankS = n$ and vectors $\rho_1, \rho_2 \in \mathbb{R}^r$, $\rho_1, \rho_2 > 0$ with positive components such that

$$S(A + BK) = HS \tag{31}$$

$$\sum_{i=1}^{n} s_{ji}C_iK = S^T D_j S \qquad j = 1, 2, \dots, r \tag{32}$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix}\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} + $$
$$+ \begin{bmatrix} D_1^{\delta^+} \odot P^M + D_1^{\mu^+} \odot P^M + D_1^{\mu^-} \odot P^m \\ \vdots \\ D_r^{\delta^+} \odot P^M + D_r^{\mu^+} \odot P^M + D_r^{\mu^-} \odot P^m \\ D_1^{\delta^-} \odot P^M + D_1^{\mu^-} \odot P^M + D_1^{\mu^+} \odot P^m \\ \vdots \\ D_r^{\delta^-} \odot P^M + D_r^{\mu^-} \odot P^M + D_r^{\mu^+} \odot P^m \end{bmatrix} < \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{33}$$

$$L\begin{bmatrix} G \\ -G \end{bmatrix} = \begin{bmatrix} S \\ -S \end{bmatrix} \tag{34}$$

$$L\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \le \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{35}$$

Proof. Setting $v_1(x) = Sx$, $v_2(x) = -Sx$ and following the demonstration procedure of Theorem 2 , we establish (28) with $h(y)$ given by (14) and H, D_j, $j = 1, 2, \dots, r$ being matrices satisfying (31) and (32) respectively. By construction, function $h(y)$ is nondecreasing. In addition, from (33) it follows that $h(r\rho) < r\rho$ for all $r \in (0, 1]$. Indeed, setting $h(y) = H^*y + g^*(y)$ where H^*y and $g^*(y)$ denote the linear and the nonlinear part of function $h(y)$ respectively, from (33) it follows that

$$h(r\rho) = H^*r\rho + g^*(r\rho) = rH^*\rho + r^2g^*(\rho) =$$
$$= r(H^*\rho + rg^*(\rho)) \le r(H^*\rho + g^*(\rho)) \le r\rho$$

for all $r \in (0, 1]$. On the other hand, the nonnegative scalar function

$$v^*(x) = \max\{(Sx)_1, ..., (Sx)_r, (-Sx)_1, ..., (-Sx)_r\}$$

is positive definite because from the hypothesis that $rankS = n$, it follows that $Sx = 0$ only if $x = 0$. Thus, function $v(x)$ defined by (30) and vector ρ satisfy all hypotheses of Lemma 3. Therefore, the set

$$Q(S, \rho_1, \rho_2) = R(v, \rho)$$

is a domain of attraction of the closed-loop system (3). On the other hand, according to Farkas Lemma relations (34)-(35) together with $L \ge 0$ imply the set relation

$$Q(G, w_1, w_2) \subseteq Q(S, \rho_1, \rho_2) = R(v, \rho).$$

Therefore, $Q(G, w_1, w_2)$ is a domain of attraction of the equilibrium $x = 0$ of the closed-loop system (3).

Many different approaches to the determination of a solution of the unconstrained stabilization problem can be developed using this result. Such an approach can be established by applying the next result which follows directly from Theorem 4 by setting $S = G, \rho_i = w_i \quad i = 1, 2$, $P^M = W^M$, $P^m = W^m$ where

$$W^M = (w_{ij}^M) \text{ with } w_{ij}^M = \max(w_{1i}w_{1j}, w_{2i}w_{2j})$$
$$W^m = (w_{ij}^m) \text{ with } w_{ij}^m = \max(w_{1i}w_{2j}, w_{2i}w_{1j})$$

that is, by applying Lemma 3 with

$$v(x) = \begin{bmatrix} Gx \\ -Gx \end{bmatrix}.$$

Then we obtain the following corollary of Theorem 4.

Corollary 5. The control law $u(t) = Kx(t)$ is a solution to the unconstrained stabilization problem if there exist matrices $H \in \mathbb{R}^{r\times r}$ and $D \in \mathbb{R}^{r\times r}$ $j = 1, 2, \dots, r$ and a real number ε, $0 \le \varepsilon < 1$ satisfying relations

$$G(A+BK) = HG \tag{36}$$

$$\sum_{i=1}^{n} g_{ji} C_i K = G^T D_j G \qquad j = 1, 2, \ldots, r \tag{37}$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \\ + \begin{bmatrix} D_1^{\delta^+} \odot W^M + D_1^{\mu^+} \odot W^M + D_1^{\mu^-} \odot W^m \\ \vdots \\ D_r^{\delta^+} \odot W^M + D_r^{\mu^+} \odot W^M + D_r^{\mu^-} \odot W^m \\ D_1^{\delta^-} \odot W^M + D_1^{\mu^-} \odot W^M + D_1^{\mu^+} \odot W^m \\ \vdots \\ D_r^{\delta^-} \odot W^M + D_r^{\mu^-} \odot W^M + D_r^{\mu^+} \odot W^m \end{bmatrix} \leq \varepsilon \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{38}$$

According to this result, a stabilizing control law $u(t) = Kx(t)$ can be obtained by applying any standard method for the determination of a solution to the set of linear algebraic relations (36)-(38). For example, a solution can be determined by applying a linear programming algorithm to solve the optimization problem with performance index:

$$\min_{K,H,D_1,\ldots,D_r,\varepsilon} \{\varepsilon\} \tag{39}$$

under linear constraints (36)-(38). If the positive optimal value ε^* is less than one, then the corresponding control law is a solution to the stabilization problem. It should be noticed that from (36)-(38) it follows that

$$v^*(x(t+1)) \leq \varepsilon v^*(x(t))$$

where $v^*(x)$ is the positive definite function

$$v^*(x) = \max \left\{ \frac{(Gx)_1}{w_1}, \ldots, \frac{(Gx)_r}{w_r}, \frac{(-Gx)_1}{w_1}, \ldots, \frac{(-Gx)_r}{w_r} \right\}$$

Therefore, minimization of ε results to a faster transient behavior of the system.

Let us now consider the case where control constraints of the form (5) are imposed. It is known from Bitsoris et al. [1995] that a linear control law $u(t) = Kx(t)$ is a solution to the constrained control problem if and only if there exists a subset Ω of the state space which is both a positively invariant set and a domain of attraction of the resulting closed-loop system and satisfies the set relation

$$Q(G, w_1, w_2) \subseteq \Omega \subseteq Q(K, u_M, u_m). \tag{40}$$

Many different approaches for the determination of such a control law can be developed by combining this result with those concerning the positive invariance of polyhedral sets. An interesting special case is when $Q(G, w_1, w_2) = \Omega$, that is when the stabilizing linear control law $u(t) = Kx(t)$ renders the desired domain of attraction positively invariant w.r.t. the closed-loop system. Then the set relation (40) becomes

$$Q(G, w_1, w_2) \subseteq Q(K, u_M, u_m)$$

and is equivalent to the existence of a nonnegative matrix $L \in \mathbb{R}^{2m \times 2r}$ such that

$$L \begin{bmatrix} G \\ -G \end{bmatrix} = \begin{bmatrix} K \\ -K \end{bmatrix} \tag{41}$$

$$L \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \leq \begin{bmatrix} u_M \\ u_m \end{bmatrix}. \tag{42}$$

Combining these relations with the conditions of positive invariance and attractivity of the set $Q(G, w_1, w_2)$ stated in Corollary 5, we establish the following result:

Theorem 6. The control law $u(t) = Kx(t)$ is a solution to the constrained stabilization problem if there exist matrices $H \in \mathbb{R}^{r \times r}$, $D_j \in \mathbb{R}^{r \times r}$ $j = 1, 2, \ldots, r$, $L \in \mathbb{R}^{2m \times 2r}$, $L \geq 0$ such that (36)-(38) and (41)-(42) are satisfied.

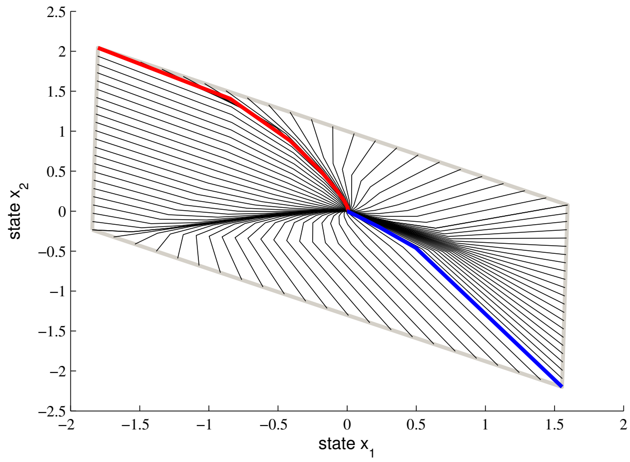

Fig. 1. Phase portrait of the closed loop system for initial states belonging in $Q(G, w_1, w_2)$.

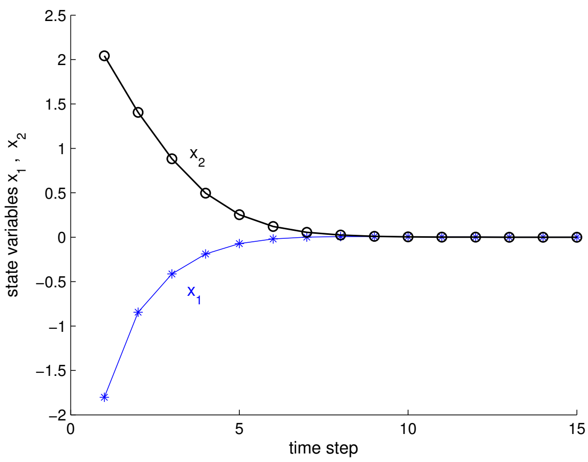

Fig. 2. State response of the closed-loop system for initial state $x_0 = [-1.80 \ 2.05]^T$.

5. NUMERICAL EXAMPLE

We consider a bilinear system (1) with

$$A = \begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.45 \\ 0.45 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}$$

The desired domain of attraction is a polyhedral set $Q(G, w_1, w_2)$ with

$$G = \begin{bmatrix} 2.28 & -0.04 \\ -1.62 & -2.79 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 3.63 \\ 3.63 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 4.20 \\ 2.80 \end{bmatrix}$$

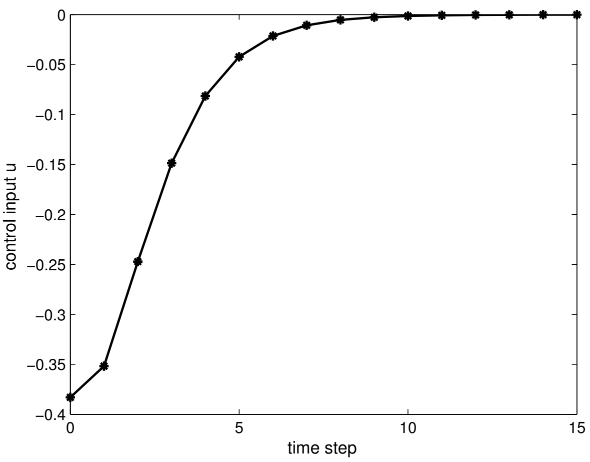

Fig. 3. Control input of the closed loop system when $x_0 = [-1.80\ 2.05]^T$.

The control input must satisfy physical constraints

$$-u_m \leq u \leq u_M$$

with $u_M = u_m = 0.5$.

The optimal values resulting from the linear programming problem with performance index (39) and constraints (37)-(38),(41)-(42) are $\varepsilon^* = 0.96$ and

$$K^* = [\,-0.22\ \ -0.38\,]\,.$$

Since $\varepsilon^* < 1$, the control law

$$u(t) = -0.22x_1(t) - 0.38x_2(t)$$

is a solution to the constrained control problem. In (Fig. 1) the phase portrait of the closed loop system for states belonging to the domain of attraction is shown. The bold curves are trajectories starting from $x_0 = [-1.8\ 2.05]^T$ and $x_0 = [1.55\ -2.20]^T$. In (Fig. 2), the time response of states x_1 and x_2 for $x_0 = [-1.8\ 2.05]^T$ is shown. Finally, in (Fig. 3), the control input for the closed-loop system when the initial state vector is $x_0 = [-1.8\ 2.05]^T$ is shown.

6. CONCLUSION

A new approach to the constrained and unconstrained stabilization of discrete-time bilinear systems by linear state-feedback has been presented. In contrast to all known Lyapunov oriented methods which are based on quadratic functions, in this paper "polyhedral" Lyapunov functions have been used. The first step to this direction has been the development of the necessary theoretical background, namely the establishment of conditions guaranteeing the positive invariance of polyhedral sets w.r.t to nonlinear systems with second order polynomial nonlinearity. Using known results on the connection between comparison systems and positively invariant sets (Bitsoris et al. [1995]), it has been shown that a polyhedral set is positively invariant w.r.t this class of nonlinear systems if an associated linear algebraic problem is feasible. Then the stabilization problem of bilinear systems is investigated. In Theorem 4 conditions for a linear state-feedback control law to be a solution to the stabilization problem have been developed. The result stated in Corollary 5 which reduces the determination of a stabilizing control law to a linear programming problem is just one of many different design approaches that can be developed using the general result stated in Theorem 4.

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