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Contractive distributed MPC for consensus in networks of single- and double-integrators ⋆

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Abstract: In this paper we propose an application of distributed model predictive control techniques to the problem of driving a group of autonomous agents towards a consensus point, *i.e.* a negotiated position in their state space. Agents are assumed to be governed by discrete-time single- or double-integrator dynamics and the communication network can be directed and time-varying. Our control protocols are called "contractive" due to a specific constraint imposed on the agents' state path. Consensus is formally proven, also in presence of bounds on the norm of the inputs, by means of a geometrical analysis of the optimal paths.

Keywords: Multi-agent systems, Model predictive control, Consensus problems, Decentralized control.

1. INTRODUCTION

The topic of cooperative control of multiple agents has gained considerable attention over the last years. In this paper we deal with a specific coordination task, called *consensus problem*, whose objective is to obtain the convergence of the states of a group of autonomous agents to a common value by means of suitable control laws. Consensus problems were historically faced in computer science (Lynch [1996]) and recently have received much attention in control engineering, due to their impact in many applicative contexts, *e.g.* unmanned autonomous vehicles and sensor networks (see Olfati-Saber [2006], Olfati-Saber et al. [2007], Ren et al. [2007] and the references therein). Many control techniques have been applied to solve this problem in presence of various models of the agents' dynamics and the communication network, see *e.g.* Moreau [2005], Olfati-Saber and Murray [2004], Tanner et al. [2003a], Tanner et al. [2003b], Cortes et al. [2006], Ferrari-Trecate et al. [2006], Bauso et al. [2006]. Most of them do not exploit optimal control ideas and, with the exception of Moreau [2005] and Cortes et al. [2006], do not account for input constraints, which in many cases have to be included in the problem formulation due to actuators limitations.

In Ferrari-Trecate et al. [2007a] we proposed an innovative solution for consensus in networks of *single-integrators*, based on Model Predictive Control (MPC). This method can be applied in a distributed fashion to the control of a group of agents by letting each agent solve, at each step, a Constrained Finite-Time Optimal Control (CFTOC) problem involving the state of neighboring agents. Moreover, following the so-called Receding-Horizon principle, at each time step the controller only applies the first input of the computed control sequence. An advantage of MPC is the built-in capability to handle control and state constraints. The MPC scheme proposed in Ferrari-Trecate et al. [2007a] applies to time-varying and undirected communication graphs, and in Ferrari-Trecate et al. [2007b] these results were extended to the case of directed graphs.

In this paper we propose alternative distributed MPC schemes for consensus. The main advantage with respect to the techniques proposed in Ferrari-Trecate et al. [2007a,b] is that the new schemes comprise also the case of networks of *double-integrators* with time-varying and directed communication graphs. Our control laws require periodical communication among agents and use a peculiar state constraint, which is called "contractive" because it mimics, in a multi-agent system domain, the state constraint proposed in De Oliveira and Morari [2000] for the control of nonlinear systems. In our schemes, because of the Receding-Horizon technique, the value of the consensus point to which agents' states converge depends on the sequence of agents' states and the communication network along time. Since the global cost to be minimized by our MPC algorithms is not monotonically decreasing, it cannot be used as a Lyapunov function for studying closed-loop

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stability, as it is done for example in Mayne et al. [2000] and De Nicolao et al. [2000]. Rather, we exploit geometric properties of the optimal path followed by individual agents and rely on the results exposed in Moreau [2005] for analyzing consensus.

The paper is organized as follows: in Section 2 we define a model of the communication network and summarize some key results on convergence in multi-agent systems presented in Moreau [2005]. Section 3 describes our contractive MPC solution for networks of single-integrators. In Section 4 its generalization to the case of double-integrators is presented. Simulation examples illustrate the results obtained by applying the presented control laws. Section 5 is devoted to conclusions.

2. BASIC NOTIONS AND PRELIMINARY RESULTS

We consider a set of n agents moving in a d-dimensional Euclidean space; the uncontrolled dynamics of each agent is described by a discrete-time single- or double-integrator model.
The communication network is represented by a directed graph (or digraph) $G = (\mathcal{N}_G, \mathcal{E}_G)$, where $\mathcal{N}_G = \{1, \dots, n\}$ is the set of nodes indexing individual agents and $\mathcal{E}_G \subseteq \{(i,j) : i,j \in \mathcal{N}_G, j \neq i\}$ is the set of edges. If, $\forall i,j \in \mathcal{N}_G$, $(i,j) \in \mathcal{E}_G$, the graph is *complete*. G is *undirected* if $(i,j) \in \mathcal{E}_G \Leftrightarrow (j,i) \in \mathcal{E}_G$. The *adjacency matrix* defined on G is the $n \times n$ matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } \exists (j,i) \in \mathcal{E}_G, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

A node $i \in \mathcal{N}_G$ is *connected* to a node $j \in \mathcal{N}_G \backslash \{i\}$ if there is a path from i to j in the graph following the orientation of the arcs. The graph G is *strongly connected* if, $\forall (i,j) \in \mathcal{N}_G \times \mathcal{N}_G$, i is connected to j. The creation and loss of communication links can be modeled by means of a time-dependent collection of graphs $\{G(k) = (\mathcal{N}_G, \mathcal{E}_G(k)), k \in \mathbb{N}\}$.

Definition 1. [Jadbabaie et al. [2003]] A collection of graphs $\{G(1), \dots, G(m)\}$ is *jointly connected* if $\cup_{k=1}^m G(i) = (\mathcal{N}_G, \cup_{k=1}^m \mathcal{E}_G(i))$ is strongly connected. The agents are *linked together* across the interval $[l,m]$, $l \leq m \leq +\infty$ if the collection of graphs $\{G(k), k = l, \dots, m\}$ is jointly connected. A node i is connected to all other nodes across a time interval $\mathcal{T} \subseteq \mathbb{N}$ if i is connected to all other nodes in the directed graph $(\mathcal{N}_G, \bigcup_{k\in\mathcal{T}} \mathcal{E}_G(k))$.

If $(j,i) \in \mathcal{E}_G$ we say that j is a *neighbor* to i and the j-th agent transmits instantaneously its state to the i-th agent. The set of neighbors to the node $i \in \mathcal{N}_G$ is $\mathcal{N}_i(G) = \{j \in \mathcal{N}_G : (j,i) \in \mathcal{E}_G\}$ and $|\mathcal{N}_i|$ is the valency of the i-th node. The *valency matrix* is $V(G) = \text{diag}\{|\mathcal{N}_1(G)|, \dots, |\mathcal{N}_n(G)|\}$. We introduce the matrix

$$\tilde{K}(G) = [V(G) + I_n]^{-1}[I_n + A(G)].$$

$\tilde{K}(G)$ is a stochastic matrix (*i.e.* it is square and nonnegative and its row sums are equal to 1, see Jadbabaie et al. [2003]), whose entry (i,j) is non null if and only if $i = j$ or $(j,i) \in \mathcal{E}_G$. It can be shown by the Gershgorin's disc theorem that all eigenvalues of $\tilde{K}(G)$ that are not unitary are within the open unit circle. Notably, if 1 is a simple eigenvalue of $\tilde{K}(G)$ and the others have modulus less than one, it results $\lim_{k\to+\infty} \tilde{K}(G)^k = \mathbf{1}\nu^T$, where ν is a column vector. This implies that, given the discrete-time system

$$x(k+1) = \tilde{K}(G)x(k)$$

with $x(\cdot) = [x_1(\cdot) \ \cdots x_n(\cdot)]$, one has $\lim_{k\to+\infty} \|x_i(k) - x_j(k)\| = 0, \forall i,j \in \{1, \dots, n\}$. Moreover, the Perron-Frobenius Theorem states that, if a stochastic matrix has a single eigenvalue in 1, the graph corresponding to the matrix is strongly connected.
We also define

$$K(G) = \tilde{K}(G) \otimes I_d$$

where $\otimes$ denotes the Kronecker product and I_d is the identity matrix of order d. The matrix $K(G)$ is stochastic as well and inherits the spectral properties of matrix $\tilde{K}(G)$ up to eigenvalue multiplicity. In the sequel, we also use the symbol $\|\cdot\|$ to denote the Euclidean norm.

For sake of completeness, now we summarize some results provided in Moreau [2005] that will enable us to prove consensus under the MPC schemes we will propose in the sequel. Assume that agents obey to the general closed-loop dynamics

$$x(k+1) = f(k, x(k)) \tag{2}$$

where $x(k) = [x_1(k)^T \cdots x_n(k)^T]^T$ and $x_i \in \mathbb{R}^d$, $\forall i \in \mathcal{N}_G$.

The nodes of the network have reached *consensus* if and only if $x_i = x_j, \forall i,j \in \mathcal{E}_G, i \neq j$. The corresponding state value is called *consensus point*. Let $\Phi \neq \emptyset$ be the set of equilibria for (2).

Definition 2. [Moreau [2005]] System (2) is globally attractive w.r.t. Φ if for each $\phi_1 \in \Phi$, $\forall c_1, c_2 > 0$ and $\forall k_0 \in \mathbb{N}$, $\exists T \geq 0$ such that every solution ζ to (2) has the following property:

$$|\zeta(k_0) - \phi_1| < c_1 \Rightarrow \exists \phi_2 \in \Phi : |\zeta(k) - \phi_2| < c_2, \forall k \geq k_0 + T.$$

The system is uniformly globally attractive w.r.t. Φ if it is globally attractive w.r.t. Φ and the constant T can be chosen independently of k_0.

Definition 3. The multi-agent system (2) asymptotically reaches consensus if it is (uniformly) globally attractive w.r.t. $\Phi = \{x \in \mathbb{R}^d : x_1 = x_2 = \cdots = x_n\}$.

The consensus results stated in Moreau [2005] hinge on the following assumption.

Assumption 1. For every graph $G(k)$, agent $i \in \mathcal{N}_G$ and state $x \in X^n, X \subseteq \mathbb{R}^d$, there is a compact set $e_i(G(k))(x) \subseteq X$ such that:

1. $f_i(x,k) \in e_i(G(k))(x), \forall k \in \mathbb{N}, \forall x \in X^n$;
2. $e_i(G(k))(x) = \{x_i\}$ if $x_i = x_j, \forall j \in \mathcal{N}_i(G(k))$;
3. whenever the states of agent i and agents $j \in \mathcal{N}_i(G(k))$ are not all equal,
   $e_i(G(k))(x) \in \text{Ri}(\text{Co}(\{x_i(k)\} \cup \{x_j(k), j \in \mathcal{N}_i(G(k))\}))$
   where $\text{Co}(A)$ and $\text{Ri}(A)$ denote the convex hull and the relative interior, respectively, of the set A;
4. the set-valued function $e_i(G(k))(x) : X^n \mapsto 2^X$ is continuous (2^X is the power set of X).

We are now in a position to state the main Theorems on consensus we will use.

Theorem 1. [Moreau [2005]] Let $\{G(k) = (\mathcal{N}_G, \mathcal{E}_G(k))$, $k \in \mathbb{N}\}$ be a collection of directed graphs and assume that

f in (2) verifies Assumption 1. Then, the system (2) is uniformly globally attractive with respect to the collection of equilibrium solutions $x_1 = x_2 = \cdots = x_n$ if and only if there exists a non-negative integer $T \geq 0$ such that, $\forall k_0 \in \mathbb{N}$, there is a node connected to all other nodes across $[k_0, k_0 + T]$.

Theorem 2. [Moreau [2005]] Let $\{G(k) = (\mathcal{N}_G, \mathcal{E}_G(k)),\ k \in \mathbb{N}\}$ be a collection of undirected graphs and assume that f in (2) verifies Assumption 1. Then, the system (2) is globally attractive with respect to the collection of equilibrium solutions $x_1 = x_2 = \cdots = x_n$ if and only if, $\forall k_0 \in \mathbb{N}$, all agents are linked together across the interval $[k_0, +\infty)$.

3. CONSENSUS FOR AGENTS WITH SINGLE-INTEGRATOR DYNAMICS

In this Section we consider a system of n agents where each agent's dynamics is described by the following discrete-time single-integrator model:

$$x_i(k+1) = x_i(k) + u_i(k) \tag{3}$$

with initial condition $x_i(0) = \tilde{x}_i$, $i = 1, \ldots, n$. The vectors $x_i(k) \in \mathbb{R}^d$ and $u_i(k) \in \mathbb{R}^d$ are the state and the control input, respectively, of agent i at time $k \in \mathbb{N}$.

Let $N \geq 1$ denote the length of the prediction horizon. We associate to the i-th agent, whose dynamics is given by (3), the input vector $U_i(k) = [u_i^T(k) \cdots u_i^T(k+N-1)]^T$ and the cost

$$J_i(x(k), U_i(k)) = J_i^x(x(k), U_i(k)) + J_i^u(U_i(k)) \tag{4}$$

where

$$J_i^x(x(k), U_i(k)) = q_i \sum_{j=1}^{N} \|x_i(k+j) - z_i(p(k)N)\|^2 \tag{5}$$

$$J_i^u(U_i(k)) = r_i \sum_{j=0}^{N-1} \|u_i(k+j)\|^2 \tag{6}$$

where $q_i, r_i > 0$ are weights, $x(k) = [x_1^T(k) \cdots x_n^T(k)]^T$ is the state of the multi-agent system at the beginning of the prediction horizon, $z_i(k) \doteq K_i(G(p(k)))x(k)$ and $K_i(G(p(k)))$ is the i-th block of the matrix $K(G(p(k)))$, partitioned as $[K_1^T(G(p(k))) \cdots K_n^T(G(p(k)))]^T$, with $K_i(G(p(k))) \in \mathbb{R}^{d \times dn}$, $i = 1, ..., n$. Notice that $z_i(k)$ is just the barycenter of $\{x_i(k)\} \cup \{x_j(k), j \in \mathcal{N}_i(k)\}$. $G(p(k)N)$ is the communication graph at time $p(k)N$, where

$$p(k) = \max_{\substack{\lambda \in \mathbb{N} \\ \lambda N \leq k}} \lambda$$

The function $p(k)$ is illustrated in Fig. 1 for $N = 4$.

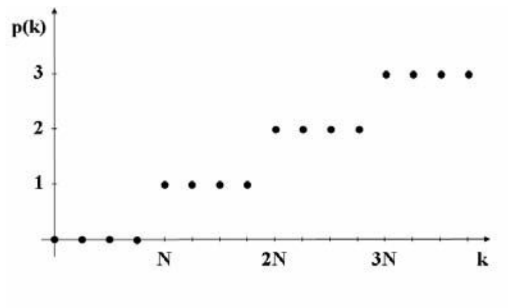

Fig. 1. Function $p(k)$ for N=4.

We associate to $k \in \mathbb{N}$ the *p-interval* $P(k) = \{p(k)N, p(k)N+1, \ldots, (p(k)+1)N-1\}$. Note that $p(k)$ is constant over each p-interval.

Now consider the following CFTOC problem for agent $i \in \mathcal{N}_G$:

$$\min_{U_i(k)} J_i(x(k), U_i(k)) \tag{7}$$

subject to the following constraints:

(A) the agent dynamics (3);

(B) the input constraint
$$\|u_i(k+j)\| \leq u_{i,max}, \tag{8}$$
with $u_{i,max} > 0, \forall i \in \mathcal{N}_G, \forall j \in \{0, \ldots, (N-1)\}$;

(C) the state constraint
$$x_i((p(k)+1)N) = z_i(p(k)N). \tag{9}$$

Constraint (9) is such that:

- it is defined on the multi-agent system state at the first time instant after the end of $P(k)$;
- it changes when the current time k switches from a p-interval to the next one;
- it is the same for all $k \in P(k)$; consequently, while k approaches the end of a p-interval, the difference between the next time at which the state is constrained and k decreases.

Therefore (9) can be interpreted as a "contractive" constraint because, under suitable conditions on the communication graph, it forces the reduction of the convex hull spanned by agents' states, as it will be shown in the sequel.

Remark 1. For $p(k)N < k < (p(k)+1)N$, the cost (4) is independent of the graph $G(k)$ and the states $x_i(k)$, $i \in \mathcal{N}_i(k)$. This implies that agents are required to transmit their state to neighbors only at times lN, $l \in \mathbb{N}$.

Remark 2. Note that if at time lN, $l \in \mathbb{N}$ the CFTOC problem (7) is feasible, then it is feasible at times $lN + j$, $\forall j \in \{1, \ldots, (N-1)\}$. Indeed let $X_i^o(lN) = [x_i(lN)^T\ x_i^o(lN+1)^T\ \cdots\ x_i^o(lN+N)^T]^T$ collect the states produced by the input sequence $U_i^o(x(lN))$. $U_i^o(lN)$ steers $x_i(lN)$ into $x_i(lN+N) = z_i(lN)$ in order to fulfill constraint (C). Hence, at time $k = lN + j$ the sequence

$$U_i(lN+j) = [u_i^o(lN+j) \cdots\ u_i^o(lN+N-1)\ 0 \cdots 0]$$

is feasible (since it steers $x_i^o(lN+j)$ into $x_i^o(lN+N) = K_i(G(lN))x(lN))$ and optimal (by Bellman's principle). This means that in the nominal case, problem (7) needs to be solved just at times lN, $l \in \mathbb{N}$. However, computing the control law in a receding-horizon fashion enhances the robustness properties of the control scheme against unmodeled perturbations.

Optimal inputs will be denoted with

$$U_i^o(x(k)) = [u_i^{oT}(k) \cdots u_i^{oT}(k+N-1)]^T$$

and we will investigate the consensus properties provided by the Receding-Horizon control law

$$u_i^{RH}(k) = \kappa_i^{RH}(x(k)),\ \kappa_i^{RH}(x(k)) = u_i^o(k). \tag{10}$$

The following result shows that an appropriate choice of the prediction horizon N always ensures the feasibility of the control problem:

Lemma 1. The CFTOC problem (7) with constraints (A), (B), (C) is feasible at all times if $N \geq \max_{i \in \mathcal{N}_G} N_i$ where

$$N_i = \left\lceil \frac{\max_{j\in\mathcal{N}_G} \|x_i(0) - x_j(0)\|}{u_{i,max}} \right\rceil$$

and $\lceil \xi \rceil$ denotes the least integer upper bound to $\xi \in \mathbb{R}^+$.

The proof of this result, that is easy but rather long, is omitted. From now on we will assume that the adopted value of N guarantees the feasibility of (7) at all times. We are now in a position to state the main results of this Section.

Theorem 3. Given the collection of directed graphs $G_S(k_0) = \{G((p(k_0)+j)N), j = 0, \dots, +\infty\}$ defined over a common node set $\mathcal{N}_G = 1, \dots, n$, the closed-loop multi-agent system given by (3) and (10) asymptotically reaches consensus if and only if there exists a non-negative integer $\tilde{T} \geq 0$ such that, $\forall k_0 \in \mathbb{N}$, there is a node connected to all the others across $\{p(k_0)N \;\; (p(k_0)+1)N \;\; \cdots \;\; (p(k_0)+\tilde{T})N\}$, with $\tilde{T} \in \mathbb{N}$.

Theorem 4. Given the collection of undirected graphs $G_S(k_0) = \{G((p(k_0)+j)N), j = 0, \dots, +\infty\}$ defined over a common node set $\mathcal{N}_G = 1, \dots, n$, the closed-loop multi-agent system given by (3) and (10) asymptotically reaches consensus if and only if, $\forall k_0 \in \mathbb{N}$, $G_S(k_0)$ is jointly connected.

Due to space limitations, we just sketch the technique used for proving Theorems 3 and 4. First, one can show that the sequence of contractive constraints, thanks to the spectral properties of matrix $K(\cdot)$, fulfills Assumption 1. Furthermore, by a geometrical analysis of the optimal state path followed by the agents, it can be proven that Assumption 1 is fulfilled also within each P-interval. Thanks to these results, the rest of the proof is a straightforward application of Theorems 1 and 2, respectively.

Remark 3. If there exists $k \in \mathbb{N}$ such that $G(p(k)N)$ is complete, consensus is reached at time $(p(k)+1)N$. In fact, in this case, $\forall i, j \in \mathcal{N}_G$

$$K_i(G(p(k)N))x(p(k)N) = K_j(G(p(k)N))x(p(k)N)$$

as it can be readily seen from the definition of the matrices $K_i(G)$.

Example 1. We consider a set of $n = 5$ agents moving in a bidimensional space, with initial states $x_1(0) = [-30\ 30]^T$, $x_2(0) = [-25\ 35]^T$, $x_3(0) = [65\ \ -75]^T$, $x_4(0) = [70\ \ -68]^T$, $x_5(0) = [100\ \ -25]^T$. The prediction horizon is $N = 10$. The weights in the cost function (4) are $q_i = 1, i = 1, \dots, 5$, $r_1 = 100$, $r_i = 1, i = 2, \dots, 5$. The communication network is described by the time-invariant undirected graph represented in Fig. 2, that corresponds to the following matrix:

$$\tilde{K}(G) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{11}$$

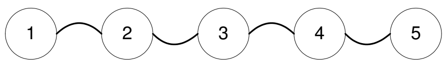

Fig. 2. Communication network.

The input constraints (8) are given by $u_{i,max} = 100$, $i = 1, \dots, 5$. Simulations depicted in Figures 3-4 confirm the expected tendency of agents to consensus. Notice that the contractive constraint (9) forces the agents to reach states which are independent of the weights r_i and q_i.

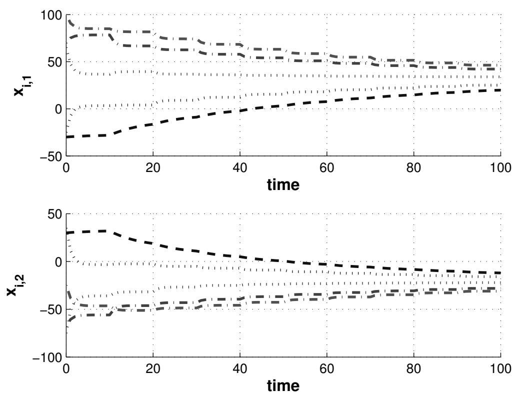

Fig. 3. Example 1: Evolution of agents' states. Dashed line: path followed by agent 1.

In this example input constraints are not active. The cost functions relative to the single agents and the global cost function have sudden growths in correspondence of the beginning of each p-interval, due to the changes in $K(G(p(\cdot)N))$.

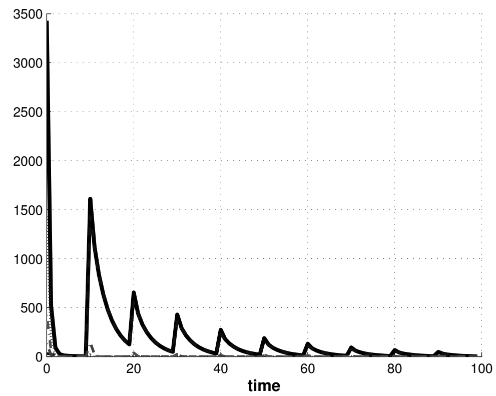

Fig. 4. Example 1: Optimal cost of individual agents $J_i(x^o(k), U_i^o(k))$ and optimal global cost $J^o(x^o(k), U^o(k)) = \sum_{i=1}^{5} J_i^o(x^o(k), U_i^o(k))$ (continuous line).

Example 2. We consider the setting of Example 1, with the difference that now the input constraint is $u_{i,max} = 20$, $\forall i \in \mathcal{N}_G$. Moreover, the collection of graphs $G_S(0)$ is structured as follows:

$$G_S(0) = \{\underbrace{G_{S1}}_{1}\ G_{S2}\ \underbrace{G_{S1}\ G_{S1}}_{2}\ G_{S2}\ \underbrace{G_{S1}\ G_{S1}\ G_{S1}}_{3}\ G_{S2}\ \cdots\} \tag{12}$$

where

$$\tilde{K}(G_{S1}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \tilde{K}(G_{S2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$G_S(0)$ is a collection of directed graphs, because G_{S1} is directed, and does not fulfill the assumptions of Theorem 3. In fact the non-connected communication graph G_{S1} is active over time intervals of increasing length. Therefore asymptotic consensus is not achieved (see Fig. 5).

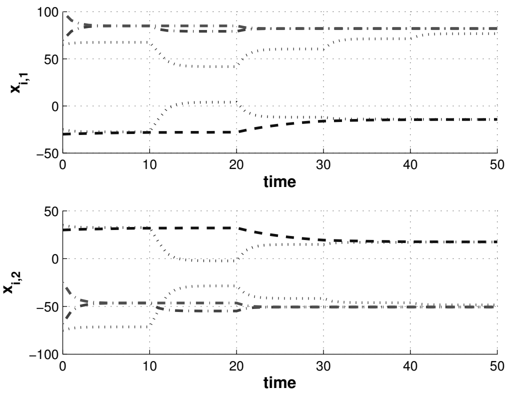

Fig. 5. Example 2: Evolution of agents' states.

4. CONSENSUS FOR AGENTS WITH DOUBLE-INTEGRATOR DYNAMICS

We now consider a set of n agents with discrete-time dynamics

$$x_i(k+1) = x_i(k) + v_i(k) \tag{13a}$$
$$v_i(k+1) = v_i(k) + u_i(k) \tag{13b}$$

$\forall i = 1, \dots, n$ and with initial conditions $x_i(0) = \tilde{x}_i$, $v_i(0) = \tilde{v}_i$. The vectors $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$ and $u_i \in \mathbb{R}^d$ can be used, for example, to describe position, velocity and control input, respectively, of vehicle i moving in a d-dimensional space. Next, we present an extension of the contractive technique proposed in Section 3 for guaranteeing consensus.

Define the following cost:

$$J_i(x(k), v(k), U_i(k)) = J_i^x(x(k), v(k), U_i(k)) + J_i^v(v(k), U_i(k)) + J_i^u(U_i(k)) \tag{14}$$

where

$$J_i^x(x(k), v(k), U_i(k)) = q_{ix} \sum_{j=2}^{N+1} \|x_i(k+j) - z_i(p(k)N)\|^2 \tag{15}$$

$$J_i^v(v(k), U_i(k)) = q_{iv} \sum_{j=1}^{N} \|v_i(k+j)\|^2 \tag{16}$$

$$J_i^u(U_i(k)) = r_i \sum_{j=0}^{N-1} \|u_i(k+j)\|^2 \tag{17}$$

and $q_{ix}, q_{iv}, r_i > 0$. We consider the optimization problem

$$\min_{U_i(k)} J_i(x(k), v(k), U_i(k)) \tag{18}$$

subject to the following constraints:

(I) the agent dynamics (13);
(II) the input constraint $\|u_i(k+j)\| \leq u_{i,max},\ u_{i,max} > 0, \forall i \in \mathcal{N}_G, \forall j \in \{0, \dots, (N-1)\}$;
(III) the contractive state constraints

$$x_i((p(k)+1)N) = z_i(p(k)N) \tag{19a}$$
$$v_i((p(k)+1)N) = 0. \tag{19b}$$

Due to the double-integrator dynamics of the agents, we require that $N \geq 2$ in order to guarantee the reachability of system (13). The constraints (19a) and (19b) imply that, during its movement, each agent is periodically forced to arrive to the position given by (19a) with zero velocity.

Denoting with $U_i^o(x(k)) = [u_i^{oT}(k) \cdots u_i^{oT}(k+N-1)]^T$ the optimal control sequence, the corresponding control law takes the form

$$u_i^{RH}(k) = \kappa_i^{RH}(x(k), v(k)),\ \kappa_i^{RH}(x(k), v(k)) = u_i^o(k). \tag{20}$$

It can be shown that by choosing N big enough, problem (18) is feasible at all time instants, as we will assume from here on.

It is now possible to state the main results on consensus of this Section:

Theorem 5. Given the collection of directed graphs $G_S(k_0) = \{G((p(k_0)+j)N), j = 0, \dots, +\infty\}$ defined over a node set $\mathcal{N}_G = 1, \dots, n$, the closed-loop multi-agent system given by (13) and (20) asymptotically reaches consensus if and only if there exists a non-negative integer $\tilde{T} \geq 0$ such that, $\forall k_0 \in \mathbb{N}$, there is a node connected to all the others across $\{p(k_0)N\ (p(k_0)+1)N\ \cdots\ (p(k_0)+\tilde{T})N\}$, with $\tilde{T} \in \mathbb{N}$.

Theorem 6. Given the collection of undirected graphs $G_S(k_0) = \{G((p(k_0)+j)N), j = 0, \dots, +\infty\}$ defined over a node set $\mathcal{N}_G = 1, \dots, n$, the closed-loop multi-agent system given by (13) and (20) asymptotically reaches consensus if and only if, $\forall k_0 \in \mathbb{N}$, $G_S(k_0)$ is jointly connected.

Proofs of Theorems 5 and 6, which are omitted, are based on a geometrical decomposition of the optimal control problem and on the properties of the sequence of contractive constraints (19a), which fulfill Assumption 1.

Remark 4. Also in this case, if there exists $k \in \mathbb{N}$ such that $G(p(k)N)$ is complete, consensus is reached at time $(p(k)+1)N$.

Example 3. The case we consider includes five agents moving in the plane with the following initial conditions: $x_1(0) = [-30\ 30]^T$, $v_1(0) = [-50\ 50]^T$. $x_2(0) = [-25\ 35]^T$, $v_2(0) = [5\ 15]^T$. $x_3(0) = [65\ \ -75]^T$, $v_3(0) = [-65\ \ -5]^T$. $x_4(0) = [70\ \ -68]^T$, $v_4(0) = [-10\ 48]^T$. $x_5(0) = [100\ \ -25]^T$, $v_5(0) = [20\ 85]^T$. The prediction horizon is $N = 20$. The weights in the cost function (14) are $q_i = 1$, $r_i = 1$, $i = 1, \dots, 5$ and the input constraint is $u_{i,max} = 200$, $i = 1, \dots, 5$. We consider a time-invariant undirected graph collection $G_S(\cdot)$ described by the matrix (11). The simulation results depicted in Fig. 6 confirm asymptotic consensus.

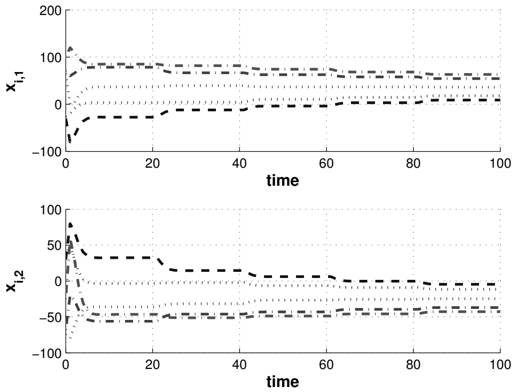

Fig. 6. Example 3: Evolution of agents' states.

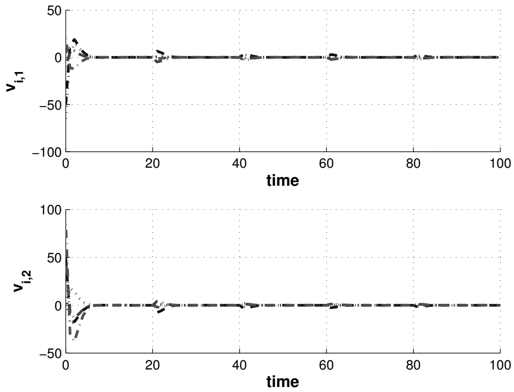

Fig. 7. Example 3: Evolution of agents' velocity.

5. CONCLUSIONS

We have proposed MPC control schemes capable of guaranteeing consensus in a multi-agent system where individual dynamics are described by single- and double-integrator models. The proof of consensus, which holds under suitable assumptions on the communication graph, relies on the convergence results presented in Moreau [2005], that are applicable because of the particular properties of optimal state trajectories.

The control techniques proposed in this paper can be blended to obtain more complex group behaviors in multiagent systems. This can be shown by an example. Take n agents with double-integrator dynamics, moving in a three-dimensional space. We apply to these agents:

- the contractive single-integrator solution (Section 3) to obtain consensus on two components of the velocity variables;
- the contractive double-integrator solution presented in Section 4 to obtain consensus on the remaining component of the position variable.

In this way we (asymptotically) obtain a planar motion of the agents with the alignment of their velocities along that plane. Further combinations of the proposed solutions can be studied, thus realizing other actions of interest.

Future extensions of this work will concern an evaluation of the effect of communication delays and/or uncertainties on the performance of the proposed control schemes.

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