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A membership-function-dependent stability analysis of Takagi-Sugeno models

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Abstract: This paper presents a new approach for stability analysis of Takagi-Sugeno (TS) models. The analysis considers information derived from existing or induced order relations among the membership functions. Partitioning of the state-space and the use of piecewise Lyapunov functions (PWLF) arise naturally as a consequence of induced order relations. Conditions under the novel approach can be expressed as linear matrix inequalities (LMIs) so they can be efficiently solved. Examples are provided to show the advantages over the classical quadratic approach.

1. INTRODUCTION

In recent years, Takagi-Sugeno (TS) models (Takagi and Sugeno, 1985) have been the subject of an intensive research by virtue of their approximation capabilities. They can represent exactly a nonlinear model in a compact set of the state variables (Taniguchi *et.al.*, 2001). TS models are constructed by a set of linear models blended together with nonlinear functions holding the convex-sum property (Tanaka and Wang, 2001). The stabilization problem is usually addressed via the so-called PDC (Parallel Distributed Compensation) control law (Wang et.al., 1996). It consists in a set of linear state feedbacks blended together using the same nonlinear functions as the TS model.

Stability and stabilization of TS models are usually investigated through the direct Lyapunov method. An LMI (Linear Matrix Inequality) formulation (Boyd et.al., 1994) of these problems is preferred, since LMIs can be easily solved by convex optimization techniques. This formulation is directly achieved by quadratic Lyapunov functions (Tanaka and Wang, 2001) and many results concerning robustness and performance under this approach have been developed (see Sala *et.al.*, 2005, and references therein). Nevertheless, quadratic-stability-based results have nearly reached their limits since they are very particular cases of stability which main drawback is the conservative behaviour of their solutions.

In order to reduce conservativeness, different Lyapunov functions have been proposed in the literature. Piecewise Lyapunov functions have been investigated (Johansson *et.al.*, 1999; Feng, 2003) as a natural option for those TS models which do not have all linear models activated at once. State space is partitioned according to linear models activation allowing the Lyapunov function to change from one region to another. Unfortunately, this assumption generally does not hold for TS models built using the sector nonlinearity approach. On the other hand, different non-quadratic Lyapunov functions have been also employed, though results in the continuous-time domain (Rhee and Won, 2006) have not been as powerful as those of the discrete case (Guerra and Vermeiren, 2004; Ding et.al., 2006; Kruszewski and Guerra, 2005). Most of these Lyapunov functions depend on the same nonlinear functions of the model (membership functions), hereby taking into account structural information otherwise ignored by the quadratic approach.

Other relaxations have been successfully employed, though they are focused on stabilization (Liu and Zhang, 2003; Tuan *et.al.*, 2001). Therefore, they are inapplicable on stability issues.

This paper presents a novel approach to cope with stability issues for TS models. By investigating the properties of TS models with order relations among different membership functions, relaxed conditions are found for TS models. The new approach allows incorporating piecewise analysis for any kind of TS fuzzy system since state-space partition is induced by the aforementioned order relations. Therefore, piecewise approach is not longer excluded for TS models obtained via sector nonlinearity.

This paper is organized as follows: Section II introduces TS models and notation. MF-dependent stability analysis is developed in Section III along with an example. Section IV shows the piecewise analysis extension of MF-dependent approach, providing examples to illustrate the advantages of the proposed method. Finally, Section V gives some conclusions and perspectives.

2. TAKAGI-SUGENO MODELS AND NOTATION

Consider a nonlinear model given by the expression:

$$\dot{x}(t) = f\left(z(t)\right)x(t) \tag{1}$$

with $f(\cdot)$ being a nonlinear function and $x(t) \in \mathbb{R}^n$ the state vector. Using the sector nonlinearity approach with bounded nonlinearities (Tanaka and Wang, 2001), a TS model can be derived from (1) as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t) \tag{2}$$

where $A_i \in \mathbb{R}^{n\times n}$, $z(t) \in \mathbb{R}^p$ is the premise vector, $r \in \mathbb{N}$ is the number of linear models blended together by nonlinear scalar functions $h_i(\cdot)$, which satisfy the convex sum property: $\sum_{i=1}^{r} h_i(\cdot) = 1$, $h_i(\cdot) \geq 0$.

Consider an order relation between two MFs such that $\forall z(t)$: $h_i\left(z(t)\right) \leq h_j\left(z(t)\right)$. A set of ordered indexes can be used to represent this relationship as follows $C_i = \{i, j\}$, where sub-index i represents the lowest end of the order relation. In the same way, a large order relation beginning in membership $h_i(\cdot)$ can be represented with a set of ordered indexes $C_i = \{c_i^1, c_i^2, \cdots\}$ representing $\forall z(t)$: $h_{c_i^1} \leq h_{c_i^2} \leq \cdots$ where $c_i^1 = i$. In case there is more than one order relation beginning in $h_i(\cdot)$ they are distinguished by another index j, like $C_i^j = \{c_{ij}^1, c_{ij}^2, \cdots\}$.

By means of the previous notation, the following sets can be defined for a given TS model (2) in order to facilitate proof construction:

Definition 1: $C_i^j = \{c_{ij}^1, c_{ij}^2, \cdots\}$, $i = 1, \cdots, r$, $j = 1, \cdots, v_i$ represents the different longest order relation chains $h_{c_{ij}^1} \leq h_{c_{ij}^2} \leq \cdots$ beginning in h_i where $c_{ij}^1 = i$.

Definition 2: $S_i = \bigcup_{j=1}^{v_i} C_i^j = \{s_1^i, \cdots, s_{r_i}^i\}$ represents all the elements which are equal or greater than h_i. It is not an ordered set.

Definition 3: $\left[C_i^j\right]$ is the set of all pairs in C_i^j with two consecutive elements.

Definition 4: $\ell = \{l : \nexists k : \forall z(t), h_l > h_k\}$ is the set of all lower-end elements.

Example: For the sake of clarity, consider the following graph representing a possible order relation among MFs of an 8-rules TS model, where upper elements are greater than lower ones. For element $i = 5$ it is clear that $C_5^1 = \{5,1,2\}$, $C_5^2 = \{5,3,2\}$, $\left[C_5^1\right] = \{\{5,1\},\{1,2\}\}$, $\left[C_5^2\right] = \{\{5,3\},\{3,2\}\}$ and $S_5 = \{5,3,1,2\}$ while $\ell = \{4,7,8\}$. Note that there could be independent graphs for non-related order relation chains. Note also that isolated terms represent membership functions that have no order relation with any other (for example h_8).

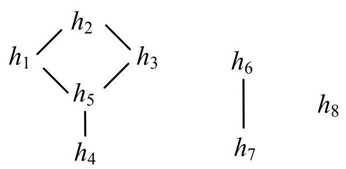

Fig. 1.Graph of MFs' order relations.

3. MF-DEPENDENT STABILITY ANALYSIS

3.1 Main result

A sufficient condition for stability of a TS model (2) is the existence of a common matrix $P > 0$ such that for $L_i = A_i^T P + PA_i$ the following holds

$$h_1 L_1 + h_2 L_2 + \cdots + h_r L_r < 0 \tag{3}$$

Classical quadratic stability consists in finding a common matrix $P > 0$ such that $L_i = A_i^T P + PA_i < 0$, so condition (3) is guaranteed since $\forall i$, $h_i \geq 0$. Nevertheless, no structural information is taken into account, i.e., quadratic stability discards MFs' information, thereby constituting a source of conservativeness.

The key idea of this paper consists in exploiting order relations among the membership functions in a TS model (2) by rewriting condition (3). For example, if $h_i \leq h_j$, then $h_i L_i + h_j L_j$ can be rewritten as follows:

$$h_i L_i + h_j L_j = h_i\left(L_i + L_j\right) + \left(h_j - h_i\right) L_j$$

allowing to write less-conservative conditions $L_i + L_j < 0$, $L_j < 0$ instead of classical $L_i < 0$, $L_j < 0$ since it is known that $h_i \geq 0$ and $h_j - h_i \geq 0$.

As multiple order relations can appear among MFs of a TS model, the previous idea can be generalized as follows.

Theorem 1: TS model (2) under order relations described by sets in Definitions 1–4 is globally asymptotically stable if there exists a common matrix $P > 0$ such that the following LMIs hold for $L_i = A_i^T P + PA_i$:

$$\frac{n_{s_1^i}}{d_{s_1^i}} L_{s_1^i} + \frac{n_{s_2^i}}{d_{s_2^i}} L_{s_2^i} + \cdots + \frac{n_{s_{r_i}^i}}{d_{s_{r_i}^i}} L_{s_{r_i}^i} < 0, \quad i = 1, \cdots, r \tag{4}$$

where $d_{s_i^k} = \sum_{i\in\ell} card\{C_i \cap s_i^k\} = \sum_{i\in\ell}\sum_{j=1}^{v_i} card\{C_i^j \cap s_i^k\}$ and $n_{s_k^i} = card\{C_i \cap s_k^i\} = \sum_{j=1}^{v_i} card\{C_i^j \cap s_k^i\}$.

Proof: Taking into account the order relations for model (2) described in Definitions 1-4, i.e, sets C_i^j, S_i, $\left[C_i^j\right]$ and ℓ, sufficient stability condition (3) can be rewritten as follows: $h_1L_1+h_2L_2+\cdots+h_rL_r$

$$=\sum_{i\in\ell}\sum_{j=1}^{v_i}\left(h_{c_{ij}^1}\frac{L_{c_{ij}^1}}{d_{c_{ij}^1}}+h_{c_{ij}^2}\frac{L_{c_{ij}^2}}{d_{c_{ij}^2}}+\cdots+h_{c_{ij}^{q_{ij}}}\frac{L_{c_{ij}^{q_{ij}}}}{d_{c_{ij}^{q_{ij}}}}\right)<0,$$

where $d_{c_{ij}^k}=\sum_{i\in\ell}card\left\{C_i\cap c_{ij}^k\right\}=\sum_{i\in\ell}\sum_{j=1}^{v_i}card\left\{C_i^j\cap c_{ij}^k\right\}$.

The latter coefficients arise since $i\in\ell$ implies that all the index-sequences (even one-element ones) beginning in a low-end element are taken into account; i.e., every h_iL_i is included, but it may be repeated as many times as d_i. Note that for a given low-end element i there are v_i index-sequences beginning at it. The latter expression can be thus rewritten as:

$$\sum_{i\in\ell}\sum_{j=1}^{v_i}h_{c_{ij}^1}\left[\left(\frac{L_{c_{ij}^1}}{d_{c_{ij}^1}}+\frac{L_{c_{ij}^2}}{d_{c_{ij}^2}}+\cdots+\frac{L_{c_{ij}^{q_{ij}}}}{d_{c_{ij}^{q_{ij}}}}\right)+\cdots\right.$$
$$\left.+\left(h_{c_{ij}^2}-h_{c_{ij}^1}\right)\left(\frac{L_{c_{ij}^2}}{d_{c_{ij}^2}}+\cdots+\frac{L_{c_{ij}^{q_{ij}}}}{d_{c_{ij}^{q_{ij}}}}\right)+\cdots+\left(h_{c_{ij}^{q_{ij}}}-h_{c_{ij}^{q_{ij}-1}}\right)\left(\frac{L_{c_{ij}^{q_{ij}}}}{d_{c_{ij}^{q_{ij}}}}\right)\right]<0$$

where every left-factor of each summand is positive. Adding terms with identical left-side factors and recalling that $\bigcup_j C_i^j=S_i$ gives:

$$\sum_{i\in\ell}h_i\left(\frac{n_{s_1^i}}{d_{s_1^i}}L_{s_1^i}+\frac{n_{s_2^i}}{d_{s_2^i}}L_{s_2^i}+\cdots+\frac{n_{s_{r_i}^i}}{d_{s_{r_i}^i}}L_{s_{r_i}^i}\right)+\cdots$$
$$+\sum_{i\notin\ell}\sum_{i,j\in[C_i^j]}\left(h_i-h_j\right)\left(\frac{n_{s_1^i}}{d_{s_1^i}}L_{s_1^i}+\frac{n_{s_2^i}}{d_{s_2^i}}L_{s_2^i}+\cdots+\frac{n_{s_{r_i}^i}}{d_{s_{r_i}^i}}L_{s_{r_i}^i}\right)<0$$

where $\forall l=1,\cdots,r$, $d_{s_k^i}=d_{c_{\alpha\beta}^\gamma}:\sum_{c_{\alpha\beta}^\gamma=l,\,\alpha\in\ell}\frac{1}{d_{c_{\alpha\beta}^\gamma}}=1$,

$$n_{s_k^i}=card\left\{C_i\cap s_k^i\right\}=\sum_{j=1}^{v_i}card\left\{C_i^j\cap s_k^i\right\}.$$

The previous hold if (4) does so, which concludes the proof.

Remark 1: Results in Theorem 1 reduce to quadratic stability if no order relation among the membership functions is taken into account.

Remark 2: Scalars $d_{s_k^i}$ can be chosen in another way as long as they hold the property $\forall l=1,\cdots,r$, $\sum_{c_{ij}^k=l,\,i\in\ell}\frac{1}{d_{c_{ij}^k}}=1$, via a two-steps algorithm:

Step 1: Fix scalars $d_{s_k^i}$ to some initial value and solve problem (5) for P.

Step 2: Fix matrix P to the value found in Step 1 and solve the same problem (5) for scalars $d_{s_k^i}$. If $\lambda<0$ then a solution has been found; otherwise, take the new values of scalars $d_{s_k^i}$ and go to Step 1. Stop if λ does not decrease significantly.

$$\min\ \lambda:\frac{n_{s_1^i}}{d_{s_1^i}}L_{s_1^i}+\frac{n_{s_2^i}}{d_{s_2^i}}L_{s_2^i}+\cdots+\frac{n_{s_{r_i}^i}}{d_{s_{r_i}^i}}L_{s_{r_i}^i}<\lambda I,\quad i=1,\cdots,r \qquad (5)$$

under definitions of Theorem 1.

3.2 Example.

Consider the following nonlinear model:

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}S_{11}&S_{12}\\ S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, where

$$S_{11}=\left(\frac{1}{1+e^{-x_1}}\right)\left(\frac{1}{1+e^{-x_1-1}}\right)-2$$

$$S_{12}=-3-2\left(1-\frac{1}{1+e^{-x_1}}\right)-\frac{1}{1+e^{-x_1-1}}$$

$$S_{21}=8-15\left(1-\frac{1}{1+e^{-x_1-1}}\right)\left(\frac{1}{1+e^{x_1}}\right)\left(\frac{1}{1+e^{-x_1}}\right)$$

$$S_{22}=-10\left(1-\frac{1}{1+e^{x_1}}\right)-9\left(\frac{1}{1+e^{-x_1}}\right)+0.1\cos x_1$$

Consider also $w_0^1=\frac{1}{1+e^{-x_1}}$, $w_0^2=\frac{1}{1+e^{-x_1-1}}$, $w_0^3=\frac{1}{1+e^{x_1}}$, $w_0^4=\frac{1+\cos x_1}{2}$, $w_1^1=1-w_0^1$, $w_1^2=1-w_0^2$, $w_1^3=1-w_0^3$ and $w_1^4=1-w_0^4$ to define $h_1=w_0^1w_0^2w_0^3w_1^4$, $h_2=w_0^1w_0^2w_1^3w_1^4$, $h_3=w_0^1w_1^2w_0^3w_1^4$, $h_4=w_0^1w_1^2w_1^3w_1^4$, $h_5=w_1^1w_0^2w_0^3w_1^4$, $h_6=w_1^1w_0^2w_1^3w_1^4$, $h_7=w_1^1w_1^2w_0^3w_1^4$, $h_8=w_1^1w_1^2w_1^3w_1^4$, $h_9=w_0^1w_0^2w_0^3w_0^4$, $h_{10}=w_0^1w_0^2w_1^3w_0^4$, $h_{11}=w_0^1w_1^2w_0^3w_0^4$, $h_{12}=w_0^1w_1^2w_1^3w_0^4$, $h_{13}=w_1^1w_0^2w_0^3w_0^4$, $h_{14}=w_1^1w_0^2w_1^3w_0^4$, $h_{15}=w_1^1w_1^2w_0^3w_0^4$, $h_{16}=w_1^1w_1^2w_1^3w_0^4$, in order to construct the following Takagi-Sugeno representation of the original model via sector nonlinearity:

$$\dot{x}(t)=A_zx(t)=\sum_{i=1}^{16}h_i\left(z(t)\right)A_ix(t), \qquad (6)$$

$$A_1=\begin{bmatrix}-1&-4\\ 8&-9.1\end{bmatrix},\quad A_2=\begin{bmatrix}-1&-4\\ 8&-19.1\end{bmatrix},\quad A_3=\begin{bmatrix}-2&-3\\ -7&-9.1\end{bmatrix},$$
$$A_4=\begin{bmatrix}-2&-3\\ 8&-19.1\end{bmatrix},\quad A_5=\begin{bmatrix}-2&-6\\ 8&-0.1\end{bmatrix},\quad A_6=\begin{bmatrix}-2&-6\\ 8&-10.1\end{bmatrix},$$
$$A_7=\begin{bmatrix}-2&-5\\ 8&-0.1\end{bmatrix},\quad A_8=\begin{bmatrix}-2&-5\\ 8&-10.1\end{bmatrix},\quad A_9=\begin{bmatrix}-1&-4\\ 8&-8.9\end{bmatrix},$$
$$A_{10}=\begin{bmatrix}-1&-4\\ 8&-18.9\end{bmatrix},\quad A_{11}=\begin{bmatrix}-2&-3\\ -7&-8.9\end{bmatrix},\quad A_{12}=\begin{bmatrix}-2&-3\\ 8&-18.9\end{bmatrix},$$

$A_{13} = \begin{bmatrix} -2 & -6 \\ 8 & 0.1 \end{bmatrix}$, $A_{14} = \begin{bmatrix} -2 & -6 \\ 8 & -9.9 \end{bmatrix}$, $A_{15} = \begin{bmatrix} -2 & -5 \\ 8 & 0.1 \end{bmatrix}$, $A_{16} = \begin{bmatrix} -2 & -5 \\ 8 & -9.9 \end{bmatrix}$.

Fig. 2. Order relations of model (6)

Ordinary stability analysis fails for this model since A_3 is unstable. Nevertheless, taking into account that $h_4 \le h_1$, $h_4 \le h_6$, $h_3 \le h_5$, $h_8 \le h_5$, $h_{12} \le h_{14}$, $h_{12} \le h_9$, $h_{11} \le h_{13}$ and $h_{16} \le h_{13}$ (see Fig. 2), the following sets can be defined $\ell = \{2,3,4,7,8,10,11,12,15,16\}$, $C_1^1 = \{1\}$, $C_2^1 = \{2\}$, $C_3^1 = \{3,5\}$, $C_4^1 = \{4,6\}$, $C_4^2 = \{4,1\}$, $C_5^1 = \{5\}$, $C_6^1 = \{6\}$, $C_7^1 = \{7\}$, $C_8^1 = \{8,5\}$, $C_9^1 = \{9\}$, $C_{10}^1 = \{10\}$, $C_{11}^1 = \{11,13\}$, $C_{12}^1 = \{12,14\}$, $C_{12}^2 = \{12,9\}$, $C_{13}^1 = \{13\}$, $C_{14}^1 = \{14\}$, $C_{15}^1 = \{15\}$, $C_{16}^1 = \{16,13\}$ to express conditions (4) in Theorem 1 as follows:

$$L_1 < 0,\ L_2 < 0,\ L_3 + \frac{1}{2}L_5 < 0,\ L_4 + L_6 + L_1 < 0,\ \frac{1}{2}L_5 < 0,$$
$$L_6 < 0,\quad L_7 < 0,\quad L_8 + \frac{1}{2}L_5 < 0,\quad L_9 < 0,\quad L_{10} < 0,$$
$$L_{11} + \frac{1}{2}L_{13} < 0,\quad L_{12} + L_{14} + L_9 < 0,\quad \frac{1}{2}L_{13} < 0,\quad L_{14} < 0,$$
$$L_{15} < 0,\ L_{16} + \frac{1}{2}L_{13} < 0.$$

LMI conditions above have a feasible solution with matrix

$$P = \begin{bmatrix} 0.2352 & 0.0093 \\ 0.0093 & 0.1594 \end{bmatrix}$$

which proves stability for TS model (6).

4. PIECEWISE ANALYSIS

4.1 Main result.

When there are no order relations among the membership functions of a TS model (2), results in Theorem 1 can not be directly applied. Nevertheless, a suitable partition of the state space could adapt them to this case. Stability analysis based in piecewise Lyapunov functions comes at hand since it allows partitioning the state space in compliance with some criteria. These criteria can be MF-dependent, i.e., state space can be partitioned in as many regions as different order relations exist among the membership functions. At each region, Theorem 1 analysis will hold since a particular order relation among membership functions will be locally valid.

Consider then a partition of the state space as a collection of regions $\{X_q\}_{q\in I} \subseteq \mathbb{R}^n$, where I is the set of region indexes. At each region X_q some particular order relations among the MFs will hold, i.e., specific sets C_i^j, S_i, $\left[C_i^j\right]$ and ℓ from Definitions 1-4 will be defined $\forall x(t) \in X_q$ in order to describe those relationships. Then, another index will be added to those sets to distinguish them from sets of another region, i.e., C_i^{jq}, S_i^q, $\left[C_i^{jq}\right]$ and ℓ^q for $q \in I$. A transition from one region to another means at least one order relation between two MFs has changed.

The best way to partition the state space is to define each region X_q such that $\forall x(t) \in X_q$: $h_{q(1)} \le h_{q(2)} \le \cdots \le h_{q(r)}$. Unfortunately, though theoretically possible, this partitioning could be hard to obtain and lead to complicated regions if MFs depend on more than one state. Moreover, complicated regions could be non-expressible as LMIs.

In order to deal with this problem, a polyhedral partition of the state space is suggested. This is always possible if MFs are expressible as the product of functions which depend at most of one state variable, i.e., $h_i(z(t)) = w_i^1(x_1)\cdots w_i^n(x_n)$. In this case, order relations among functions $w_i^j(x_j)$, $i = 1,\cdots,r$ induce partitions in each state variable x_j, $j = 1,\cdots,n$ and, therefore, in the overall state space. Order relations among functions $w_i^j(x_j)$ will naturally induce order relations among MFs $h_i(z(t))$, $i = 1,\cdots,r$ in each region or cell X_q. These induced order relations will allow to define sets C_i^{jq}, S_i^q, $\left[C_i^{jq}\right]$ and ℓ_q for each region X_q, $q \in I$.

Definition 5: C_i^{jq}, S_i^q, $\left[C_i^{jq}\right]$ and ℓ^q are equivalent to sets C_i^j, S_i, $\left[C_i^j\right]$ and ℓ from Definitions 1–4, though locally valid $\forall x(t) \in X_q$.

Example: To better illustrate the partitioning described above, consider a four-rules two-states TS model with MFs $h_1 = w_1(x_1)w_3(x_2)$, $h_2 = w_1(x_1)w_4(x_2)$, $h_3 = w_2(x_1)w_3(x_2)$ and $h_4 = w_2(x_1)w_4(x_2)$ where w_1 and w_2 (depending on x_1) and w_3 and w_4 (depending on x_2) are shown in Fig. 3.

Two possible order relations in x_1 ($w_1 > w_2$ or $w_2 > w_1$) and other two in x_2 ($w_3 > w_4$ and $w_4 > w_3$) induce the four-regions partitioning shown in Fig. 3. In region 2, for example, where $w_1 > w_2$ and $w_3 > w_4$, two order relations are induced among MFs: $h_1 > h_2 > h_4$, $h_1 > h_3 > h_4$.

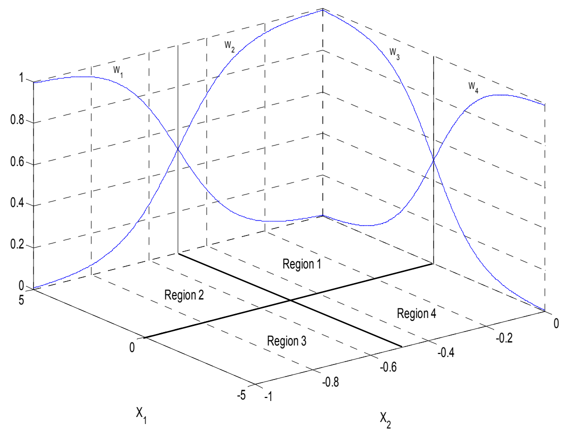

Fig. 3. State space partitioning.

As in (Johansson et.al., 1999) piecewise Lyapunov function candidates $V(x)=x^T P_q x, \quad x\in X_q$ are parameterized to be continuous across cell boundaries. Continuity is fulfilled by means of constraint matrices F_q satisfying

$$F_i x = F_j x\,,\ x\in X_i \cap X_j \tag{7}$$

so Lyapunov functions are parameterized as $P_q = F_q^T T F_q$, where free parameters are collected in symmetric matrix T, allowing an LMI formulation. Moreover, since matrix P_q is only used to describe the Lyapunov function in cell X_q then it can be restricted to that cell by means of matrices E_q satisfying

$$E_q x \succeq 0\,,\ x\in X_q \tag{8}$$

where the vector inequality $\succeq$ means that each entry of the vector is nonnegative.

Theorem 2: TS model (2) under order relations described by sets in Definition 5 for regions X_q, $q\in I$, tends to zero exponentially for any continuous C^1 piecewise trajectory in $\bigcup_{q\in I} X_q$ if there exists symmetric matrices T, U_q and W_q, U_q and W_q with nonnegative entries, such that the following LMIs hold for $P_q = F_q^T T F_q$ and $L_i^q = A_i^T P_q + P_q A_i + E_q^T W_q E_q$ for each $q\in I$:

$$\begin{gathered} P_q - E_q^T U_q E_q > 0 \\ \frac{n^q_{s_1^i}}{d^q_{s_1^i}} L^q_{s_1^i} + \frac{n^q_{s_2^i}}{d^q_{s_2^i}} L^q_{s_2^i} + \cdots + \frac{n^q_{s_{r_i}^i}}{d^q_{s_{r_i}^i}} L^q_{s_{r_i}^i} < 0, \quad i=1,\cdots,r \end{gathered} \tag{9}$$

where $d^q_{s_k^i} = \sum_{i\in\ell^q} card\left\{C_i^q \cap s_i^{kq}\right\} = \sum_{i\in\ell^q}\sum_{j=1}^{v_i^q} card\left\{C_i^{jq}\cap s_i^{kq}\right\}$ and $n^q_{s_k^i} = card\left\{C_i^q \cap s_k^{iq}\right\} = \sum_{j=1}^{v_i^q} card\left\{C_i^{jq}\cap s_k^{iq}\right\}$.

Proof: It follows immediately from proofs in Appendix A of (Johansson et.al., 1999) and Theorem's 1 proof.

Remark 3: A systematic procedure to construct non-unique matrices E_q and F_q can be found in (Johansson et.al., 1999).

Remark 4: The results here provided can be applied to affine TS models straightforwardly, with proper modifications of the Lyapunov function and partitioning matrices. Details can be found also in (Johansson et.al., 1999).

Remark 5: As in Section 3, a two-step algorithm can be used to determine coefficients $d^q_{s_k^i}$ in another way.

4.2 Example.

Consider the following TS model:

$$\dot{x}(t) = A_z x(t) = \sum_{i=1}^{4} h_i\left(z(t)\right) A_i x(t) \tag{10}$$

$$A_1=\begin{bmatrix}-10 & -11\\ 0 & 1\end{bmatrix},\quad A_2=\begin{bmatrix}-1 & -2\\ 2 & -8\end{bmatrix},\quad A_3=\begin{bmatrix}-10 & -10\\ 0 & -5\end{bmatrix},$$
$$A_4=\begin{bmatrix}-1 & -1\\ 2 & -14\end{bmatrix},\ w_0^1=\frac{1}{1+e^{x_1}},\ w_0^2=\frac{1}{1+e^{-10x_1}},\ w_1^1=1-w_0^1,$$
$$w_1^2=1-w_0^2,\ h_1=w_0^1 w_0^2,\ h_2=w_0^1 w_1^2,\ h_3=w_1^1 w_0^2,\ h_4=w_1^1 w_1^2.$$

Note that model A_1 is unstable, thus ordinary stability analysis fails for model (10). Since no order relation can be found among their membership functions, piecewise analysis proceeds. State space is split in two as shown in Fig. 4 because MFs have only two possible order relations:

Region 1: $h_2 < h_4 < h_1 < h_3$ for $x_1 < 0$.

Region 2: $h_3 < h_1 < h_4 < h_2$ for $x_1 > 0$.

Note that in this case there is no more than one relationship per element i per region j, which fixes coefficients in LMIs (9) to $d^q_{s_k^i} = n^q_{s_k^i} = 1$. Matrices $E_1=\begin{bmatrix}-1 & 0\\ -3 & 0\end{bmatrix}$, $E_2=\begin{bmatrix}11 & 0\\ 33 & 0\end{bmatrix}$, $F_1=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $F_2=\begin{bmatrix}11 & 2\\ 33 & 4\end{bmatrix}$ are constructed to satisfy properties (7)-(8).

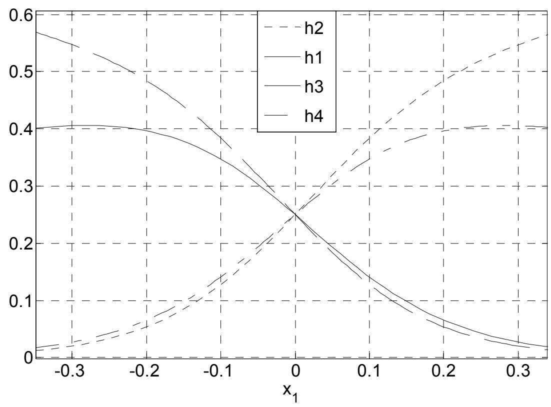

Fig. 4. Membership functions for TS model (10).

Then, LMIs in (9) are feasible with $T = \begin{bmatrix} 0.3710 & -0.1291 \\ -0.1291 & 0.0452 \end{bmatrix}$

and piecewise Lyapunov function $V\left(x(t)\right) = \begin{cases} x^T P_1 x, & x_1 < 0 \\ x^T P_2 x, & x_1 \geq 0 \end{cases}$.

In Fig. 5 some TS-model trajectories and level curves of the piecewise Lyapunov function $V\left(x(t)\right)$ are shown.

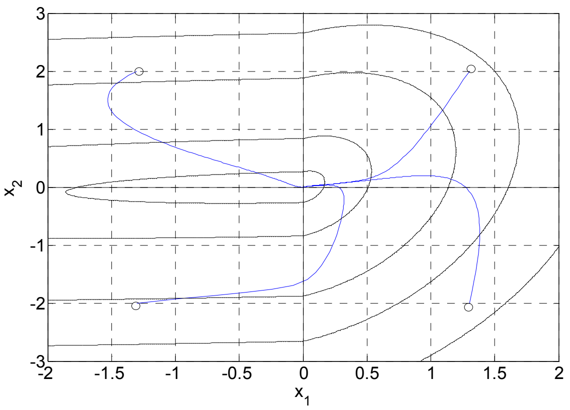

Fig. 5. PWLF curve levels.

5. CONCLUSIONS

This paper presented a novel approach for stability analysis of TS models based on existent or induced order relations among the membership functions of the model. This approach outperforms the classical quadratic stability analysis and allows employing piecewise Lyapunov functions on TS models that have been obtained via the sector nonlinearity technique. Stability conditions can be expressed as linear matrix inequalities (LMIs) which can be efficiently solved by available software. Examples were provided that illustrate the advantages of the proposed method.

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