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Robust Stability of Nonlinear Systems

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Abstract: In recent years the main focus of research in the area of robust stability of systems was on linear systems. In this paper the question of robust stability for nonlinear systems is adressed. We are mainly interested in global asymptotic stability and the main tools to solve these problems are methods based on Lyapunov functions. Using an appropriate Lyapunov function and an exact linearization of the system we are able to derive a sufficient condition for global asymptotic stability of a nonlinear system. This sufficient condition is known in the literature as robust linear matrix inequality. The main contribution of this paper is a new relaxation for robust linear matrix inequalities which avoids vertexization and leads to a computationally efficient procedure.

Keywords: Robust linear matrix inequalities; Robustness analysis; Sum-of-squares

1. INTRODUCTION

In this paper we will investigate nonlinear systems given by the state space representation

$$\dot{z}(t) = f(z(t)) \ , \ z(0) = z_0 \tag{1}$$

where $z \in \mathbb{R}^n$ represents the state vector and the nonlinear function f satisfies $f(0) = 0$. Thus, $z = 0$ is a stationary point of the autonomous system. We assume that the system under investigation results from the application of a controller to a nonlinear system in such a way that the closed loop system is independent of external inputs. As a result of this it is also natural to assume that $z = 0$ is asymptotically stable (cf. Hahn [1967]). Based on the previous considerations we are able to transform (1) into the formally linear expression

$$\dot{z}(t) = A\big(z(t)\big)\ z(t)\,,\ z(0) = z_0 \tag{2}$$

where the $n \times n$ matrix function $A(z)$ is not unique. One convenient choice of $A(z)$ is given by

$$A(z) = \int_0^1 g(sz)\,ds\,, \quad g(z) = \frac{\partial f}{\partial z}(z) \tag{3}$$

where it is assumed that the function $f(z)$ is differentiable.

In controller synthesis it is guaranteed, as already stated before, that the stationary point $z = 0$ is asymptotically stable. But it is very difficult to estimate or compute the region of attraction of this stationary point.

We will approach this problem for the case of global asymptotic stability. In order to have a chance to find sufficient conditions we assume that all entries of $A(z)$ are bounded functions of z. The formal linearity of (2) enables us to use as a Lyapunov function (cf. Hahn [1967]) the following quadratic form

$$V(z) = z^T P\ z\,,\ P = P^T > 0 \tag{4}$$

for which the time derivative along the trajectories of (2) is computed as

$$\dot{V}(z) = z^T \Big\{ A^T(z)P + PA(z) \Big\}\ z \tag{5}$$

which leads to the condition

$$A^T(z)\,P + P\,A(z) \ < \ 0 \quad \forall\, z \tag{6}$$

for global asymptotic stability. Due to the assumed boundedness of the elements of $A(z)$ we have an embedding

$$A(z) \ \in \ \big\{ M \in \mathbb{R}^{n\times n} \ \big| \ m_{ij} \in [\min_z a_{ij}(z)\,, \max_z a_{ij}(z)]\ , \ i,j = 1,\ldots,n \big\} \ = \ \tilde{A} \tag{7}$$

for all z where $\tilde{A}$ is an interval matrix (cf. Moore [1966]). The stability condition (6) can now be relaxed to

$$A^T P + P\ A \ < \ 0 \qquad \forall\, A \in \tilde{A} \tag{8}$$

which represents a robust linear matrix inequality (robust LMI). In the following sections an efficient and fast numerical solution of this condition for P will be presented.

This article is an expanded version of Tibken [2002]. For a different approach to the same problem using the theorem of Ehlich and Zeller see Warthenpfuhl and Tibken [2008].

2. LINEAR MATRIX INEQUALITIES

In the last decade, LMI conditions and efficient numerical software for solving them played a major role in the development of control theory. A typical LMI is given by

$$F(x) = F_0 \ + \ F_1\,x_1 \ + \ \ldots \ + \ F_r\,x_r \ \geq \ 0 \tag{9}$$

where $x \in \mathbb{R}^r$ represents the vector of decision variables while $F_i \in \mathbb{R}^{n\times n},\ i = 0,\ldots,r$ are given real symmetric matrices. The feasible set

$$\Omega = \big\{ \, x \in \mathbb{R}^r \ \big| \ F(x) \ \geq \ 0 \, \big\} \tag{10}$$

is either empty or convex. Due to the convexity of Ω, very efficient software for the computation of a point in Ω is available (interior-point methods, cf. Nesterov and Nemirovski [1994]).

A very important generalization of LMIs are robust LMIs which are represented as

$$\begin{aligned}\tilde{F}(x,\delta) = \tilde{F}_0(\delta) \ + \ \tilde{F}_1(\delta)\, x_1 \ + \ \ldots \\ + \ \tilde{F}_r(\delta)\, x_r \ \geq \ 0 \qquad \forall\, \delta \in \Delta\end{aligned} \tag{11}$$

where $\delta \in \Delta \subset \mathbb{R}^m$ is a vector of parameters lying in the bounded set Δ and the matrices $\tilde{F}_i$ depend on these parameters. In this case the feasible set is defined by

$$\tilde{\Omega} = \left\{\, x \in \mathbb{R}^r \;\middle|\; \tilde{F}(x,\delta) \ \geq \ 0 \quad \forall\, \delta \in \Delta \,\right\} \tag{12}$$

and a robust LMI represents an uncountable number of ordinary LMI conditions if the set Δ contains uncountably many points. The stability condition (8) represents a robust LMI if we make the following identifications

$$\begin{aligned}\text{decision variables } x \ &= \ \text{independent entries of } P \ , &(13)\\ r \ &= \ \frac{1}{2}\, n\,(n+1) \ , &(14)\\ \text{parameter vector } \delta \ &= \ \text{independent entries of } \tilde{A} \ , &(15)\\ m \ &= \ n^2 \ . &(16)\end{aligned}$$

It is important to observe that in this case the dependence on the parameter δ is linear, which will simplify the following analysis. It is convenient to introduce the abbreviations

$$\begin{aligned}\bar{A} \ &= \ \text{midpoint matrix of } \tilde{A} \text{ and} &(17)\\ S \ &= \ \text{radius matrix of } \tilde{A} &(18)\end{aligned}$$

which lead to the equivalent characterization

$$\begin{aligned}\tilde{A} = \big\{ M \in \mathbb{R}^{n\times n} \;\big|\; m_{ij} \in [\bar{a}_{ij} - s_{ij}\,,\, \bar{a}_{ij} + s_{ij}]\ , \\ i,j = 1,\ldots,n \big\}\,.\end{aligned} \tag{19}$$

Due to its linearity in both the decision variables x and the uncertain parameters δ, it is easy to convert the robust LMI (8) into a set of ordinary LMIs. In order to do this, we first define the set

$$\operatorname{vert}(\tilde{A}) = \text{set of vertices of } \tilde{A} \tag{20}$$

and immediately arrive at the equivalent stability condition

$$\hat{A}^T P \ + \ P\, \hat{A} < 0 \quad \forall\, \hat{A} \in \operatorname{vert}(\tilde{A}) \ , \tag{21}$$

which represents a finite number of ordinary LMIs. The major problem with this approach is that the vertex set consists of $2^{(n^2)}$ matrices, in general. This number grows rapidly with increasing n, so we get 512 LMI conditions for $n = 3$ and 65,536 LMI conditions for for $n = 4$. Thus, with available LMI solvers this problem cannot be solved, even for $n = 4$. Considering that most practical problems are of much higher dimension, the vertexization method is useless for solving the stability problem in general. In the next section we will use recent results from real algebra in order to find new sufficient conditions to replace (8).

3. REFORMULATION OF THE PROBLEM

In this section we assume a robust LMI with linear dependence on the uncertain parameters δ (cf. Apkarian and Tuan [2000]). This robust LMI, with the linearity in δ made explicit, is given by

$$\begin{aligned}G(x,\delta) = G_0(x) \ + \ G_1(x)\,\delta_1 \ + \ \ldots \\ + \ G_m(x)\,\delta_m \ \geq \ 0 \qquad \forall\, \delta \in \Delta \ .\end{aligned} \tag{22}$$

The matrices G_i also depend linearly on the decision variables x. In order to transform the problem from a matrix to a polynomial formulation we first define the quadratic form

$$q(x,\delta,z) \ = \ z^T G(x,\delta)\ z \ .$$

We then have the following equality of solution sets

$$\begin{aligned}\Omega &= \{\, x \in \mathbb{R}^r \ | \ G(x,\delta) \ \geq 0 \quad \forall\, \delta \in \Delta \,\} \\ &= \{\, x \in \mathbb{R}^r \ | \ q(x,\delta,z) \geq 0 \quad \forall\, \delta \in \Delta \text{ and } \forall\, z \in \mathbb{R}^n \,\},\end{aligned} \tag{23}$$

where the second characterization of Ω consists solely of a polynomial condition. We then interpret $q(x,\delta,z)$ as a polynomial in δ and z with coefficients which are linear in x. Our goal is now to find a set of sufficient conditions which are linear in x and which represent ordinary LMI conditions.

If we omit the variable x for clarity, $q(\delta,z)$ can be written as

$$q(\delta,z) = q_0(z) \ + \ \delta_1\, q_1(z) \ + \ \ldots + \ \delta_m\, q_m(z) \tag{24}$$

and we would like to test if

$$q(\delta,z) > 0 \quad \forall\, \delta \in \Delta \ \text{ and } \ \forall\, z \in \mathbb{R}^n \tag{25}$$

holds with

$$\Delta = \{\, \delta \ | \ -1 \leq \delta_i \leq 1\,, \quad i = 1,\ldots,m \,\} \ . \tag{26}$$

We may assume this convenient structure for Δ as an interval vector without loss of generality (cf. (33)).

At the heart of our approach lies the equivalent alternative representation

$$\begin{aligned}q(\delta,z) = \tilde{q}_0(z) \ + \ \sum_{i=1}^{m} (1 \ + \ \delta_i)\ \tilde{q}_{i1}(z) \\ + \ \sum_{i=1}^{m} (1 \ - \ \delta_i)\ \tilde{q}_{i2}(z)\end{aligned} \tag{27}$$

which was chosen to have a structure similar to the representation described in Theorem 4.2 in Jacobi and Prestel [2001]. Because of the structure of (27), the set of sufficient conditions for positivity

$$\begin{aligned}&\tilde{q}_0(z) > 0 \\ &\tilde{q}_{i1}(z) > 0 \ , \quad i = 1,\ldots,m \\ &\tilde{q}_{i2}(z) > 0 \ , \quad i = 1,\ldots,m\end{aligned} \tag{28}$$

can be derived immediately. A comparison of coefficients leads to the set of equations

$$\begin{aligned}q_0(z) &= \tilde{q}_0(z) \ + \ \sum_{i=1}^{m} \Big(\tilde{q}_{i2}(z) \ + \ \tilde{q}_{i1}(z)\Big) \\ q_i(z) &= \tilde{q}_{i1}(z) \ - \ \tilde{q}_{i2}(z)\end{aligned}$$

which relate the polynomials from (24) to the polynomials occuring in (28). The general solution of these equations is given by

$$\begin{aligned}\tilde{q}_0(z) &= q_0(z) \ - \ \sum_{i=1}^{m} \Big(q_i(z) \ + 2p_i(z)\Big) \\ \tilde{q}_{i1}(z) &= q_i(z) \ + \ p_i(z) \\ \tilde{q}_{i2}(z) &= p_i(z)\end{aligned} \tag{29}$$

where the $p_i(z), i = 1,\ldots,m$ are new free variables which have to be determined to eventually fulfill the conditions (28).

Note that while the $q_i(z)$ have to be quadratic forms, the maximum degree of the $p_i(z)$ is unrestricted and also determines the maximum degree of the $\tilde{q}_{ij}(z), j = 1, 2$. In the following we assume that the $p_i(z)$ are quadratic forms. Under this assumption we can derive the relaxation

$$G_0(x) - \sum_{i=1}^{m} (\, G_i(x) + 2P_i \,) \; > \; 0 \tag{30}$$

$$G_i(x) + P_i \; > \; 0 \,, \; i = 1, \ldots, m \tag{31}$$

$$P_i \; > \; 0 \,, \; i = 1, \ldots, m \tag{32}$$

for (22). Here the $G_i(x)$ and P_i are Gramian matrices for the quadratic forms $q_i(x, z)$ and $p_i(z)$ and thus are symmetric. The P_i represent new additional decision variables introduced by the relaxation.

The main advantage of this relaxation is that the number of LMI conditions is reduced from 2^m to $2m + 1$ for the price of some additional decision variables represented by the matrices P_i. Due to this drastic reduction of the number of LMI conditions, the stability problem (8) can be solved with available LMI solvers for larger m. In the following section we will give a benchmark example from literature and will compute a realistic robustness bound for this example.

4. EXAMPLE

The example in this section is widely used in literature as a benchmark example, see e.g. Calafiore and Polyak [2001] and the references cited there. The matrices $\bar{A}, S \in \mathbb{R}^{3\times3}$ $(n = 3)$ defining the problem are given by

$$\bar{A} = \begin{pmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0.6510 & 0.9394 & 0.5691 \\ 0.2451 & 0.4727 & 0.1457 \\ 0.7004 & 0.4014 & 0.3141 \end{pmatrix}$$

and the problem to be solved is to determine the largest $r > 0$ such that the interval matrix $\tilde{A}$ given by the pair $\bar{A}$, rS is asymptotically stable. In Calafiore and Polyak [2001], a stochastic approach is used to show $0.5 < r < 1$ with very high probability.

Starting from the stability condition (8)

$$A^T P + P\,A \; < \; 0 \qquad \forall\, A \in \tilde{A}$$

where $\tilde{A}$ is defined by (19), we have to make the description of $\tilde{A}$ compliant to the box constraints (25). If we let $[A(\delta)]_{ij}$ denote the element (i, j) of the matrix $A(\delta)$, we can accomplish that by introducing δ_{ij} as

$$[A(\delta)]_{ij} = \bar{a}_{ij} + \delta_{ij}\, r\, s_{ij} \,, \; -1 \le \delta_{ij} \le 1 \,, \tag{33}$$

so that they parameterize the whole interval in each component of $\tilde{A}$. The δ_{ij} are then collected into the vector $\delta = (\delta_{11}, \delta_{12}, \ldots, \delta_{1n}, \delta_{21}, \ldots, \delta_{nn})$. Replacing $A \in \tilde{A}$ with $A(\delta)$, $\delta \in \Delta$, we can write

$$A^T(\delta)\, P + P\; A(\delta) < 0 \quad \forall\, \delta \in \Delta \,, \tag{34}$$

$$\Delta = \{\, \delta \mid -1 \le \delta_{ij} \le 1 \,\}$$

as a new condition equivalent to (8).

Defining the $n \times n$ matrices

$$E_{ij} = \begin{cases} 1 & \text{in component } (i, j) \,, \\ 0 & \text{in all other components} \,, \end{cases} \tag{35}$$

and $\tilde{E}_{ij} = s_{ij} E_{ij}$, as well as the notation

$$\sum_{ij} = \text{sum over all indices } \; ij \text{ with } i, j = 1, \ldots, n$$

allows us to write

$$A(\delta) = \sum_{ij} E_{ij}\, (a_{ij} + r\delta_{ij} s_{ij}) \tag{36}$$

$$= \bar{A} + r \sum_{ij} s_{ij} E_{ij} \delta_{ij} \tag{37}$$

$$= \bar{A} + r \sum_{ij} \tilde{E}_{ij} \delta_{ij} \;. \tag{38}$$

Thus, we get

$$\begin{aligned} &A^T(\delta)\, P + P\; A(\delta) \\ &= \bar{A}^T P + P\, \bar{A} + r \sum_{ij} \Big(\tilde{E}_{ij}^T P + P \tilde{E}_{ij} \Big) \delta_{ij} \end{aligned} \tag{39}$$

by using (38) in (34). After multiplying (39) by -1 to invert the inequality relation, we can identify

$$G_0(P) = -\left(\bar{A}^T P + P\, \bar{A} \right) \text{ and} \tag{40}$$

$$G_{ij}(P, r) = -r \left(\tilde{E}_{ij}^T P + P \tilde{E}_{ij} \right) \tag{41}$$

and so derive the LMI

$$G_0(P) + \sum_{ij} G_{ij}(P, r)\; \delta_{ij} > 0 \quad \forall\, \delta \in \Delta \tag{42}$$

which corresponds to (22).

Using the relaxation described in the previous section, we get a set of LMIs

$$G_0(P) - \sum_{ij} \Big(G_{ij}(P, r) + 2P_{ij} \Big) \; > \; 0 \tag{43}$$

$$G_{ij}(P, r) + P_{ij} \; > \; 0 \quad \forall\, i, j \tag{44}$$

$$P_{ij} \; > \; 0 \quad \forall\, i, j \tag{45}$$

which corresponds to (30)–(32). Note that the P_{ij} are new matrix decision variables introduced by the relaxation, while P without indices is the original matrix decision variable already present in (8) and (42).

The G_{ij} are only linearly dependent on r and we also have $r > 0$. Thus, we can now substitute

$$G_{ij}(P, r) \Rightarrow r\tilde{G}_{ij}(P) \text{ and also} \tag{46}$$

$$P_{ij} \Rightarrow r\tilde{P}_{ij} \;, \tag{47}$$

as this does not change the solution set of (43)–(45).

We then get a final set of LMIs,

$$\frac{1}{r} G_0(P) \; > \; \sum_{ij} \Big(\tilde{G}_{ij}(P) + 2\tilde{P}_{ij} \Big) \tag{48}$$

$$\tilde{G}_{ij}(P) + \tilde{P}_{ij} \; > \; 0 \quad \forall\, i, j \tag{49}$$

$$\tilde{P}_{ij} \; > \; 0 \quad \forall\, i, j \tag{50}$$

$$\bar{A}^T P + P \bar{A} \; < \; 0 \tag{51}$$

$$P \; > \; 0 \tag{52}$$

$$P \; < \; I \;. \tag{53}$$

This set of LMIs can be solved as a generalized eigenvalue problem (GEVP) with $\lambda = \frac{1}{r}$ by software such as the MATLAB Robust Control Toolbox (cf. Balas et al. [2005]). The inequality (51) results from a constraint of the MATLAB GEVP algorithm, which requires $G_0(P) > 0$ in (48). This is no actual constraint, since (51) holds true if (8) does. The last two inequalities, (52) and (53), come from (4) and an attempt to be more numerically stable, respectively.

For the given problem the solution of (48)–(53) with GEVP results in $r = 0.60253002735799$, which is a guaranteed lower bound for the maximal r. Since this is a small example, it was possible to also compute the exact solution, $r = 0.68743994643239$, using the vertexization method. The gap between our lower bound and the exact solution is caused mainly by our restrictive assumption about the maximum degree of the $p_i(z)$ in (29).

5. CONCLUSIONS AND OUTLOOK

In this paper we have presented a new relaxation for robust LMIs with linear parameter dependence. These new results have been applied to the robust stability of a special class of nonlinear systems and a benchmark problem from literature has been solved completely. The exponential complexity of the exact reformulation through vertexization has been avoided and replaced by a linear complexity for the LMI relaxation. Due to this drastic reduction of the complexity, the solution of problems with $n \geq 4$ will be possible using the Robust Control Toolbox for MATLAB. Future research will concentrate on introducing this method into controller design and on using higher order relaxations in order to compute better sufficient conditions.

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