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New Results on the Generalized Frequency Response functions of Nonlinear Volterra Systems Described by NARX model

Xing Jian Jing, Zi Qiang Lang and Stephen A. Billings

Department of Automatic Control and Systems Engineering,
University of Sheffield, Mappin Street, Sheffield, S1 3JD, U.K.
(e-mail: xingjian.jing@googlemail.com)

Abstract: In order that the nth-order Generalized Frequency Response Function (GFRF) for nonlinear systems described by a NARX model can be directly written into a more straightforward and meaningful form in terms of the first order GFRF and model parameters, the nth-order GFRF is now determined by a new mapping function based on a parametric characteristic. This can explicitly unveil the linear and nonlinear factors included in the GFRFs, reveal clearly the relationship between the nth-order GFRF and the model parameters, and also the relationship between the nth-order GFRF and the first order GFRF. Some new properties of the GFRFs can consequently be developed. These new results provide a novel and useful insight into the frequency domain analysis of nonlinear systems.

1. INTRODUCTION

It was showed by Boyd and Chua (1985) that nonlinear systems, which are causal and have fading memory, can be approximated in the neighbourhood of the zero equilibrium by a Volterra series of finite order. Based on a Volterra series approximation, the frequency domain analysis of nonlinear systems can be conducted (Bedrosian and Rice 1971, Brilliant 1958, Kotsios 1997, Rugh 1981, Volterra 1959).The nth-order Generalized Frequency Response Function (GFRF) of nonlinear Volterra systems was defined in George (1959). By applying the probing method (Rugh 1981), a recursive algorithm to compute the GFRFs for nonlinear Volterra systems described by a NARX model was derived in Peyton-Jones and S. A. Billings (1989). These results play a fundamental role in many important results achieved latterly for the frequency domain analysis of nonlinear Volterra systems such as those in Billings and Lang (1996), Jing, et al (2007, 2008). Although significant results have been achieved, many problems still remain unsolved regarding how the frequency response functions are influenced by the parameters of the underlying system, and the connection to complex non-linear behaviours. The existing recursive algorithms in Peyton-Jones and Billings (1989) for the computation of the GFRFs can not explicitly reveal the analytical relationship between system time domain model parameters and system frequency response functions in a straightforward manner. In order to solve these problems, the parametric characteristics of the GFRFs for nonlinear Volterra systems described by a NARX model were studied in Jing et al (2006), which effectively builds up a mapping from the model parameters to the parametric characteristics of the GFRFs and provides an explicit expression for the analytical relationship between the GFRFs and the system time-domain model parameters. Based on the results in Jing et al (2006), an inverse mapping function from the parametric characteristics of the GFRF to the GFRF itself is established for nonlinear Volterra systems described by a NARX model in this study. The nth-order GFRF can directly be determined as an n-degree polynomial function of the first order GFRF according to its parametric characteristic by using this new mapping function. Compared with the existing recursive algorithm for the computation of the GFRFs, the new mapping function enables the nth-order GFRF for a NARX model to be determined in a more straightforward and meaningful structure in terms of the first order GFRF and model parameters without recursive relationship with the lower order GFRFs, and unveils some new properties of the nth-order GFRF. These results facilitate the frequency domain analysis and design of nonlinear systems based on the GFRFs.

2. BACKGROUND

Nonlinear systems can be approximated by a Volterra series up to a maximum order N under certain conditions (Boyd and Chua, 1985). Consider nonlinear Volterra systems described by the following NARX model

$$y(t)=\sum_{m=1}^{M} y_m(t)$$

$$y_m(t)=\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p} y(t-k_i)\prod_{i=p+1}^{p+q} u(t-k_i) \tag{1}$$

where $y_m(t)$ is the mth-order output of the NARX model, $p+q=m$, k_i=1,…, K, M is the maximum degree of nonlinearity in terms of y(t) and u(t), and $\sum_{k_1,k_{p+q}=1}^{K}(\cdot)=\sum_{k_1=1}^{K}(\cdot)\cdots\sum_{k_{p+q}=1}^{K}(\cdot)$. $p+q$ is referred to as the nonlinear degree of parameter $c_{pq}(\cdot)$, which corresponds to the (p+q)-degree nonlinear terms

$\prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i)$. $c_{0,1}(.)$ and $c_{1,0}(.)$ of degree 1 are referred to as linear parameters, and all the other model parameters are referred to as nonlinear parameters. A recursive algorithm to compute the nth-order GFRF for nonlinear Volterra systems described by A NARX model (1) is given in Peyton-Jones and Billings (1989). Let

$$H_{0,0}(\cdot)=1,\ H_{n,0}(\cdot)=0 \text{ for } n>0,\ H_{n,p}(\cdot)=0 \text{ for } n<p \tag{2a}$$

$$\exp(\sum_{i=1}^{q} \upsilon(p)) = \begin{cases} 1 & q=0, p>1 \\ 0 & q=0, p\le 1 \end{cases} \tag{2b}$$

and let

$$L_n(\omega_{r_1},\cdots,\omega_{r_n}) = 1-\sum_{k_1=1}^{K} c_{1,0}(k_1)\exp(-j(\omega_{r_1}+\cdots+\omega_{r_n})k_1) \tag{3}$$

Then the recursive algorithm can be written as

$$\begin{aligned} &H_n(j\omega_1,\cdots,j\omega_n) \\ &= \frac{1}{L_n(\omega_1\cdots\omega_n)} \sum_{q=0}^{n}\sum_{p=0}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}\Big[c_{p,q}(k_1,\cdots,k_{p+q}) \\ &\quad \cdot e^{-j\sum_{i=1}^{q}(\omega_{n-q+i}k_{p+i})} H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\Big] \end{aligned} \tag{4}$$

where,

$$\begin{aligned} H_{n,p}(\cdot) = \sum_{i=1}^{n-p+1}\big(&H_i(j\omega_1,\cdots,j\omega_i) \\ &\cdot H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_i)k_p)\big) \end{aligned} \tag{5}$$

$$\begin{aligned} &H_{n,1}(j\omega_1,\cdots,j\omega_n) \\ &\quad = H_n(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \end{aligned} \tag{6}$$

$H_n(j\omega_1,\cdots,j\omega_n)$ is the nth-order GFRF for NARX model (1). Note that the terms for cross input-output nonlinearities in (4) are corrected (This should be compared with the original results in Peyton-Jones and Billings (1989)). From (2-6), it can be seen that, although $H_n(j\omega_1,\cdots,j\omega_n)$ can be effectively computed by this recursive algorithm, the relationship between $H_n(j\omega_1,\cdots,j\omega_n)$ and the model parameters is not straightforward, and it is not clear about how the nonlinear parameters, which define the system nonlinearity, and the linear parameters, which define the first order GFRF of the NARX model, affect the GFRFs. This inhibits the understanding of the frequency domain characteristics of the GFRFs, and their connection to complex nonlinear behaviours. To solve this problem, the parametric characteristic analysis of the GFRFs for NARX models was studied by the authors (Jing et al 2006). Thus the nth-order GFRF can be expressed into a polynomial form as

$$H_n(j\omega_1,\cdots,j\omega_n) = CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot\overline{\varphi}_n(j\omega_1,\cdots,j\omega_n) \tag{7}$$

where $\overline{\varphi}_n(j\omega_1,\cdots,j\omega_n)$ is a complex valued function vector with an appropriate dimension, which is a function of the frequency variables and $H_1(j\omega_1)$, and referred to as the correlative function of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ in this paper; $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is referred to as the parametric characteristic of $H_n(j\omega_1,\cdots,j\omega_n)$, which can be recursively determined by

$$\begin{aligned} &CE(H_n(j\omega_1,\cdots,j\omega_n)) \\ &= C_{0,n} \oplus\left(\underset{q=1}{\overset{n-1}{\oplus}}\underset{p=1}{\overset{n-q}{\oplus}} C_{p,q}\otimes CE(H_{n-q-p+1}(\cdot))\right) \\ &\quad \oplus\left(\underset{p=2}{\overset{n}{\oplus}} C_{p,0}\otimes CE(H_{n-p+1}(\cdot))\right) \end{aligned} \tag{8}$$

where $C_{p,q} = [c_{p,q}(0,\cdots,0), c_{p,q}(0,\cdots,1),\cdots,c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})]$, $CE(.)$ is a novel coefficient extraction operator which has two basic operations "$\oplus$" and "$\otimes$".The detailed definition and operation rules for $CE(.)$ can be referred to Jing et al (2006, 2008). Note that from (8), elements of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ are monomial functions of the nonlinear parameters of degree from 2 to n.

Equation (7) provides a straightforward insight into the analytical relationship between the GFRFs and the system time-domain model parameters, and facilitates the frequency domain analysis of the nonlinear system characteristics (Jing et al 2006). In this study, a mapping function from $CE(H_n(j\omega_1,\cdots,j\omega_n))$ to $H_n(j\omega_1,\cdots,j\omega_n)$ is established in order to completely determine equation (7). Therefore, the complex valued correlative function $\overline{\varphi}_n(j\omega_1,\cdots,j\omega_n)$ in (7) can directly be determined in terms of the first order GFRF $H_1(j\omega_1)$ based on the parametric characteristic vector $CE(H_n(j\omega_1,\cdots,j\omega_n))$. That is, the nth-order GFRF can directly be written into the parametric characteristic function (7) in its detailed and analytical form by using this mapping function, and consequently some new properties of the GFRFs are revealed. For further derivations, the following result is needed.

Lemma 1 (Jing et al 2006). The elements of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ include and only include the nonlinear parameters in C_{0n} and all the nonlinear parameter monomial functions in $C_{pq}\otimes C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$ for $0\le k\le n-2$, where the subscripts satisfy

$$\left.\begin{aligned} 1\le p\le n-k \\ p+q+\sum_{i=1}^{k}(p_i+q_i)=n+k \\ 2\le p+q\le n-k,\ 2\le p_i+q_i\le n-k \end{aligned}\right\} \tag{9}$$

Lemma 1 provides a sufficient and necessary condition for which nonlinear parameters and how these parameters are included in $CE(H_n(j\omega_1,\cdots,j\omega_n))$. $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (5) can also be written as

$$\begin{aligned} &H_{n,p}(j\omega_1,\cdots,j\omega_n) \\ &= \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p} H_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})\exp(-j(\omega_{X+1}+\cdots+\omega_{X+r_i})k_i) \end{aligned} \tag{10}$$

where $X=\sum_{x=1}^{i-1} r_x$.

3. A NEW MAPPING FUNCTION

Definition 1. Let $S_C(n)$ be a set composed of all the elements in $CE(H_n(j\omega_1,\cdots,j\omega_n))$, and let $S_f(n)$ be a set composed of all the elements in $f_n(j\omega_1,\cdots,j\omega_n)$. Then there is a mapping

$$\varphi_n : S_C(n) \to S_f(n) \quad (11a)$$

such that in $\omega_1,\cdots,j\omega_n$,

$$\varphi_n(CE(H_n(j\omega_1,\cdots,j\omega_n))) = \overline{\varphi}_n(j\omega_1,\cdots,j\omega_n) \quad (11b)$$

(11ab) define a new mapping function in the parametric characteristics of the nth-order GFRF, which will be determined in this section. From Lemma 1, a monomial in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is either a single parameter coming from a pure input nonlinearity e.g. $c_{0n}(.)$, or a nonlinear parameter function of the form $C_{pq} \otimes C_{p_1q_1} \otimes C_{p_2q_2} \otimes \cdots \otimes C_{p_kq_k}$ satisfying (9), and the first parameter of $C_{pq} \otimes C_{p_1q_1} \otimes C_{p_2q_2} \otimes \cdots \otimes C_{p_kq_k}$ must come from a pure output nonlinearity or input-output cross nonlinearity, *i.e.*, $c_{pq}(.)$ with $p \geq 1$ and $p+q>1$.

Definition 2. A parameter monomial of the form $C_{pq} \otimes C_{p_1q_1} \otimes C_{p_2q_2} \otimes \cdots \otimes C_{p_kq_k}$ with $k \geq 0$ and $p+q>1$ is effective for $CE(H_n(j\omega_1,\cdots,j\omega_n))$ if the involved nonlinear parameters satisfy $p+q=n(>1)$ for $k=0$, or (9) is satisfied for $k>0$. □

All the parameter monomials in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ are effective for $CE(H_n(j\omega_1,\cdots,j\omega_n))$.

Lemma 2. For a monomial $c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ with k>0, then it is effective for the Z^{th}-order GFRF if and only if there is at least one parameter $c_{p_iq_i}(.)$ with $p_i>0$, where $Z=\sum_{i=0}^{k}(p_i+q_i)-k$.

Proof. This follows directly from Definition 2. Z can be computed from Lemma 1. This completes the proof. □

Lemma 3. For a monomial $c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ with $k>0$, if there are l different parameters with $p_i>0$, then there are l different cases in which this monomial is produced in the recursive computation of the ($\sum_{i=0}^{k}(p_i+q_i)-k$)th-order GFRF.

Proof. This can be concluded from the recursive algorithm (4). The proof is omitted. □

Definition 3. A (p,q)-partition of $H_n(j\omega_1,\cdots,j\omega_n)$ is a combination $H_{r_1}(w_{r_1})H_{r_2}(w_{r_2})\cdots H_{r_p}(w_{r_p})$ satisfying $\sum_{i=1}^{p} r_i = n-q$, where $1 \leq r_i \leq n-p-q+1$, and w_{r_i} is a set consisting of r_i different frequency variables such that $\bigcup_{i=1}^{p} w_{r_i} = \{\omega_1,\omega_2,\cdots,\omega_n\}$ and $w_{r_i} \cap w_{r_j} = \phi$ for $i \neq j$. □

From (10), it can be obtained that

$$H_{n-q,p}(j\omega_1,\cdots,j\omega_n) = \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i = n-q}}^{n-q-p+1} \prod_{i=1}^{p} H_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})\exp(-j(\omega_{X+1}+\cdots+\omega_{X+r_i})k_i) \quad (12)$$

The parameter $c_{p,q}(k_1,\cdots,k_{p+q})$ is the coefficient of $H_{n-q,p}(j\omega_1,\cdots,j\omega_n)$ in (4). From Definition 3, a (p,q)-partition of $H_n(j\omega_1,\cdots,j\omega_n)$ corresponds to a combination of $H_{r_1}(w_{r_1})H_{r_2}(w_{r_2})\cdots H_{r_p}(w_{r_p})$ in (12), which also corresponds to an effective parameter monomial initialized by the parameter $c_{p,q}(k_1,\cdots,k_{p+q})$. Note that $H_{n-q,p}(j\omega_1,\cdots,j\omega_n)$ includes all the possible permutations of $H_{r_1}(w_{r_1})H_{r_2}(w_{r_2})\cdots H_{r_p}(w_{r_p})$, hence it includes all the (p,q)-partitions.

Definition 4. A p-partition of an effective monomial $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ is a combination $s_{x_1}s_{x_2}\cdots s_{x_p}$ of the involved parameters such that $s_{x_i}s_{x_2}\cdots s_{x_p} = c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, where s_{x_i} is a monomial of x_i parameters in $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, $0 \leq x_i \leq k$, $s_0=1$, and each non-unitary s_{x_i} is an effective monomial.□

Suppose that a p-partition for 1 is $\underbrace{1\cdot 1\cdots 1}_{p} = 1$. The sub-monomial s_{x_i} in a p-partition of an effective monomial $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ is denoted by $s_{x_i}(c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot))$. Obviously, $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot) = s_k(c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot))$.

Lemma 4. If a monomial $c_{pq}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ is effective, and $c_{pq}(\cdot)$ is the initial parameter directly produced in the Zth-order GFRF and $p>0$, then (1) $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ comes from (p,q)-partitions of the Zth-order GFRF, where $Z= p+q+\sum_{i=1}^{k}(p_i+q_i)-k$; (2) if additionally s_0 comes from $H_1(.)$, then each p-partition of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ corresponds to a (p,q)-partition of the Zth-order GFRF, and each (p,q)-partition of the Zth-order GFRF produces at least one p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$.

Proof. The results follow from Lemma 2 and Definition 3. The detailed proof is omitted. □

Lemma 5. $c_{pq}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ is an effective monomial for the Zth-order GFRF, and $c_{pq}(\cdot)$ is the initial parameter satisfying $p>0$, then the correlative function of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ are the summation of the correlative functions from all the (p,q)-partitions of the Zth-order GFRF which produces $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, and therefore are the summation of the correlative functions corresponding to all the p-partition of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$.

Proof. It follows from Lemma 4. The detailed is omitted. □

From Lemma 5, all the (p,q)-partitions of the Zth-order GFRF which produce $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ are all the (p,q)-partitions corresponding to all the p-partitions for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$.

Proposition 1. For an effective nonlinear parameter monomial $c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, let

$$\bar{s} = c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot),\ n(s_x(\bar{s})) = \sum_{i=1}^{x}(p_i+q_i) - x + 1,$$

where x is the number of the parameters in s_x, $\sum_{i=1}^{x}(p_i+q_i)$ is the summation of the subscripts of all the parameters in s_x, and let $\sum_{i=1}^{x}(.)=0$ if $x<1$ and $n(1)=1$. Then for $0\le k\le n(\bar{s})-2$,

$$\begin{aligned}&\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\ &= \sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{pq}(\cdot) \text{ and } p>0}} \Big\{ f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\ &\cdot \sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot): s_{x_1}\cdots s_{x_p}}} \ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } s_{x_1}\cdots s_{x_p}}} \Big[f_2(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})\\ &\cdot\prod_{i=1}^{p}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot)))}(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})\Big]\Big\}\end{aligned} \quad (13)$$

The recursive computation stops at k=0 and $\varphi_1(1;\omega_i)=H_1(j\omega_i)$ or $\varphi_{p+q}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(p+q)})$, where,

$$\bar{X}(i) = \sum_{j=1}^{i-1} n(s_{\bar{x}_j}(\bar{s}/c_{pq}(\cdot))) \quad (14a)$$

$$f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) = \exp(-j\sum_{i=1}^{q}\omega_{l(n(\bar{s})-q+i)}k_{p+i}) \Big/ L_{n(\bar{s})}(\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \quad (14b)$$

$$f_2(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) = \exp(-j\sum_{i=1}^{p}k_i(\omega_{l(\bar{X}(i)+1)}+\cdots+\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})) \quad (14c)$$

Moreover, $s_{\bar{x}_1}\cdots s_{\bar{x}_p}$ is a permutation of $s_{x_1}\cdots s_{x_p}$, $\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}$ represents the frequency variables involved in the corresponding functions, $l(i)$ for $i=1\ldots n(\bar{s})$ is a positive integer representing the index of the frequency variables.

Proof. The recursive structure in (13) is directly followed from Lemma 3 and Lemma 5 based on the recursive Equation (4). The correlative function of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ are the summation of the correlative functions with respect to all the cases by which this monomial is produced in the same $n(\bar{s})$th-order GFRF. In each case it should include all the correlative functions corresponding to all the p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, and for each p-partition of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, the correlative function should include all the different permutations of $s_{x1}s_{x2}\ldots s_{xp}$, since the correlative function $f_2(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})$ is different with each different permutation which can be verified by Equation (12). $f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$ is a part of the correlative function for $c_{p,q}(k_1,\cdots,k_{p+q})$ except for $H_{n(\bar{s})-q,p}(j\omega_1,\cdots,j\omega_{n(\bar{s})-q})$, which directly follows from (4). $f_2(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})$ is a part of the correlative function with respect to a permutation of a p-partition $s_{x_1}\cdots s_{x_p}(\bar{s}/c_{pq}(\cdot))$ of the monomial $\bar{s}/c_{pq}(\cdot)$ which corresponds to a (p,q)-partition for the $n(\bar{s})$th-order GFRF, and it is followed from (12). This completes the proof. □

Remark 1. Equation (13) is recursive. The terminating condition is k=0, which is also included in (13). For k=0, it can be derived from (13) that

$$\begin{aligned}&\varphi_{n(\bar{s})}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\ &= f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\ &\quad\cdot\sum_{\substack{\text{all the p-partitions}\\ \text{for } 1}}\ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{0,\cdots,0\}}}\Big[f_2(1;\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})\\ &\quad\cdot\prod_{i=1}^{p}\varphi_{n(1)}(1;\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(1))})\Big]\\ &= f_1(c_{p,q}(\cdot),p+q;\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\cdot f_2(1;\omega_{l(1)}\cdots\omega_{l(p)})\\ &\quad\cdot\prod_{i=1}^{p}\varphi_{n(1)}(1;\omega_{l(i)})\\ &= \frac{\exp(-j\sum_{i=1}^{q}\omega_{l(p+i)}k_{p+i})\exp(-j\sum_{i=1}^{p}k_i\omega_{l(i)})}{L_{n(\bar{s})=p+q}(\omega_{l(1)}\cdots\omega_{l(p+q)})}\prod_{i=1}^{p}H_1(j\omega_{l(i)})\end{aligned} \quad (15)$$

Let $\prod_{i=1}^{p}(\cdot)=1$ if p=0 in (15). Note that in the derivation above, $n(\bar{s})$=p+q, and $\bar{s}=c_{p,q}(\cdot)$. □

Remark 2. Consider the three summation operations in (13). $\sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{pq}(\cdot) \text{ and } p>0}}\{\cdot\}$ follows from Lemma 3, which includes l cases for a parameter monomial having l different parameters satisfying $p>0$. For example, for $c_{1,1}(.)c_{2,0}(.)c_{1,1}(.)$, this summation includes the following cases: $s_1(c_{1,1}(.))s_2(c_{2,0}(.)c_{1,1}(.))$, $s_1(c_{2,0}(.))s_2(c_{1,1}(.)c_{1,1}(.))$

$\sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot):s_{x_1}\cdots s_{x_p}}}$ and $\sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } s_{x_1}\cdots s_{x_p}}}[\cdot]$ follow from Lemma 5. $\sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot):s_{x_1}\cdots s_{x_p}}}$ includes all the cases of the *p*-partitions denoted by $s_{x_1}\cdots s_{x_p}$ for the monomial $\bar{s}/c_{pq}(\cdot)$, and in a *p*-partition $\sum_{\substack{\text{all the different permutations}\\ \text{of } s_{x_1}\cdots s_{x_p}}}[\cdot]$ includes all the different permutations of the involved parameters satisfying the *p*-partition. □

Example 1. For a monomial $c_{1,1}(.)c_{0,2}(.)c_{2,0}(.)$ which is an effective monomial for the 4th-order GFRF, it can be obtained from Proposition 1 that

$$\begin{aligned}
&\varphi_4(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_4)\\
&=\frac{e^{-j\omega_4k_2}e^{-j\omega_3k_2}}{L_4(\omega_1\cdots\omega_4)L_3(\omega_1\cdots\omega_3)}\cdot\left\{\frac{e^{-j(k_1\omega_1+k_2(\omega_2+\omega_3))}e^{-j(\omega_2k_1+\omega_3k_2)}}{L_2(\omega_2,\omega_3)}H_1(j\omega_1)+\frac{e^{-j(k_1(\omega_1+\omega_2)+k_2\omega_3)}e^{-j(\omega_1k_1+\omega_2k_2)}}{L_2(\omega_1,\omega_2)}H_1(j\omega_3)\right\}\\
&+\frac{1}{L_4(\omega_1\cdots\omega_4)}\cdot\left\{\begin{aligned}&\frac{e^{-j(k_1\omega_1+k_2(\omega_2+\cdots+\omega_4))}e^{-j\omega_4k_2}e^{-jk_1(\omega_2+\omega_3)}e^{-j(\omega_2k_1+\omega_3k_2)}}{L_3(\omega_2\cdots\omega_4)L_2(\omega_2,\omega_3)}H_1(j\omega_1)\\
&+\frac{e^{-j(k_1(\omega_1+\cdots+\omega_3)+k_2\omega_4)}e^{-j\omega_3k_2}e^{-jk_1(\omega_1+\omega_2)}e^{-j(\omega_1k_1+\omega_2k_2)}}{L_3(\omega_1\cdots\omega_3)L_2(\omega_1,\omega_2)}H_1(j\omega_4)\\
&+\frac{e^{-j(k_1(\omega_1+\omega_2)+k_2(\omega_3+\omega_4))}e^{-j(\omega_1k_1+\omega_2k_2)}e^{-j\omega_4k_2}e^{-jk_1\omega_3}}{L_2(\omega_1,\omega_2)L_2(\omega_3,\omega_4)}H_1(j\omega_3)\\
&+\frac{e^{-j(k_1(\omega_1+\omega_2)+k_2(\omega_3+\omega_4))}e^{-j(\omega_3k_1+\omega_4k_2)}e^{-j\omega_2k_2}e^{-jk_1\omega_1}}{L_2(\omega_1,\omega_2)L_2(\omega_3,\omega_4)}H_1(j\omega_1)\end{aligned}\right\}
\end{aligned}\tag{16}$$

□

4. SOME PROPERTIES

4.1 Properties of the mapping function φ_n

$\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$ in Proposition 1 is asymmetric. Different permutation of $\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}$ may result in different value. The symmetric result can be obtained by

$$\begin{aligned}&{}^{sym}\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\&=\frac{1}{n(\bar{s})!}\sum_{\substack{\text{all the permutations}\\ \text{of }\{\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}\}}}\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\end{aligned}$$

Corollary 1. For an effective parametric monomial $c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, its correlative function is a ρ-degree function of $H_1(j\omega_{l(1)})$ which can be written as

$$\begin{aligned}&{}^{sym}\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\&=C^{\rho}_{n(\bar{s})}\sum_{\substack{\text{all the combinations of }\rho\text{ integers}\\ \{r_1,r_2,\cdots,r_\rho\}\text{taken from}\{1,2,\cdots,n(\bar{s})\}\\ \text{without repetition}\\ j\text{ is for different combination}}}\mu_j(\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\prod_{i=1}^{\rho}H_1(j\omega_{\bar{l}(i)})\end{aligned}\tag{17a}$$

The magnitude bound of (22a) can be evaluated by

$$\begin{aligned}&\left|{}^{sym}\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\right|\\&=\frac{\aleph(\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}))}{\underline{L}^{k+1}}\\&\quad\cdot C^{\rho}_{n(\bar{s})}\cdot\sum_{\substack{\text{all the combinations of}\\ \rho\text{ integers}\{r_1,r_2,\cdots,r_\rho\}\text{taken from}\\ \{1,2,\cdots,n(\bar{s})\}\text{without repetition}}}\prod_{i=1}^{\rho}\left|H_1(j\omega_{\bar{l}(i)})\right|\end{aligned}\tag{17b}$$

where $\rho=n(\bar{s})-\sum_{i=0}^{k}q_i=\sum_{i=0}^{k}p_i-k$, $\bar{l}=[r_1,r_2,\cdots,r_\rho]$, $C^{\rho}_{n(\bar{s})}=\frac{(n(\bar{s})-\rho)!\rho!}{n(\bar{s})!}$, $\mu_j(\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$ can be determined by equations (13-14), $\underline{L}=\inf_{\omega}(|L(\omega)|)$ and $\aleph(\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}))$ is a positive integer which can be determined by

$$\begin{aligned}&\aleph(\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}))\\&=\sum_{\substack{\text{all the 2-partitions}\\ \text{for }\bar{s}\text{ satisfying}\\ s_1(\bar{s})=c_{pq}(\cdot)\text{ and }p>0}}\ \sum_{\substack{\text{all the p-partitions}\\ \text{for }\bar{s}/c_{pq}(\cdot):s_{x_1}\cdots s_{x_p}}}\ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } s_{x_1}\cdots s_{x_p}}}\left(\prod_{i=1}^{p}\aleph(\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot)))}(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot));W^*))\right)\end{aligned}$$

and $\aleph(\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}))=1$ when k=0 or $n(\bar{s})=1$, where $W^*=\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))}$.

Proof. The proof is omitted. □

Recalling Equation (7), Corollary 2 shows that, the *n*th-order GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ can be expressed as an *n*-degree polynomial function of the first order GFRF $H_1(j\omega_i)$

$$\begin{aligned}&{}^{sym}H_n(j\omega_1,\cdots,j\omega_n)\\&=CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot{}^{sym}\varphi_n(j\omega_1,\cdots,j\omega_n)\end{aligned}$$

and its magnitude bound can be simply evaluated by

$$\begin{aligned}&\left|{}^{sym}H_n(j\omega_1,\cdots,j\omega_n)\right|\\&\le CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot\left|{}^{sym}\varphi_n(j\omega_1,\cdots,j\omega_n)\right|\end{aligned}$$

These results reveal how the first order GFRF, which represents the linear part of system model, affects the higher order GFRFs, together with the nonlinear dynamics. The conclusion in Corollary 2 can also be verified by Example 1.

4.2 Magnitude characteristic of the nth-order GFRF

By using Lemma 1 and Proposition 1, Equation (7) can now be determined definitely. Let $CE_n = CE(H_n(\cdot))$, $\Theta_n = \varphi_n(CE(H_n(\cdot)))\cdot\varphi_n(CE(H_n(\cdot)))^*$, $\varphi_n = \varphi_n(CE(H_n(\cdot)))$, and $\Lambda_n = CE(H_n(\cdot))^T CE(H_n(\cdot))$, it can be derived that

$$\begin{aligned}|H_n(j\omega_1,\cdots,j\omega_n)|^2 &= H_n(j\omega_1,\cdots,j\omega_n)\cdot H_n^*(j\omega_1,\cdots,j\omega_n)\\ &= CE(H_n(\cdot))\cdot\varphi_n(CE(H_n(\cdot)))\cdot(CE(H_n(\cdot))\cdot\varphi_n(CE(H_n(\cdot))))^*\\ &= CE(H_n(\cdot))\cdot\left(\varphi_n(CE(H_n(\cdot)))\cdot\varphi_n(CE(H_n(\cdot)))^*\right)\cdot CE(H_n(\cdot))^T\\ &= CE_n\Theta_n CE_n^T \qquad (18a)\end{aligned}$$

Similarly, it also holds that

$$|H_n(j\omega_1,\cdots,j\omega_n)|^2 = \varphi_n^*\Lambda_n\varphi_n \qquad (18b)$$

Denote $\lambda_M(A)$ to be the maximum eigenvalue of matrix A. The following result can be obtained.

Proposition 2.

$$\sup_{\omega_1,\cdots,\omega_n}|H_n(j\omega_1,\cdots,j\omega_n)| \le \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|h_n(\tau_1,\cdots,\tau_n)|d\tau_1\cdots d\tau_n \le \sup_{\omega_1,\cdots,\omega_n}(\lambda_M(\Theta_n))\cdot\|CE_n\|^2 \qquad (19a)$$

$$\sup_{\omega_1,\cdots,\omega_n}|H_n(j\omega_1,\cdots,j\omega_n)| \le \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|h_n(\tau_1,\cdots,\tau_n)|d\tau_1\cdots d\tau_n \le \lambda_M(\Lambda_n)\cdot\sup_{\omega_1,\cdots,\omega_n}(\|\varphi_n\|^2) \qquad (19b)$$

Proof. The proof is omitted. □

From Equations (18-19), it can be seen that the squared magnitude of the *n*th-order GFRF is proportional to a quadratic function of the parametric characteristic and also proportional to a quadratic function of the corresponding correlative function. These results demonstrate a new property of the *n*th-order GFRF, which reveals the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and its nonlinear parametric characteristic, and also the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and the correlative functions which include the linear (the first order GFRF) and the nonlinear behaviour. Proposition 2 also shows that the absolute integral of the *n*th-order Volterra kernel function in the time domain is bounded by a quadratic function of the parameter characteristic.

5. CONCLUSIONS

A mapping function from the parametric characteristics of the GFRFs to the GFRFs is established, such that the nth-order GFRF can directly be written into a more straightforward and meaningful form in terms of the first order GFRF and the model parameters. The new mapping function enables the linear and nonlinear factors included in the GFRFs to be unveiled explicitly, thus some new properties of the GFRFs can be obtained, which reveals clearly the relationship between the nth-order GFRF and its parametric characteristic, and also the relationship between the nth-order GFRF and the first order GFRF. These results provide a novel insight into the frequency domain analysis and design of nonlinear Volterra systems described by a NARX model based on the GFRFs.

REFERENCES

E. Bedrosian and S. O. Rice (1971), "The output properties of Volterra systems driven by harmonic and Gaussian inputs". *Proceedings of the Institute of Electrical and Electronics Engineering*, 59, 1688-1707

S.A. Billings and Z.Q. Lang (1996), "A bound of the magnitude characteristics of nonlinear output frequency response functions", *International Journal of Control*, Part 1, 65(2) (1996) 309-328 and Part 2, 65(3), pp365-380

S. Boyd and L. O. Chua (1985), "Fading memory and the problem of approximating nonlinear operators with Volterra series", *IEEE Trans. Circuits Syst.*, CAS-32 pp1150-1161

M.B. Brilliant (1958), "Theory of the analysis of non-linear systems", *Technical Report 345*, MIT, Research Laboratory of Electronics, Cambridge, Mass, Mar. 3

L.O. Chua and C.Y. Ng (1979), "Frequency domain analysis of nonlinear systems: general theory", *IEE Journal of Electronic Circuits and Systems*, 3(4), pp. 165–185

D.A. George (1959), "Continuous nonlinear systems", *Technical Report 355*, MIT Research Laboratory of Electronics, Cambridge, Mass. Jul. 24

X.J. Jing, Z.Q. Lang and S.A. Billings (2007), "New Bound Characteristics of NARX Model in the Frequency Domain", *International Journal of Control*, 80(1), pp140-149

X.J.Jing, Z.Q.Lang and S.A.Billings (2008), Frequency domain analysis for nonlinear volterra systems with a general nonlinear output function. *International Journal of Control,* 81(2), pp 235-251

X.J. Jing, Z.Q. Lang, S. A. Billings and G. R. Tomlinson (2006), "The Parametric Characteristics of Frequency Response Functions for Nonlinear Systems", *International Journal of Control*, Vol. 79, No. 12, December, pp 1552–1564

S. Kotsios (1997), "Finite input/output representative of a class of Volterra polynomial systems". *Automatica*, 33, pp257-262

J. C. Peyton-Jones and S. A. Billings (1989). "Recursive algorithm for computing the frequency response of a class of nonlinear difference equation models". *International Journal of Control*, 50(5), pp1925-1940

W. J. Rugh (1981). *Nonlinear system theory --- the Volterra/Wiener approach*. Baltimore and London: the Johns Hopkins University Press

V. Volterra (1959), Theory of Functionals and of Integral and Integro-differential Equations, Dover, New York