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$\mathcal{H}^\infty$ Control for Linear Discrete-Time Descriptor Systems: State Feedback and Full Information Cases

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Abstract: This paper addresses the $\mathcal{H}^\infty$ state feedback (SF) and full information (FI) control problems for linear discrete-time descriptor systems. It is shown that, under some rank assumptions, necessary and sufficient conditions for the solution to the problems can be characterized by an invertible symmetric solution to a certain generalized discrete-time algebraic Riccati inequality (GDARI) involving only one unknown parameter. When the problems have solutions, one such SF controller and one such FI controller are also given, expressed in terms of an invertible symmetric solution to the above-mentioned GDARI. It is also shown that the SF controller given coincides with the FI controller given in the presence of the worst disturbance input.

1. INTRODUCTION

Since the success of the celebrated paper written by Doyle et al. [1989], much effort has been devoted to extending the $\mathcal{H}^\infty$ control theory to linear discrete-time system; see for example, Stoorvogel [1989], Stoorvogel [1992a], Stoorvogel [1992b] and references therein. Lately, the solution to the $\mathcal{H}^\infty$ control problem for linear continuous-time descriptor systems has been shown that it can be characterized by two generalized algebraic Riccati equations (GARE), see Wang et al. [1998] and Takaba et al. [1994], or equivalently by two LMIs [Masubuchi et al., 1997].

Most recently, a necessary and sufficient condition has been given for the solution to the $\mathcal{H}_\infty$ state feedback control problem for linear discrete-time descriptor systems [Xu and Yang, 2000]. The condition given in Xu and Yang [2000] was expressed in terms of certain matrix inequalities involving several parameters.

The present paper continues this line of research to study the $\mathcal{H}^\infty$ control problem for linear discrete-time descriptor systems. More precisely, two kinds of $\mathcal{H}^\infty$ control problems will be addressed: the $\mathcal{H}^\infty$ state feedback (SF) and full information (FI) control problems for linear discrete-time descriptor systems. It will be shown that, under some rank assumptions, necessary and sufficient conditions for the solution to the problems can be characterized by an invertible symmetric solution of a certain generalized discrete-time algebraic Riccati inequality (GDARI) involving only one unknown parameter. When the system is in the state-space model, that is, the matrix $\boldsymbol{E} = \boldsymbol{I}$, the identity matrix, this GDARI exactly corresponds to the inequality version of discrete-time algeberic Riccati equation (DARE) given in Stoorvogel [1992b]. When the problems have solutions, one such SF controller and one such FI controller are also given, expressed in terms of an invertible symmetric solution to the above-mentioned GDARI. Relationship between $\mathcal{H}^\infty$ SF and FI controllers given from the viewpoint of game theory is also explored. It will be shown that the SF controller given coincides with the FI controller given in the presence of the worst disturbance input.

This paper is organized as follows: In Section 2, we briefly review some preliminary results concerning descriptor systems. The $\mathcal{H}^\infty$ SF and FI control problems are then formulated. Section 3 contains main results of the paper. Finally, in Section 4 we give some concluding remarks.

2. PRELIMINARY AND PROBLEM FORMULATION

In this section, we briefly summarize some basic definitions and preliminary results concerning descriptor systems; see Dai [1989], Verghese et al. [1981] and Lewis [1986], for example, for more details. Let $\boldsymbol{A}$ and $\boldsymbol{E}$ be $n \times n$ constant real matrices. Assume that rank$\boldsymbol{E} = r \le n$. The (ordered) pair $(\boldsymbol{E}, \boldsymbol{A})$ is said to be *regular* if there exists a scalar λ (may be real or complex) such that $\det(\lambda\boldsymbol{E} - \boldsymbol{A}) \neq 0$. Clearly, if $\det \boldsymbol{E} \neq 0$, $(\boldsymbol{E}, \boldsymbol{A})$ is regular. A scalar λ is called a finite eigenvalue of $(\boldsymbol{E}, \boldsymbol{A})$ if $\det(\lambda\boldsymbol{E} - \boldsymbol{A}) = 0$. Let $q \triangleq \deg\det(\lambda\boldsymbol{E} - \boldsymbol{A})$. Then it is quite well known that $(\boldsymbol{E}, \boldsymbol{A})$ has q finite dynamic modes, $r - q$ noncausal modes (called impulsive modes for continuous-time case) and $n - r$ nondynamic modes. Furthermore, if $r = q$, there exist no noncausal modes and in this case the system is said to be *causal* (impulse-free for continuous-time case). $(\boldsymbol{E}, \boldsymbol{A})$ is called *stable* if all the finite eigenvalues of $(\boldsymbol{E}, \boldsymbol{A})$ lie within the open unit disk. $(\boldsymbol{E}, \boldsymbol{A})$ is called *admissible* if $(\boldsymbol{E}, \boldsymbol{A})$ is regular, causal and stable.

In this paper, we shall be studying the $\mathcal{H}^\infty$ control problem for linear discrete-time descriptor systems. Consider the standard feedback configuration shown in Figure 1. Let the plant $\boldsymbol{\Sigma}$ be described by the dynamic equations:

$$\boldsymbol{\Sigma} : \begin{cases} \boldsymbol{E}\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{w}(k) + \boldsymbol{B}_2\boldsymbol{u}(k), \\ \boldsymbol{z}(k) = \boldsymbol{C}_1\boldsymbol{x}(k) + \boldsymbol{D}_{11}\boldsymbol{w}(k) + \boldsymbol{D}_{12}\boldsymbol{u}(k), \end{cases} \tag{1}$$

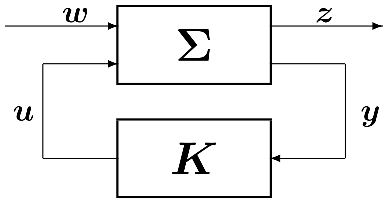

Fig. 1. Standard Feedback Configuration

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, and $\boldsymbol{w} \in \mathbb{R}^{m_1}$ represents a set of exogenous inputs which includes disturbances to be rejected and/or reference commands to be tracked. $\boldsymbol{z} \in \mathbb{R}^p$ is the output to be controlled. $\boldsymbol{u} \in \mathbb{R}^{m_2}$ is the control input. The matrices $\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C}_1, \boldsymbol{D}_{11}$, and $\boldsymbol{D}_{12}$ are constant matrices with compatible dimensions. $\boldsymbol{E} \in \mathbb{R}^{n\times n}$ with rank$\boldsymbol{E} \leq n$.

Two types of $\mathcal{H}^\infty$ control problem will be considered: $\mathcal{H}^\infty$ state feedback(SF) and full information(FI) control problems.

$\mathcal{H}^\infty$ SF control problem: Find a static state feedback of the form $\boldsymbol{u}(k) = \boldsymbol{F}\boldsymbol{x}(k)$ such that the resulting closed-loop system

$$\boldsymbol{\Sigma}_{\mathbf{SF}} : \begin{cases} \boldsymbol{E}\boldsymbol{x}(k+1) = (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{F})\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{w}(k), \\ \boldsymbol{z}(k) = (\boldsymbol{C}_1 + \boldsymbol{D}_{12}\boldsymbol{F})\boldsymbol{x}(k) + \boldsymbol{D}_{11}\boldsymbol{w}(k), \end{cases} \tag{2}$$

is admissible and $\boldsymbol{T_{zw}}$, the transfer matrix of the closed-loop system from $\boldsymbol{w}$ to $\boldsymbol{z}$, has $\mathcal{H}^\infty$ norm strictly less than a prescribed positive number γ.

$\mathcal{H}^\infty$ FI control problem: Find a full information of the form $\boldsymbol{u}(k) = \boldsymbol{F}\boldsymbol{x}(k) + \boldsymbol{H}\boldsymbol{w}(k)$ such that the resulting closed-loop system

$$\boldsymbol{\Sigma}_{\mathbf{FI}} : \begin{cases} \boldsymbol{E}\boldsymbol{x}(k+1) = & (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{F})\boldsymbol{x}(k) + (\boldsymbol{B}_1 \\ & +\boldsymbol{B}_2\boldsymbol{H})\boldsymbol{w}(k), \\ \boldsymbol{z}(k) = & (\boldsymbol{C}_1 + \boldsymbol{D}_{12}\boldsymbol{F})\boldsymbol{x}(k) + (\boldsymbol{D}_{11} \\ & +\boldsymbol{D}_{12}\boldsymbol{H})\boldsymbol{w}(k), \end{cases} \tag{3}$$

is admissible and $\boldsymbol{T_{zw}}$ has $\mathcal{H}^\infty$ norm strictly less than a prescribed positive number γ.

The following lemma, which is an extension version (for the case of $\boldsymbol{D} \neq 0$) of bounded real lemma taken from Xu and Yang [2000], is needed in our later development.

Lemma 1. (bounded real lemma) Consider a discrete-time descriptor system described by the following equations:

$$\begin{aligned} \boldsymbol{E}\boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k), \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k). \end{aligned}$$

Let $\boldsymbol{T_{yu}}(z) \triangleq \boldsymbol{C}(z\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$. Then, $(\boldsymbol{E}, \boldsymbol{A})$ is admissible and $\|\boldsymbol{T_{yu}}\|_\infty < \gamma$ if and only if there exists an invertible symmetric matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ satisfying the following:

(1) $\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} \geq 0$,
(2) $\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{D}^T\boldsymbol{D} - \gamma^2\boldsymbol{I} < 0$,
(3) $\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}^T\boldsymbol{C} - (\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{C}^T\boldsymbol{D})(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{D}^T\boldsymbol{D} - \gamma^2\boldsymbol{I})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{D}^T\boldsymbol{C}) < 0$.

□

3. MAIN RESULTS

3.1 State feedback case

The main result of this subsection is given in the following statements.

Theorem 2. Consider the plant $\boldsymbol{\Sigma}$ given in (1). Suppose that the following assumptions hold.

A1) rank$\boldsymbol{D}_{12} = m_2$.
A2) rank $[\,\boldsymbol{E}\ \ \boldsymbol{B}_2\,] =$ rank$\boldsymbol{E}$.

Then, the following statements are equivalent:

(1) The $\mathcal{H}^\infty$ SF control problem is solvable.
(2) There exists an invertible symmetric matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ satisfying the following:
  (a) $\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} \geq 0$,
  (b) $\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{11} - \gamma^2\boldsymbol{I} < 0$,
  (c) $\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{12} > 0$,
  (d) $\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}_1^T\boldsymbol{C}_1 - (\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{S})(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T) < 0$, where

$$\boldsymbol{B} \triangleq [\,\boldsymbol{B}_1\ \ \boldsymbol{B}_2\,], \boldsymbol{S} \triangleq \left[\,\boldsymbol{C}_1^T\boldsymbol{D}_{11}\ \ \boldsymbol{C}_1^T\boldsymbol{D}_{12}\,\right],$$

and

$$\boldsymbol{R} \triangleq \begin{bmatrix} \boldsymbol{D}_{11}^T\boldsymbol{D}_{11} - \gamma^2\boldsymbol{I} & \boldsymbol{D}_{11}^T\boldsymbol{D}_{12} \\ \boldsymbol{D}_{12}^T\boldsymbol{D}_{11} & \boldsymbol{D}_{12}^T\boldsymbol{D}_{12} \end{bmatrix}.$$

Moreover, when these conditions hold, one solution is given by $\boldsymbol{u}(k) = \boldsymbol{F}_2\boldsymbol{x}(k)$, where

$$\boldsymbol{F}_2 \triangleq -[\,\mathbf{0}\ \ \boldsymbol{I}_{m_2}\,](\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T), \tag{4}$$

and $\boldsymbol{I}_{m_2}$ is the $m_2 \times m_2$ identity matrix. □

Proof: 2) ⇒ 1). Set $\boldsymbol{M}_1 \triangleq \gamma^2\boldsymbol{I} - \boldsymbol{D}_{11}^T\boldsymbol{D}_{11} - \boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_1 > 0$. Note that the matrix

$$\begin{aligned} \boldsymbol{M} &\triangleq -(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}) \\ &= \begin{bmatrix} \boldsymbol{M}_1 & -\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 - \boldsymbol{D}_{11}^T\boldsymbol{D}_{12} \\ -\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 - \boldsymbol{D}_{12}^T\boldsymbol{D}_{11} & -\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_2 - \boldsymbol{D}_{12}^T\boldsymbol{D}_{12} \end{bmatrix} \end{aligned}$$

is invertible by Hypotheses 2b) and 2c), provided by Schur theorem [Horn and Johnson, 1990]. Now consider the plant $\boldsymbol{\Sigma}$ with the state feedback $\boldsymbol{u}(k) = \boldsymbol{F}_2\boldsymbol{x}(k)$. Then the closed-loop system becomes

$$\boldsymbol{\Sigma}_1 : \begin{cases} \boldsymbol{E}\boldsymbol{x}(k+1) = (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{F}_2)\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{w}(k), \\ \boldsymbol{z}(k) = (\boldsymbol{C}_1 + \boldsymbol{D}_{12}\boldsymbol{F}_2)\boldsymbol{x}(k) + \boldsymbol{D}_{11}\boldsymbol{w}(k), \end{cases} \tag{5}$$

Let

$$\boldsymbol{F}_1 \triangleq [\,\boldsymbol{I}_{m_1}\ \ \mathbf{0}\,]\boldsymbol{M}^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T). \tag{6}$$

Then, we have

$$\begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T).$$

Thus,

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix} = \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T, \tag{7}$$

or equivalently,

$$\boldsymbol{F}_1^T\boldsymbol{M}_1 = (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{F}_2)^T\boldsymbol{P}\boldsymbol{B}_1 + (\boldsymbol{C}_1 + \boldsymbol{D}_{12}\boldsymbol{F}_2)^T\boldsymbol{D}_{11},$$

and

$$\boldsymbol{F}_1^T(\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{11}^T\boldsymbol{D}_{12}) = -((\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F}_2)^T\boldsymbol{P}\boldsymbol{B}_2 + (\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F}_2)^T\boldsymbol{D}_{12}).$$

Using the above equalities, it is straightforward to show that

$$\begin{aligned}&(\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F}_2)^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F}_2)-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+\\&(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F}_2)^T(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F}_2)\\&+((\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F}_2)^T\boldsymbol{P}\boldsymbol{B}_1+(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F}_2)^T\boldsymbol{D}_{11})\boldsymbol{M}_1^{-1}\\&(\boldsymbol{B}_1^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F}_2)+\boldsymbol{D}_{11}^T(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F}_2))\\&=\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+\boldsymbol{C}_1^T\boldsymbol{C}_1-(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{S})(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}+\\&\boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{S}^T)<0.\end{aligned}$$

Then, by Lemma 1, it is concluded that the closed-loop system (5) is admissible and its transfer matrix $\boldsymbol{T_{zw}}$ has $\mathcal{H}^\infty$ norm strictly less than γ.

1) ⇒ 2). Let $\boldsymbol{u}(k)=\boldsymbol{F}\boldsymbol{x}(k)$ be a static state feedback such that the resulting closed-loop system (2) is admissible and $\|\boldsymbol{T_{zw}}\|_\infty<\gamma$. Then, by Lemma 1, there exists an invertible symmetric matrix $\boldsymbol{P}$ satisfying Conditions 2a) and 2b), and $\mathcal{R}ic_1(\boldsymbol{P})<0$, where

$$\begin{aligned}&\mathcal{R}ic_1(\boldsymbol{P})\triangleq(\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F})^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F})\\&-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F})^T(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F})\\&+((\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F})^T\boldsymbol{P}\boldsymbol{B}_1+(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F})^T\boldsymbol{D}_{11})\boldsymbol{M}_1^{-1}\\&(\boldsymbol{B}_1^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{F})+\boldsymbol{D}_{11}^T(\boldsymbol{C}_1+\boldsymbol{D}_{12}\boldsymbol{F})),\end{aligned}$$

with $\boldsymbol{M}_1$ defined as above. By Assumption A2), there exists a matrix $\boldsymbol{N}$ satisfying $\boldsymbol{B}_2=\boldsymbol{E}\boldsymbol{N}$. Together with Assumption A1) and Condition 2a), we have $\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{12}^T\boldsymbol{D}_{12}=\boldsymbol{N}^T\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}\boldsymbol{N}+\boldsymbol{D}_{12}^T\boldsymbol{D}_{12}>0$. Thus, Condition 2c) is also satisfied. It remains to verify that $\boldsymbol{P}$ satisfies Condition 2d). To do this, let $\boldsymbol{M}$ be defined as above. Recall that $\boldsymbol{M}$ is invertible by Conditions 2b) and 2c). Let

$$\boldsymbol{\Phi}\triangleq\begin{bmatrix}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{M}_1^{-1}\boldsymbol{B}_1^T\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B}_1\boldsymbol{M}_1^{-1}\boldsymbol{D}_{11}^T\\ \boldsymbol{D}_{11}\boldsymbol{M}_1^{-1}\boldsymbol{B}_1^T\boldsymbol{P} & \boldsymbol{I}+\boldsymbol{D}_{11}\boldsymbol{M}_1^{-1}\boldsymbol{D}_{11}^T\end{bmatrix}$$

and

$$\boldsymbol{\Theta}\triangleq\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}_2\\ \boldsymbol{C}_1 & \boldsymbol{D}_{12}\end{bmatrix}^T\boldsymbol{\Phi}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}_2\\ \boldsymbol{C}_1 & \boldsymbol{D}_{12}\end{bmatrix}\triangleq\begin{bmatrix}\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12}\\ \boldsymbol{\Theta}_{12}^T & \boldsymbol{\Theta}_{22}\end{bmatrix},$$

where

$$\begin{aligned}\boldsymbol{\Theta}_{11}&\triangleq\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{C}_1^T\boldsymbol{C}_1+(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}_1+\boldsymbol{C}_1^T\boldsymbol{D}_{11})\boldsymbol{M}_1^{-1}\\&\quad(\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{11}^T\boldsymbol{C}_1),\\ \boldsymbol{\Theta}_{12}&\triangleq(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}_1+\boldsymbol{C}_1^T\boldsymbol{D}_{11})\boldsymbol{M}_1^{-1}(\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\\&\quad+\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{C}_1^T\boldsymbol{D}_{12},\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\Theta}_{22}&\triangleq((\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{12}^T\boldsymbol{D}_{12})+(\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1+\boldsymbol{D}_{12}^T\boldsymbol{D}_{11})\boldsymbol{M}_1^{-1}\\&\quad(\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{11}^T\boldsymbol{D}_{12}))>0.\end{aligned}$$

A little bit of algebra shows that

$$\begin{aligned}\mathcal{R}ic_1(\boldsymbol{P})&=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{F}^T\end{bmatrix}\boldsymbol{\Theta}\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{F}\end{bmatrix}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}\\&=(\boldsymbol{\Theta}_{11}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E})+\boldsymbol{\Theta}_{12}\boldsymbol{F}\\&\quad+\boldsymbol{F}^T\boldsymbol{\Theta}_{12}^T+\boldsymbol{F}^T\boldsymbol{\Theta}_{22}\boldsymbol{F}\\&=\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+\boldsymbol{C}_1^T\boldsymbol{C}_1\\&\quad+(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{S})\boldsymbol{M}^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{S}^T)\\&\quad+(\boldsymbol{F}^T\boldsymbol{\Theta}_{22}+\boldsymbol{\Theta}_{12})\boldsymbol{\Theta}_{22}^{-1}(\boldsymbol{F}^T\boldsymbol{\Theta}_{22}+\boldsymbol{\Theta}_{12})^T.\end{aligned}$$

Since $\mathcal{R}ic_1(\boldsymbol{P})<0$ and $\boldsymbol{\Theta}_{22}>0$, this implies that

$$\begin{aligned}&\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+\boldsymbol{C}_1^T\boldsymbol{C}_1+\\&(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{S})\boldsymbol{M}^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{S}^T)<0.\end{aligned}$$

This completes the proof. ■

3.2 Full information case

We now turn our attention to the $\mathcal{H}^\infty$ FI control problem. The main result is summarized in the following theorem.

Theorem 3. Consider the plant $\boldsymbol{\Sigma}$ given in (1). Suppose that Assumptions A1) and A2) in Theorem 2 hold. Then, the following statements are equivalent:

(1) The $\mathcal{H}^\infty$ FI control problem is solvable.
(2) There exists an invertible symmetric matrix $\boldsymbol{P}\in\mathbb{R}^{n\times n}$ satisfying the following:
  (a) $\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}\geq 0$,
  (b) $\boldsymbol{M}_2\triangleq\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{12}^T\boldsymbol{D}_{12}>0$,
  (c)
$$\begin{aligned}\boldsymbol{\Lambda}&\triangleq\gamma^2\boldsymbol{I}-\boldsymbol{D}_{11}^T\boldsymbol{D}_{11}-\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_1\\&\quad+(\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2+\boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\boldsymbol{M}_2^{-1}\\&\quad(\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1+\boldsymbol{D}_{12}^T\boldsymbol{D}_{11})>0,\end{aligned}\tag{8}$$
  (d)
$$\begin{aligned}&\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+\boldsymbol{C}_1^T\boldsymbol{C}_1-(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{S})\\&(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{S}^T)<0,\end{aligned}$$
  where $\boldsymbol{B},\boldsymbol{S}$ and $\boldsymbol{R}$ are defined as in Theorem 2.

Moreover, when these conditions hold, one solution is given by $\boldsymbol{u}(k)=\widehat{\boldsymbol{F}}_1\boldsymbol{w}(k)+\widehat{\boldsymbol{F}}_2\boldsymbol{x}(k)$, where

$$\widehat{\boldsymbol{F}}_1=-\boldsymbol{M}_2^{-1}(\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1+\boldsymbol{D}_{12}^T\boldsymbol{D}_{11}),\tag{9}$$

and

$$\widehat{\boldsymbol{F}}_2=-\boldsymbol{M}_2^{-1}(\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{D}_{12}^T\boldsymbol{C}_1).\tag{10}$$

□

Proof. 2) ⇒ 1). As in Theorem 2, we set $\boldsymbol{M}_1\triangleq\gamma^2\boldsymbol{I}-\boldsymbol{D}_{11}^T\boldsymbol{D}_{11}-\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_1$ and $\boldsymbol{M}\triangleq-(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{R})$. Although $\boldsymbol{M}_1$ is not necessarily positive definite in the FI case, it still holds that the matrix $\boldsymbol{M}$ is invertible by Hypotheses 2b) and 2c). Now consider the plant $\boldsymbol{\Sigma}$ with the full information feedback $\boldsymbol{u}(k)=\widehat{\boldsymbol{F}}_1\boldsymbol{w}(k)+\widehat{\boldsymbol{F}}_2\boldsymbol{x}(k)$. Then the closed-loop system becomes

$$\boldsymbol{\Sigma}_2:\begin{cases}\boldsymbol{E}\boldsymbol{x}(k+1)=\boldsymbol{A}_F\boldsymbol{x}(k)+\boldsymbol{B}_{1F}\boldsymbol{w}(k)\\ \boldsymbol{z}(k)=\boldsymbol{C}_{1F}\boldsymbol{x}(k)+\boldsymbol{D}_{1F}\boldsymbol{w}(k),\end{cases}\tag{11}$$

where

$$\begin{aligned} \boldsymbol{A}_F &\triangleq \boldsymbol{A} + \boldsymbol{B}_2\widehat{\boldsymbol{F}}_2, \\ \boldsymbol{B}_{1F} &\triangleq \boldsymbol{B}_1 + \boldsymbol{B}_2\widehat{\boldsymbol{F}}_1, \\ \boldsymbol{C}_{1F} &\triangleq \boldsymbol{C}_1 + \boldsymbol{D}_{12}\widehat{\boldsymbol{F}}_2, \\ \boldsymbol{D}_{1F} &\triangleq \boldsymbol{D}_{11} + \boldsymbol{D}_{12}\widehat{\boldsymbol{F}}_1. \end{aligned}$$

Then it is straightforward to show that

$$\boldsymbol{B}_{1F}^T\boldsymbol{P}\boldsymbol{B}_{1F} + \boldsymbol{D}_{1F}^T\boldsymbol{D}_{1F} - \gamma^2\boldsymbol{I} = -\boldsymbol{\Lambda},$$

which is negative definite by Hypothesis 2c), and

$$\begin{aligned} &\boldsymbol{A}_F^T\boldsymbol{P}\boldsymbol{A}_F - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}_{1F}^T\boldsymbol{C}_{1F} \\ &- (\boldsymbol{A}_F^T\boldsymbol{P}\boldsymbol{B}_{1F} + \boldsymbol{C}_{1F}^T\boldsymbol{D}_{1F})(\boldsymbol{B}_{1F}^T\boldsymbol{P}\boldsymbol{B}_{1F} + \boldsymbol{D}_{1F}^T\boldsymbol{D}_{1F} \\ &-\gamma^2\boldsymbol{I})^{-1}(\boldsymbol{B}_{1F}^T\boldsymbol{P}\boldsymbol{A}_F + \boldsymbol{D}_{1F}^T\boldsymbol{C}_{1F}) \\ &= \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}_1^T\boldsymbol{C}_1 \\ &\quad -(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{S})(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T) \end{aligned}$$

which is negative definite by Hypothesis 2d). Together with $\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} \geq 0$, by Hypothesis 2a), it follows from Lemma 1 that the closed-loop system (11) is admissible and $\|\boldsymbol{T}_{\boldsymbol{zw}}\|_\infty < \gamma$.

1) $\Rightarrow$ 2). Let $\boldsymbol{u}(k) = \boldsymbol{F}\boldsymbol{x}(k) + \boldsymbol{H}\boldsymbol{w}(k)$ be a full information controller such that the closed-loop system (3) is admissible and $\|\boldsymbol{T}_{\boldsymbol{zw}}\|_\infty < \gamma$. Setting $\boldsymbol{v}(k) = \boldsymbol{u}(k) - \boldsymbol{H}\boldsymbol{w}(k) = \boldsymbol{F}\boldsymbol{x}(k)$ gets $\boldsymbol{u}(k) = \boldsymbol{v}(k) + \boldsymbol{H}\boldsymbol{w}(k)$. Thus, the static state feedback $\boldsymbol{v}(k) = \boldsymbol{F}\boldsymbol{x}(k)$ is a solution to the $\mathcal{H}^\infty$ SF control problem for the following plant:

$$\boldsymbol{\Sigma}_{\mathbf{EQ}}: \begin{cases} \boldsymbol{E}\boldsymbol{x}(k+1) = & \boldsymbol{A}\boldsymbol{x}(k) + (\boldsymbol{B}_1 + \boldsymbol{B}_2\boldsymbol{H})\boldsymbol{w}(k) \\ & +\boldsymbol{B}_2\boldsymbol{v}(k) \\ \boldsymbol{z}(k) = & \boldsymbol{C}_1\boldsymbol{x}(k) + (\boldsymbol{D}_{11} \\ & +\boldsymbol{D}_{12}\boldsymbol{H})\boldsymbol{w}(k) + \boldsymbol{D}_{12}\boldsymbol{v}(k). \end{cases}$$

It follows from Theorem 2 that there exists an invertible symmetric matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ such that $\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} \geq 0$ and $\boldsymbol{M}_2 \triangleq \boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{12} > 0$ (Thus Conditions 2a) and 2b) hold), and the matrix

$$\begin{aligned} \boldsymbol{Q} &\triangleq (\boldsymbol{B}_1 + \boldsymbol{B}_2\boldsymbol{H})^T\boldsymbol{P}(\boldsymbol{B}_1 + \boldsymbol{B}_2\boldsymbol{H}) \\ &+(\boldsymbol{D}_{11} + \boldsymbol{D}_{12}\boldsymbol{H})^T(\boldsymbol{D}_{11} + \boldsymbol{D}_{12}\boldsymbol{H}) - \gamma^2\boldsymbol{I} < 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}_1^T\boldsymbol{C}_1 - (\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}_H + \boldsymbol{S}_H) \\ &\quad (\boldsymbol{B}_H^T\boldsymbol{P}\boldsymbol{B}_H + \boldsymbol{R}_H)^{-1}(\boldsymbol{B}_H^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}_H^T) < 0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \boldsymbol{B}_H &\triangleq [\, \boldsymbol{B}_1 + \boldsymbol{B}_2\boldsymbol{H} \;\; \boldsymbol{B}_2 \,], \\ \boldsymbol{S}_H &\triangleq [\, \boldsymbol{C}_1^T\boldsymbol{D}_{11} + \boldsymbol{C}_1^T\boldsymbol{D}_{12}\boldsymbol{H} \;\; \boldsymbol{C}_1^T\boldsymbol{D}_{12} \,], \end{aligned}$$

and

$$\boldsymbol{R}_H \triangleq \begin{bmatrix} (\boldsymbol{D}_{11} + \boldsymbol{D}_{12}\boldsymbol{H})^T(\boldsymbol{D}_{11} + \boldsymbol{D}_{12}\boldsymbol{H}) - \gamma^2\boldsymbol{I} & (\boldsymbol{D}_{11} + \boldsymbol{D}_{12}\boldsymbol{H})^T\boldsymbol{D}_{12} \\ \boldsymbol{D}_{12}^T(\boldsymbol{D}_{11} + \boldsymbol{D}_{12}\boldsymbol{H}) & \boldsymbol{D}_{12}^T\boldsymbol{D}_{12} \end{bmatrix}$$

Let $\boldsymbol{M}_1$ be defined as above, and write $\boldsymbol{Q}$ as

$$\begin{aligned} \boldsymbol{Q} &= [\, \boldsymbol{I} \;\; \boldsymbol{H}^T \,] \begin{bmatrix} \boldsymbol{B}_1^T \\ \boldsymbol{B}_2^T \end{bmatrix} \boldsymbol{P} [\, \boldsymbol{B}_1 \;\; \boldsymbol{B}_2 \,] \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} \\ &+ \;\; [\, \boldsymbol{I} \;\; \boldsymbol{H}^T \,] \begin{bmatrix} \boldsymbol{D}_{11}^T \\ \boldsymbol{D}_{12}^T \end{bmatrix} [\, \boldsymbol{D}_{11} \;\; \boldsymbol{D}_{12} \,] \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} \\ &- [\, \boldsymbol{I} \;\; \boldsymbol{H}^T \,] \begin{bmatrix} \gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} \\ &= [\, \boldsymbol{I} \;\; \boldsymbol{H}^T \,] \begin{bmatrix} -\boldsymbol{M}_1 & \boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{12} \\ \boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{11} & \boldsymbol{M}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} \\ &= -\boldsymbol{M}_1 + \boldsymbol{H}^T(\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{11}) \\ &\quad +(\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\boldsymbol{H} + \boldsymbol{H}^T\boldsymbol{M}_2\boldsymbol{H} \\ &= -\boldsymbol{M}_1 - (\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\boldsymbol{M}_2^{-1} \\ &\quad (\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{11}) \\ &\quad + (\boldsymbol{H}^T\boldsymbol{M}_2 + \boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\boldsymbol{M}_2^{-1} \\ &\quad (\boldsymbol{M}_2\boldsymbol{H} + \boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{11}). \end{aligned}$$

Since $\boldsymbol{Q} < 0$, we have

$$\begin{aligned} &\boldsymbol{M}_1 + (\boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\boldsymbol{M}_2^{-1} \\ &(\boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{11}) \\ &> (\boldsymbol{H}^T\boldsymbol{M}_2 + \boldsymbol{B}_1^T\boldsymbol{P}\boldsymbol{B}_2 + \boldsymbol{D}_{11}^T\boldsymbol{D}_{12})\boldsymbol{M}_2^{-1} \\ &(\boldsymbol{M}_2\boldsymbol{H} + \boldsymbol{B}_2^T\boldsymbol{P}\boldsymbol{B}_1 + \boldsymbol{D}_{12}^T\boldsymbol{D}_{11}) \geq 0. \end{aligned}$$

Consequently, Condition 2c) holds. Finally, by use of

$$\boldsymbol{B}_H = \boldsymbol{B}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{H} & \boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{S}_H = \boldsymbol{S}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{H} & \boldsymbol{I} \end{bmatrix},$$

and

$$\boldsymbol{R}_H = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{H} & \boldsymbol{I} \end{bmatrix}^T \boldsymbol{R} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{H} & \boldsymbol{I} \end{bmatrix},$$

it is easy to verify that

$$\begin{aligned} &\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}_1^T\boldsymbol{C}_1 \\ &\quad -(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}_H + \boldsymbol{S}_H)(\boldsymbol{B}_H^T\boldsymbol{P}\boldsymbol{B}_H + \boldsymbol{R}_H)^{-1}(\boldsymbol{B}_H^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}_H^T) \\ &= \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E} + \boldsymbol{C}_1^T\boldsymbol{C}_1 \\ &\quad -(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{S})(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} + \boldsymbol{S}^T). \end{aligned}$$

By inequality (12), this completes the proof. ∎

Remark: In fact, the rank conditions A1) and A2) are used only in the necessity i.e., 1) $\Rightarrow$ 2) proof in both Theorems 2 and 3.

3.3 Relationship between SF and FI controllers

The following theorem connects SF and FI controllers in an elegant way.

Theorem 4. Consider the plant $\boldsymbol{\Sigma}$ given in (1). Suppose that Assumptions A1) and A2) in Theorem 2 hold. Then,

(1) If the $\mathcal{H}^\infty$ SF control problem is solvable, then the $\mathcal{H}^\infty$ FI control problem is solvable.

(2) Suppose the $\mathcal{H}^\infty$ SF control problem is solvable. Let $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ be an invertible symmetric matrix satisfying Conditions 2a)-2d) in Theorem 2. Then,
  (a) $\boldsymbol{P}$ also satisfies Conditions 2a)-2d) in Theorem 3.
  (b) Let $\boldsymbol{F}_1, \boldsymbol{F}_2, \widehat{\boldsymbol{F}}_1$, and $\widehat{\boldsymbol{F}}_2$ be the matrices given in (6), (4), (9), (10) respectively. Then,

$$\boldsymbol{F}_2 - \widehat{\boldsymbol{F}}_2 = \widehat{\boldsymbol{F}}_1\boldsymbol{F}_1. \qquad (13)$$

□

Proof. 1) and 2a): Trival.
2b): Note that the matrix $\boldsymbol{M}$ can be written as $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 & \widehat{\boldsymbol{F}}_1^T\boldsymbol{M}_2 \\ \boldsymbol{M}_2\widehat{\boldsymbol{F}}_1 & -\boldsymbol{M}_2 \end{bmatrix}$. Multiplying out (7) gets

$$\boldsymbol{M}_2\widehat{\boldsymbol{F}}_1\boldsymbol{F}_1 - \boldsymbol{M}_2\boldsymbol{F}_2 = -\boldsymbol{M}_2\widehat{\boldsymbol{F}}_2,$$

from which (13) immediately follows. ■

We shall now give an interpretation of (13) from the viewpoint of game theory. Let $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ be an invertible symmetric matrix satisfying Conditions 2a)-2d) in Theorem 2. It is well known that the $\mathcal{H}^\infty$ control problem can be viewed as a two players, zero-sum game described by (1), in which the minimizing player controls the input $\boldsymbol{u}$ and the maximizing player controls the disturbance $\boldsymbol{w}$. Associated with the game, we define a Hamiltonian function

$$\begin{aligned}\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{u}) = &(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}_1\boldsymbol{w}+\boldsymbol{B}_2\boldsymbol{u})^T\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}_1\boldsymbol{w}+\boldsymbol{B}_2\boldsymbol{u})\\ &-\boldsymbol{x}^T\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}\boldsymbol{x}\\ &+\|\boldsymbol{C}_1\boldsymbol{x}+\boldsymbol{D}_{11}\boldsymbol{w}+\boldsymbol{D}_{12}\boldsymbol{u}\|^2-\gamma^2\|\boldsymbol{w}\|^2.\end{aligned}$$

Let $\boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{F}_2\boldsymbol{x}, \boldsymbol{w}^*(\boldsymbol{x}) = \boldsymbol{F}_1\boldsymbol{x}, \widehat{\boldsymbol{u}}^*(\boldsymbol{x},\boldsymbol{w}) = \widehat{\boldsymbol{F}}_1\boldsymbol{w}+\widehat{\boldsymbol{F}}_2\boldsymbol{x}$, and

$$\begin{aligned}\widehat{\boldsymbol{w}}^*(\boldsymbol{x}) = &\boldsymbol{\Lambda}^{-1}((\boldsymbol{B}_1+\boldsymbol{B}_2\widehat{\boldsymbol{F}}_1)^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}_2\widehat{\boldsymbol{F}}_2)\\ &+(\boldsymbol{D}_{11}+\boldsymbol{D}_{12}\widehat{\boldsymbol{F}}_1)^T(\boldsymbol{C}_1+\boldsymbol{D}_{12}\widehat{\boldsymbol{F}}_2))\boldsymbol{x},\end{aligned}$$

where $\boldsymbol{F}_1, \boldsymbol{F}_2, \widehat{\boldsymbol{F}}_1, \widehat{\boldsymbol{F}}_2$ and $\boldsymbol{\Lambda}$ are the matrices given in (6), (4), (9), (10), and (8) respectively. Then, it is straightforward to show that

$$\min_{\boldsymbol{u}}\mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*(\boldsymbol{x}),\boldsymbol{u}) = \mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*(\boldsymbol{x}),\boldsymbol{u}^*(\boldsymbol{x}))$$

and

$$\max_{\boldsymbol{w}}\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{u}^*(x)) = \mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*(\boldsymbol{x}),\boldsymbol{u}^*(\boldsymbol{x})).$$

Consequently,

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{u}^*) \le \mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*,\boldsymbol{u}^*) \le \mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*,\boldsymbol{u}). \qquad (14)$$

Thus, $(\boldsymbol{w}^*, \boldsymbol{u}^*)$ constitutes a saddle point of the game. In view of (14), $\boldsymbol{w}^*(\boldsymbol{x})$ can be interpreted as the worst possible disturbance input affecting the plant and $\boldsymbol{u}^*(\boldsymbol{x})$ is the best strategy for counteracting the influence of the worst disturbance. Moreover, it can be easily verified that

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*(\boldsymbol{x}),\boldsymbol{u}^*(\boldsymbol{x})) = \boldsymbol{x}^T\mathcal{R}ic_2(\boldsymbol{P})\boldsymbol{x},$$

where

$$\begin{aligned}\mathcal{R}ic_2(\boldsymbol{P}) \triangleq &\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}-\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{E}+\boldsymbol{C}_1^T\boldsymbol{C}_1\\ &-(\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{S})(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{R})^{-1}(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}+\boldsymbol{S}^T),\end{aligned}$$

and $\boldsymbol{B}, \boldsymbol{S}, \boldsymbol{R}$ are defined as in Theorem 2. It is easily shown that Condition 2d) in Theorem 2 leads to

$$\sum_{k=0}^{N}(\|\boldsymbol{z}(k)\|^2-\gamma^2\|\boldsymbol{w}(k)\|^2) \le 0$$

for all nonnegative integers N, which in turn implies $\|\boldsymbol{T}_{\boldsymbol{zw}}\|_\infty \le \gamma$.

On the other hand, it is straightforward to show that

$$\min_{\boldsymbol{u}}\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{u}) = \mathcal{K}(\boldsymbol{x},\boldsymbol{w},\widehat{\boldsymbol{u}}^*(\boldsymbol{x},\boldsymbol{w}))$$

and

$$\begin{aligned}\max_{\boldsymbol{w}}\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\widehat{\boldsymbol{u}}^*(\boldsymbol{x},\boldsymbol{w})) &= \max_{\boldsymbol{w}}\min_{\boldsymbol{u}}\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{u})\\ &= \mathcal{K}(\boldsymbol{x},\widehat{\boldsymbol{w}}^*(\boldsymbol{x}),\widehat{\boldsymbol{u}}^*(\boldsymbol{x},\widehat{\boldsymbol{w}}^*(\boldsymbol{x}))).\end{aligned}$$

Furthermore, by using (13) and

$$\boldsymbol{\Lambda} = \boldsymbol{M}_1 + \widehat{\boldsymbol{F}}_1^T\boldsymbol{M}_2\widehat{\boldsymbol{F}}_1,$$

it is straightforward to show that

$$\boldsymbol{w}^*(\boldsymbol{x}) = \widehat{\boldsymbol{w}}^*(\boldsymbol{x}),$$

and

$$\boldsymbol{u}^*(\boldsymbol{x}) = \widehat{\boldsymbol{u}}^*(\boldsymbol{x},\widehat{\boldsymbol{w}}^*(\boldsymbol{x})) = \widehat{\boldsymbol{u}}^*(\boldsymbol{x},\boldsymbol{w}^*(\boldsymbol{x})). \qquad (15)$$

As a result, we have

$$\begin{aligned}\mathcal{K}(\boldsymbol{x},\widehat{\boldsymbol{w}}^*(\boldsymbol{x}),\widehat{\boldsymbol{u}}^*(\boldsymbol{x},\widehat{\boldsymbol{w}}^*(\boldsymbol{x}))) &= \mathcal{K}(\boldsymbol{x},\boldsymbol{w}^*(\boldsymbol{x}),\boldsymbol{u}^*(\boldsymbol{x}))\\ &= \boldsymbol{x}^T\mathcal{R}ic_2(\boldsymbol{P})\boldsymbol{x}.\end{aligned}$$

In view of (15), it is thus concluded that the SF controller $\boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{F}_2\boldsymbol{x}$ coincides with the FI controller $\widehat{\boldsymbol{u}}^*(\boldsymbol{x},\boldsymbol{w}) = \widehat{\boldsymbol{F}}_1\boldsymbol{w}+\widehat{\boldsymbol{F}}_2\boldsymbol{x}$ in the presence of the worst disturbance input $\boldsymbol{w}^*(\boldsymbol{x})$, that is,

$$\boldsymbol{F}_2\boldsymbol{x} = \widehat{\boldsymbol{F}}_1\boldsymbol{w}^*(\boldsymbol{x}) + \widehat{\boldsymbol{F}}_2\boldsymbol{x}.$$

4. CONCLUSIONS

This paper has extensively addressed the $\mathcal{H}^\infty$ SF and FI control problems for linear discrete-time descriptor systems. Under some rank assumptions, necessary and sufficient conditions for solution to the problems have been given in terms of an invertible symmetric solution of a certain generalized discrete-time algebraic Riccati inequality (GDARI) involving only one unknown parameter. When the system is in the state-space model, i.e., the matrix $\boldsymbol{E} = \boldsymbol{I}$, this GDARI exactly corresponds to the inequality version of discrete-time algeberic Riccati equation (DARE) given in the literature. When the problems have solutions, one such SF controller and one such FI controller have also been given, expressed in terms of an invertible symmetric solution of the above-mentioned GDARI. Finally, it has also been shown that the SF controller given coincides with the FI controller given in the presence of the worst disturbance input.

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