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Autonomous linear lossless systems

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Abstract: We define a lossless autonomous system as one having a quadratic differential form associated with it called an energy function, which is positive and which is conserved. We define an oscillatory system as one which has all its trajectories bounded on the entire time axis. In this paper, we show that an autonomous system is lossless if and only if it is oscillatory. Next we discuss a few properties of energy functions of autonomous lossless systems and a suitable way of splitting a given energy function into its kinetic and potential energy components.

1. INTRODUCTION

In most of the work done so far in the area of lossless systems, characterisation of losslessness is done assuming a given supply rate or the rate at which external work is done. An example for such a characterisation is the one by Willems [1972], in which losslessness was defined with respect to a given supply rate. The system is called lossless if the given supply rate is the derivative of another function, known as the storage function, along the trajectories of the system.

A lot of research has been carried out in the area of characterisation of lossless systems in the state-space. Weiss et al. [2001] and Weiss and Tucsnak [2003] have given algebraic characterizations of energy preserving and of conservative linear systems based on a state space description of the system. Here, a system is called energy preserving if the rate of change of a positive definite function defined on its state space, is equal to the difference between an incoming power and an outgoing power, which are respectively assumed to be the square of the norms of the input signal u and the output signal y. Note that in the sense of Willems [1972], if a system is energy preserving, then it is lossless with respect to the difference between the incoming and outgoing power. For a given energy preserving system, a related system known as its dual is defined by Weiss et al. [2001]. Here, a system is called conservative if both the system and its dual are energy preserving. In addition, Weiss et al. [2001] also give results about the stability, controllability and observability of conservative systems and illustrate these with the help of a model of a controlled beam.

The purpose of this paper is to give a definition of linear lossless systems which agrees with the basic intuition, derived from physics, that the external work done on such a system is equal to the difference between the final and initial values of the total energy for the system. We also make use of the fact that the total energy of such a system is a quadratic functional in the system variables and their derivatives that is positive for all infinitely differentiable non-zero trajectories.

An autonomous system is a system with no inputs or free variables. For such a system, the future of every trajectory is completely determined by its past. We characterize autonomous lossless systems based on the observation that the external work done on such a system is zero, because of the absence of inputs and hence total energy of such a system remains a constant.

The main aim of this paper is to give a characterisation of higher order autonomous lossless systems. We now explain this using an example of a mechanical system.

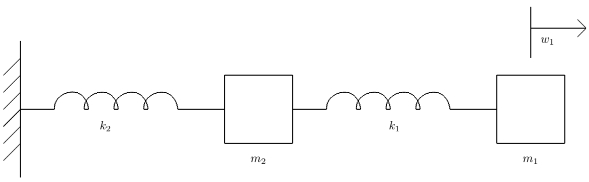

Fig. 1. A mechanical example

Example 1. Consider two masses m_1 and m_2 attached to springs with constants k_1 and k_2. The first mass is connected to the second one via the first spring, and the second mass is connected to the wall with the second spring as shown in Figure 1. Denote by w_1 and w_2, the positions of the first and the second mass respectively. We first obtain the equations of motion of the two masses as

$$m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_1 w_2 = 0 \qquad (1)$$

$$-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 = 0 \qquad (2)$$

Assume that in this case, we are interested only in the evolution of w_1. Via the process of elimination, we can lump equations (1) and (2) to obtain the differential equation governing w_1 as

$$r\left(\frac{d}{dt}\right) w_1 = \frac{d^4}{dt^4} w_1 + \left(\frac{k_1 + k_2}{m_2} + \frac{k_1}{m_1}\right) \frac{d^2}{dt^2} w_1 + \left(\frac{k_1 k_2}{m_1 m_2}\right) w_1 = 0$$

The above is a first principles model for the system. Note that mathematical modeling of a system in general, does not automatically lead to first order equations or a state space model for the system. A state space model, in many cases needs artificial construction of states from the given model. This calls for a need to deduce the properties of a system using its higher order governing differential equations. In the case of our example, we obtain a fourth order governing differential equation. Can we deduce directly from the higher order governing differential equations, whether the system is lossless or not? Can we obtain expressions for the total energy of the system, and its kinetic and potential energy components directly from the higher order governing differential equations of a lossless system? These are some of the questions that we answer in this paper.

We assume that the reader is familiar with the calculus of quadratic differential forms (QDFs), and with the behavioral framework, and we refer to respectively Willems and Trentelman [1998] and Polderman and Willems [1997] for a thorough exposition of the concepts and mathematical techniques.

The structure of the paper is as follows: In section 2, we discuss properties of oscillatory systems and the notions of conserved quantities for oscillatory systems and of $\mathfrak{B}$-canonicity and positivity of QDFs. In section 3, we prove the equivalence of autonomous lossless and oscillatory systems.

The notation used in this paper is standard: we denote the space of $\mathtt{n}$ dimensional real and complex vectors by $\mathbb{R}^{\mathtt{n}}$ and $\mathbb{C}^{\mathtt{n}}$ respectively, the space of $\mathtt{m}\times\mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ and the space of $\mathtt{m}\times\mathtt{n}$ symmetric real matrices, by $\mathbb{R}_s^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{R}^{\bullet\times\mathtt{n}}$ denotes the set of real matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^{\bullet}$ whose elements are commonly denoted with w, we use the notation $\mathbb{R}^{\mathtt{w}}$ (note the typewriter font type!). The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$. We denote with $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$. $\mathbb{R}^{+}$ denotes the set of positive real numbers. $0_{\mathtt{p}\times\mathtt{q}}$ denotes a matrix of size $\mathtt{p}\times\mathtt{q}$ consisiting of zeroes. If L_1 and L_2 are matrices having the same number of columns, then $\mathrm{col}(L_1,L_2)$ denotes the matrix obtained by stacking the matrix L_1 over L_2. $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real and imaginary parts of a complex quantity λ. $\mathrm{diag}(a_1,\ldots,a_n)$ denotes the diagonal matrix whose diagonal entries are $a_1,\ldots,a_n$ in the given order.

2. PRELIMINARIES

In this section, we illustrate the basic definitions and concepts of Rapisarda and Willems [2005] necessary to understand the results illustrated in this paper.

2.1 Oscillatory systems

Definition 2. A behavior $\mathfrak{B}$ is a linear differential behavior if $\mathfrak{B}$ is the set of solutions of a system of linear constant-coefficient differential equations

$$R\left(\frac{d}{dt}\right)w=0,\qquad R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi];$$

We denote the class of linear differential behaviors with $\mathtt{w}$ external variables by $\mathcal{L}^{\mathtt{w}}$.

Definition 3. A behavior $\mathfrak{B}$ defines a linear oscillatory system if

- $\mathfrak{B}\in\mathcal{L}^{\mathtt{w}}$.
- Every solution $w:\mathbb{R}\to\mathbb{R}^{\mathtt{w}}$ is bounded on $(-\infty,\infty)$.

From the definition, it follows that an oscillatory system is necessarily autonomous: if there were any input variables in w, then those components of w could be chosen to be unbounded.

In the following, the case of multivariable ($\mathtt{w}>1$) oscillatory systems will be often reduced to the scalar case by using the Smith form of a polynomial matrix. Consequently, we now examine in more detail the properties of scalar oscillatory systems and of their representation.

It was proved in proposition 2 of Rapisarda and Willems [2005] that any behavior $\mathfrak{B}$ is oscillatory if and only if every non-zero invariant polynomial of $\mathfrak{B}$ has distinct and purely imaginary roots. Consequently, if $r\in\mathbb{R}[\xi]$ then $\mathfrak{B}=\ker\left(r(\frac{d}{dt})\right)$ defines an oscillatory system if and only if all the roots of r are distinct and on the imaginary axis. From this it follows that r has one of the following two forms.

$$\begin{aligned}&r(\xi)=(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)\qquad\text{or}\\&r(\xi)=\xi(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)\end{aligned}$$

where $\omega_0,\ldots,\omega_{n-1}\in\mathbb{R}^{+}$. Recall from Polderman and Willems [1997], p. 69 that the dimension of $\ker\left(r(\frac{d}{dt})\right)$ as a linear subspace of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})$ equals the degree of the polynomial r and that the roots of r are called the *characteristic frequencies* of $\ker\left(r(\frac{d}{dt})\right)$.

In the following, a polynomial matrix will be called oscillatory if all its invariant polynomials have distinct and purely imaginary roots.

2.2 Quadratic differential forms (QDFs)

Consider the set of bilinear functionals acting on an infinitely differentiable trajectory w of the form

$$Q_{\Phi}(w)=\sum_{h,k=0}^{N}\left(\frac{d^hw}{dt^h}\right)^{\top}\Phi_{h,k}\left(\frac{d^kw}{dt^k}\right)\qquad(3)$$

where $\Phi_{h,k}$ are $\mathtt{w}\times\mathtt{w}$-dimensional real matrices, and N is a non-negative integer. Such a functional is called a *quadratic differential form* (QDF). With the QDF given by equation (3), we associate a two-variable polynomial matrix $\Phi(\zeta,\eta)$, which is given by

$$\Phi(\zeta,\eta)=\sum_{h,k=0}^{N}\Phi_{h,k}\zeta^h\eta^k$$

2.3 $\mathfrak{B}$-canonicity of QDFs

Consider a behavior $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Consider the equivalence relation between QDFs Q_Φ, Q_Ψ ($\Phi, \Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$), defined by

$$Q_\Phi \overset{\mathfrak{B}}{\sim} Q_\Psi \Leftrightarrow Q_\Phi(w) = Q_\Psi(w) \qquad \forall w \in \mathfrak{B}$$

It is easy to see that the set of equivalence classes under $\overset{\mathfrak{B}}{\sim}$ is a linear vector space over $\mathbb{R}$. Let $\mathfrak{B} = \ker R(\frac{d}{dt})$ be a kernel representation of a given autonomous behavior $\mathfrak{B}$. With every equivalence class of QDFs associated with $\mathfrak{B}$, we associate a certain representative known as the R-canonical representative. Below, we define the notion of R-canonicity of QDFs.

Definition 4. Consider $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, with $\det(R(\xi)) \neq 0$. Then the QDF Q_Φ is R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ is strictly proper.

If $R \in \mathbb{R}[\xi]$ and has degree n, then from the definition, it follows that the two-variable polynomials associated with R-canonical QDFs are spanned by monomials $\zeta^k\eta^j$, with $k, j \leq n-1$. It is easy to see that every QDF has an R-canonical representative.

2.4 Conserved quantities associated with an oscillatory behavior

Definition 5. Consider $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. A QDF Q_Φ is a conserved quantity associated with $\mathfrak{B}$ if

$$\frac{d}{dt}Q_\Phi(w) = 0 \quad \forall \quad w \in \mathfrak{B}.$$

Note that the trivial QDF $Q_\Phi = 0$ is always conserved. Any conserved QDF which is identically not equal to zero will be called "nontrivial conserved quantity" in the following. Consider an oscillatory behavior $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$, where $r \in \mathbb{R}[\xi]$. If r is an even polynomial of degree $2n$, then it can be shown (see p. 188 of Rapisarda and Willems [2005]) that the two-variable polynomials $\gamma_i(\zeta,\eta)$ given by

$$\gamma_i(\zeta,\eta) = \frac{r(\zeta)\eta^{2i+1} + r(\eta)\zeta^{2i+1}}{\zeta+\eta}$$

$i = 0, 1, \ldots, n-1$, induce a basis for the space of r-canonical conserved quantities over $\mathfrak{B}$. If r is an odd polynomial of degree $2n+1$, then it can be shown that a basis of r-canonical conserved quantities over $\mathfrak{B}$ is induced by the set $\{\gamma_i'(\zeta,\eta)\}_{i=0,1,\ldots,n}$, where

$$\gamma_i'(\zeta,\eta) = \frac{r(\zeta)\eta^{2i} + r(\eta)\zeta^{2i}}{\zeta+\eta}$$

2.5 Positivity of QDFs

Definition 6. Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Q_Φ is said to be positive denoted by $Q_\Phi > 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, and $Q_\Phi(w) = 0$ implies $w = 0$.

It can shown (see p. 1712 of Willems and Trentelman [1998]) that a QDF Q_Φ, where $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is positive if $\exists\ D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\Phi(\zeta,\eta) = D(\zeta)^T D(\eta)$, and $D(\lambda)$ has full column rank $\mathtt{w}$ for all $\lambda \in \mathbb{C}$.

3. AUTONOMOUS CONSERVATIVE SYSTEMS

In this section, we define an autonomous lossless system as an autonomous system for which there exists a positive conserved quantity. We then prove the equivalence between autonomous lossless and oscillatory systems. This is the main result of the paper. This is first done for the case of scalar systems (Theorem 12) and then extended to the case of multivariable systems (Theorem 16). We also discuss a few properties of energy functions of scalar lossless systems.

We begin with the following definition for autonomous lossless systems.

Definition 7. A linear autonomous behavior $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is *lossless* if there exists a conserved quantity Q_E associated with $\mathfrak{B}$, such that $Q_E > 0$. Such a Q_E is called an *energy function* for the system.

Remark 8. The total energy of any physical system does not have an absolute measure as such. It is always defined with respect to an arbitrary choice of a reference level, which is hence indeterminate. However this indeterminacy is not important as in any physical application, it is always the difference between the initial and final values of energy that matters, and this difference is independent of the reference level. Hence it is convenient to define the reference level for the total energy of a system as its lower bound. This point has been elaborated upon in Sears [1946], pp. 128-129. While defining lossless systems, we fix the reference level or lower bound of the energy functions for the system at zero, which leads to positivity of energy functions. We implicitly assume that an energy function of a lossless system is bounded from below.

For proving that lossless autonomous systems are necessarily oscillatory, we examine all the linear autonomous scalar systems, for which conserved QDFs exist. To this end, we first determine the conditions under which a linear system has conserved QDFs associated with it, and the dimension of the space of conserved QDFs for such systems. We begin with the following definition.

Definition 9. Let $r \in \mathbb{R}[\xi]$. The *maximal even polynomial factor* of r is its monic even factor polynomial of maximal degree.

For any given polynomial $r \in \mathbb{R}[\xi]$, it is easy to see that there exists a unique maximal even polynomial factor. In the next proposition, we examine the conditions under which a linear behavior $\mathfrak{B}$ has conserved QDFs associated with it.

Proposition 10. Consider a linear behaviour $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$, where $r \in \mathbb{R}[\xi]$. There exists a nontrivial conserved quantity for $\mathfrak{B}$ if and only if either r has a non-unity maximal even polynomial factor r_e or $r(\xi) = \xi$. Moreover if $p := \frac{r}{r_e}$ is such that $p(0) \neq 0$, then the dimension of the space of conserved QDFs is $\frac{\deg(r_e)}{2}$, otherwise it is equal to $\frac{\deg(r_e)}{2} + 1$.

Proof. Let the degree of r be equal to n. Let $r = r_e p$. Assume that $\mathfrak{B}$ has a conserved QDF whose two-variable polynomial representation is $\phi(\zeta,\eta)$. Then

$$\phi(\zeta,\eta) = \frac{r(\zeta)f_1(\zeta,\eta) + r(\eta)f_1(\eta,\zeta)}{\zeta+\eta}$$

for some $f_1 \in \mathbb{R}[\zeta,\eta]$. It is easy to see that since ϕ is r-canonical, f_1 is independent of ζ and is of degree less than or equal to $n-1$ in η. Let $f(\eta) = f_1(\zeta,\eta)$. Since ϕ exists, the numerator is divisible by $\zeta+\eta$. Consequently $r(-\xi)f(\xi) + r(\xi)f(-\xi) = 0$. This implies that $g(\xi) = r(\xi)f(-\xi) = r_e(\xi)p(\xi)f(-\xi)$ is an odd function. Hence

$$p(\xi)f(-\xi) = -p(-\xi)f(\xi) \tag{4}$$

Two cases arise.

- Case 1: $p(\xi)$ is not divisible by ξ. In this case, for equation (4) to hold, it is easy to see that f should be of the form
$$f(\xi) = p(\xi)f_o(\xi)$$
where $f_o(\xi)$ is an odd function such that $\deg(f) \leq n-1$. Hence, we obtain
$$\deg(f_o) \leq \deg(r_e) - 1 \tag{5}$$
From property (5), it follows that the dimension of the space of all possible polynomials $f_o(\xi)$ for a given even polynomial $r_e(\xi)$ and hence that of the space of conserved QDFs in this case is $\frac{\deg(r_e(\xi))}{2}$.
- Case 2: $p(\xi)$ is divisible by ξ. The proof for this case is very similar to that of Case 1, and will not be given explicitly.

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For proving that lossless autonomous systems are necessarily oscillatory, we make use of the following proposition.

Proposition 11. Let $r'(\xi^2)$ and $\xi r''(\xi^2)$, where $r', r'' \in \mathbb{R}[\xi]$, respectively be the even and odd parts of $r(\xi)$, where $r \in \mathbb{R}[\xi]$. If r is Hurwitz then r' and r'' have distinct roots on the negative real axis.

The above property can be deduced from Theorem 1, p.106, Holtz [2003]. In order to prove the equivalence between oscillatory systems and autonomous lossless systems, we first consider the case of scalar behaviors.

Theorem 12. A behavior $\mathfrak{B} \in \mathcal{L}^1$ is lossless if and only if it is oscillatory.

Proof. (*If*) We consider the two forms of scalar oscillatory behaviors mentioned in section 2.1. For each of these forms of oscillatory behavior, we construct an energy function that is positive.

- Case 1: The oscillatory behavior is of the form $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$, where $r(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$ and $\omega_0,\ldots,\omega_{n-1} \in \mathbb{R}^+$. From the discussion of section 2.4, it can be said that the two-variable polynomial associated with a general r-canonical conserved quantity for this case has the form
$$\phi(\zeta,\eta) = \frac{\eta r(\zeta)f_e(\eta) + \zeta r(\eta)f_e(\zeta)}{\zeta+\eta} \tag{6}$$
where f_e is an even function of degree less than or equal to $2n-2$. Define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$. It can be seen that the set $\{v_p(\xi)\}_{p=0,\ldots,n-1}$ is a basis of even polynomials of degree less than or equal to $2n-2$. It follows that there exist $b_p \in \mathbb{R}$, $p = 0,\ldots,n-1$, such that $f_e(\xi) = \sum_{p=0}^{n-1} b_p v_p(\xi)$. Now
$$\phi(\zeta,\eta) = \sum_{p=0}^{n-1} b_p\left[\frac{\eta r(\zeta)v_p(\eta) + \zeta r(\eta)v_p(\zeta)}{\zeta+\eta}\right] = \sum_{p=0}^{n-1} b_p v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$$
Define $\phi_p(\zeta,\eta) := v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$. From equation (6), it can be seen that linearly independent f_e's produce linearly independent ϕ's. Hence $\{\phi_p(\zeta,\eta)\}_{p=0,\ldots,n-1}$ is a basis of the space of two-variable polynomials that induce r-canonical conserved quantities. Now consider $E(\zeta,\eta) = \sum_{p=0}^{n-1} a_p^2\phi_p(\zeta,\eta) = D(\zeta)^T D(\eta)$, where $a_p \in \mathbb{R}\setminus\{0\}$ for $p = 0,\ldots,n-1$ and
$$D(\xi) = \text{ col }(a_0\xi v_0(\xi), a_0\omega_0 v_0(\xi), a_1\xi v_1(\xi), a_1\omega_1 v_1(\xi), \ldots, a_{n-1}\xi v_{n-1}(\xi), a_{n-1}\omega_{n-1}v_{n-1}(\xi))$$
It can be verified that $D(\lambda) \neq 0_{2n\times 1}$ for any $\lambda \in \mathbb{C}$. This proves that E induces an energy function for Case 1 oscillatory systems.
- Case 2: The oscillatory behavior is of the form $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$ where $r(\xi) = \xi(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2) = \xi r_e(\xi)$ and $\omega_0,\ldots,\omega_{n-1} \in \mathbb{R}^+$. The proof for this case is very similar the one for Case 1 oscillatory systems and will not be given explicitly.

(*Only if*) Assume that $\mathfrak{B}$ has the kernel representation $r(\frac{d}{dt})w = 0$. Let $r(\xi) = r_e(\xi)p(\xi)$ where r_e is the maximal even polynomial factor of r. If $p(\xi)$ is not a constant and $p(\xi) \neq a\xi$, where $a \in \mathbb{R}$, then it has at least one root, say $\lambda \in \mathbb{R}\setminus\{0\}$ or two roots, say $\lambda, \bar{\lambda} \in \mathbb{C}\setminus\mathbb{R}$. From the proof of Proposition 10, depending on whether $p(\xi)$ is divisible by ξ or not, any two-variable polynomial inducing conserved QDF over $\mathfrak{B}$ can either have the form
$$\phi_1(\zeta,\eta) = \frac{r(\zeta)p_1(\eta)f_e(\eta) + r(\eta)p_1(\zeta)f_e(\zeta)}{\zeta+\eta}$$
where $p_1(\xi) = \frac{p(\xi)}{\xi}$ and f_e is an even function, or the form
$$\phi_2(\zeta,\eta) = \frac{r(\zeta)p(\eta)f_o(\eta) + r(\eta)p(\zeta)f_o(\zeta)}{\zeta+\eta}$$
where $f_o(\xi)$ is an odd function. It can be seen that both ϕ_1 and ϕ_2 are divisible by $(\zeta-\lambda)(\eta-\lambda)$ if $\lambda \in \mathbb{R}$ and divisible by $(\zeta-\lambda)(\zeta-\bar{\lambda})(\eta-\lambda)(\eta-\bar{\lambda})$ if $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Hence along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda}t} \in \mathfrak{B}$, the QDFs induced by ϕ_1 and ϕ_2 are equal to zero. This implies that $\mathfrak{B}$ does not have a positive conserved QDF. This eliminates all scalar systems except those for which the kernel representation is $r(\frac{d}{dt})w = 0$, such that either $r(\xi)$ is even, or $r(\xi) = \xi r_e(\xi)$, where $r_e(\xi)$ is an even function. We now consider two cases.

- *Case 1*: r is even. Define $r'(\xi^2) := r(\xi)$. In this case, from the proof of proposition 10, any conserved quantity for $\mathfrak{B}$ has its associated two-variable polynomial of the form
$$\Phi(\zeta,\eta) = \frac{\eta r'(\zeta^2)r''(\eta^2) + \zeta r'(\eta^2)r''(\zeta^2)}{\zeta+\eta}$$
where r'' has degree less than that of r'. Assume that $Q_\Phi > 0$. Define $r_1(\xi) := r'(\xi^2) + \xi r''(\xi^2)$ and $\mathfrak{B}' := \ker\left(r_1(\frac{d}{dt})\right)$. Let $\dot{\Phi}(\zeta,\eta)$ denote the two-variable polynomial that induces the derivative of Q_Φ. Then

$$\dot{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$
$$= r_1(\zeta)r''(\eta^2) + r_1(\eta)r''(\zeta^2) - 2\zeta\eta r''(\zeta^2)r''(\eta^2)$$

Hence $Q_{\dot{\Phi}} \overset{\mathfrak{B}'}{=} Q_{\Phi_1}$, where

$$\Phi_1(\zeta,\eta) = -2\zeta\eta r''(\zeta^2)r''(\eta^2)$$

This implies that $Q_{\dot{\Phi}} \overset{\mathfrak{B}'}{<} 0$. Hence Q_Φ is a Lyapunov function for $\mathfrak{B}'$, which implies that $\mathfrak{B}'$ is asymptotically stable or that r_1 is Hurwitz. From Proposition 11, it follows that r is oscillatory.

- *Case 2*: r is odd. The proof for this case is very similar to the one for case 1 and will not be given explicitly.

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We now discuss a few properties of energy functions for scalar oscillatory behaviors. We first present an analysis of the conditions under which a conserved quantity for a scalar oscillatory behavior is positive. The following lemma can be used to construct an energy function for a scalar oscillatory behavior.

Lemma 13. Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2 + \omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$, where $\omega_0,\dots,\omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $v_p(\xi) := \frac{r_1(\xi)}{\xi^2+\omega_p^2}$, $p = 0,\dots,n-1$. Define $r_2(\xi) := \xi r_1(\xi)$. Then the following hold:

(1) Let $\mathfrak{B}_1 = \ker\left(r_1(\frac{d}{dt})\right)$. If the conserved quantity for $\mathfrak{B}_1$ induced by $\phi_1(\zeta,\eta) = \sum_{p=0}^{n-1} b_p v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$ is positive, then $b_p > 0$ for $p = 0,\dots,n-1$.

(2) Let $\mathfrak{B}_2 = \ker\left(r_2(\frac{d}{dt})\right)$. If the conserved quantity for $\mathfrak{B}_2$ induced by $\phi_2(\zeta,\eta) = \sum_{p=0}^{n-1} b_p \zeta\eta v_p(\zeta)v_p(\eta)(\zeta\eta + \omega_p^2) + b_n r_1(\zeta)r_1(\eta)$ is positive, then $b_p > 0$ for $p = 0,\dots,n$.

Proof. Assume that $b_i \leq 0$ for some $i \in \{0,\dots,n-1\}$. Consider a trajectory $w(t) = ke^{j\omega_i t} + \bar{k}e^{-j\omega_i t} \in \mathfrak{B}_1, \mathfrak{B}_2$. Along this trajectory, $v_p(\frac{d}{dt})w = 0$ for $p \in \{0,\dots,n-1\}\setminus\{i\}$. Since $\phi_p(\zeta,\eta) = v_p(\zeta)v_p(\eta)(\zeta\eta + \omega_p^2)$ and $\zeta\eta\phi_p(\zeta,\eta)$ are non-negative, the QDF induced by $\phi_1(\zeta,\eta)$ and $\phi_2(\zeta,\eta)$ over $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively along this trajectory turns out to be non-positive. Hence by contradiction, $b_p > 0$ for $p = 0,\dots,n-1$ in both cases.

In order to complete the proof, consider now statement 2 of the Lemma and assume by contradiction that $b_n \leq 0$. Consider a trajectory $w(t) = k \in \mathfrak{B}_2$. Along this trajectory $v_p(\frac{d}{dt})w = 0$ for $p \in \{0,\dots,n-1\}$. Since $r_1(\zeta)r_1(\eta)$ is non-negative, the QDF induced by $\phi_2(\zeta,\eta)$ over $\mathfrak{B}_2$ turns out to be non-positive. Hence, $b_n > 0$. This concludes the proof. ■

The next Theorem relates the positivity of a conserved quantity to an important property known as *interlacing property*, which also arises in applications like electrical network theory.

Theorem 14. Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2 + \omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$, where $\omega_0 < \omega_1 \dots < \omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $r'(\xi^2) := r_1(\xi)$; $r_2(\xi) := \xi r_1(\xi)$ and $\check{r}(\xi) := \xi r'(\xi)$. Then the following hold:

(1) Let $\mathfrak{B}_1 = \ker\left(r_1(\frac{d}{dt})\right)$. Let $f_1(\xi)$ be a polynomial of degree less than or equal to $n-1$. A conserved quantity for $\mathfrak{B}_1$ induced by

$$\phi_1(\zeta,\eta) = \frac{\eta r'(\zeta^2)f_1(\eta^2) + \zeta r'(\eta^2)f_1(\zeta^2)}{\zeta+\eta} \tag{7}$$

is positive if and only if $f_1(-\omega_0^2) > 0$ and the roots of f_1 are interlaced between those of r', i.e along the real axis, exactly one root of f_1 occurs between any two consecutive roots of r'.

(2) Let $\mathfrak{B}_2 = \ker\left(r_2(\frac{d}{dt})\right)$. Let $f_2(\xi)$ be a polynomial of degree less than or equal to n. A conserved quantity associated with $\mathfrak{B}_2$ induced by

$$\phi_2(\zeta,\eta) = \frac{\zeta r'(\zeta^2)f_2(\eta^2) + \eta r'(\eta^2)f_2(\zeta^2)}{\zeta+\eta}$$

is positive if and only if $f_2(0) > 0$ and the roots of f_2 are interlaced between those of $\check{r}$.

In the next corollary, we give the general expression for an energy function of a scalar conservative behavior that has no characteristic frequency at zero.

Corollary 15. Let $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$ be an oscillatory behavior, where $r(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$. Define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$, $V(\xi) := \mathrm{col}(v_0(\xi), v_1(\xi), \dots, v_{n-1}(\xi))$ and Ω :=diag$(\omega_0,\omega_1,\dots,\omega_{n-1})$. A two-variable polynomial E induces an energy function for $\mathfrak{B}$, if and only if there exists a diagonal matrix $C \in \mathbb{R}^{n\times n}$ with positive diagonal entries, such that

$$E(\zeta,\eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) + V(\zeta)^T C^2\Omega^2 V(\eta) \tag{8}$$

Proof. The proof follows from the (*if*) part of the proof of Theorem 12 and the proof of Lemma 13. ■

We now describe a suitable method of splitting a given energy function of a scalar autonomous lossless system into its kinetic and potential energy components. We show that with suitable choices for mass and stiffness matrices and a suitable choice for a generalized position as a function of the external variables of a scalar oscillatory system, we can obtain equations that are very similar to the equations describing a second order mechanical system. We use this idea to obtain a splitting of the total energy into its kinetic and potential energy components. Thus, with reference to the previous Corollary, if we interpret $q = V(\frac{d}{dt})w$ as a generalized position, then $\frac{dq}{dt} = \frac{d}{dt}V(\frac{d}{dt})w$ is a generalized velocity. Define $M := 2C^2$ and $K := 2C^2\Omega^2$. Using these expressions the system equations can be written in a way similar to the equations describing a second order mechanical system as

$$M\frac{d^2q}{dt^2} + Kq = 0$$
$$C^2(I\frac{d^2}{dt^2} + \Omega^2)V(\frac{d}{dt})w = 0$$

which reduces to $\mathrm{col}(r(\frac{d}{dt}), r(\frac{d}{dt}),\dots)w(t) = 0$. Thus M and K can be interpreted as the mass and the stiffness matrix respectively. This leads to the two-variable polynomials K and P corresponding to the kinetic energy $(\frac{1}{2}M(\frac{dq}{dt})^2)$ and potential energy $(\frac{1}{2}Kq^2)$ respectively being given by

$$K(\zeta,\eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) \tag{9}$$
$$P(\zeta,\eta) = V(\zeta)^T C^2\Omega^2 V(\eta) \tag{10}$$

We now illustrate the concepts discussed so far in this section using the example of a mechanical system that was introduced earlier.

Example 1 revisited: With reference to example 1, let $m_1 = m_2 = 1$, $k_1 = 2$ and $k_2 = 3$. Then $r(\xi) = \xi^4 + 7\xi^2 + 6 = (\xi^2+6)(\xi^2+1)$. This system is oscillatory and hence lossless. The natural frequencies of the system are given by $\omega_0 = \sqrt{6}$ and $\omega_1 = 1$. The total kinetic energy and the total potential energy for the system can be expressed as QDFs in terms of only w_1. The two variable polynomials corresponding to these are

$$K(\zeta,\eta) = \frac{1}{8}[\zeta^3\eta^3 + 2(\zeta\eta^3 + \zeta^3\eta) + 8\zeta\eta] \tag{11}$$

$$P(\zeta,\eta) = \frac{1}{8}[5\zeta^2\eta^2 + 6(\zeta^2+\eta^2) + 12] \tag{12}$$

The total energy of the system is a positive conserved quantity and hence from Lemma 13 will correspond to the two-variable polynomial of the form

$$E(\zeta,\eta) = a_0^2(\zeta\eta+6)(\zeta^2+1)(\eta^2+1) + a_1^2(\zeta\eta+1)(\zeta^2+6)(\eta^2+6)$$

Indeed by comparison with equations (11) and (12), we obtain real values for a_0 and a_1 as

$$a_0 = \sqrt{0.1} \qquad a_1 = \sqrt{0.025}$$

In this case, with $C = \text{diag}(a_0, a_1)$ and $\Omega = \text{diag}(\omega_0, \omega_1)$, it can be verified that equations (9) and (10) reduce to equations (11) and (12) respectively.

We now build upon the result of Theorem 12 and extend it to the multivariable case.

Theorem 16. A linear autonomous system $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is lossless if and only if it is oscillatory.

Proof. We proceed by reduction of the multivariable case to the scalar case by use of the Smith form. Consider a kernel representation of $\mathfrak{B}$ given by $\mathfrak{B} = \ker\left(R(\frac{d}{dt})\right)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $\det(R(\xi)) \neq 0$. Let $R = U\Delta V$ be the Smith form decomposition of R. Let the behavior $\mathfrak{B}'$ be given by $\mathfrak{B}' = \ker\left(\Delta(\frac{d}{dt})\right)$. Denote the number of invariant polynomials of R equal to one with $\mathtt{w}_1$ and let $\{r_i(\xi)\}_{i=\mathtt{w}_1+1,\ldots,\mathtt{w}}$ be the set consisting of the remaining invariant polynomials of R. Let $\mathfrak{B}'_i = \ker\left(r_i(\frac{d}{dt})\right)$.

(*Only If*): We assume that $\mathfrak{B}$ and hence $\mathfrak{B}'$ are lossless. Consider a trajectory $w' \in \mathfrak{B}'$. Let $\{w'_i\}_{i=1,\ldots,\mathtt{w}}$ be the components of w'. Consider an energy function Q_ϕ of $\mathfrak{B}$ acting on w. Let $\phi'(\zeta,\eta) = (V(\zeta))^{-\top}\phi(\zeta,\eta)(V(\eta))^{-1}$. Let $\phi'_i(\zeta,\eta)$ be the i^{th} diagonal entry and $\phi'_{ik}(\zeta,\eta)$ be the entry corresponding to the i^{th} row and k^{th} column of the polynomial matrix $\phi'(\zeta,\eta)$. Then

$$Q_\phi(w) = Q_{\phi'}(w') = \sum_{i=1}^{\mathtt{w}} Q_{\phi'_i}(w'_i) + \sum_{i\neq k} L_{\phi'_{ik}}(w'_i, w'_k) \tag{13}$$

Since $Q_\phi > 0$, also $Q_{\phi'} > 0$. Since each component of w' can be chosen independently of each other, it follows that $Q_{\phi'_i} > 0$ and is conserved over $\mathfrak{B}'_i$ for $i = 1, 2, \ldots, \mathtt{w}$. This is possible only if each of $\mathfrak{B}'_i$ is oscillatory for $i = 1, 2, \ldots, \mathtt{w}$, which implies that $\mathfrak{B}'$ and hence $\mathfrak{B}$ is oscillatory.

(*If*): Assume that $\mathfrak{B}$ and hence $\mathfrak{B}'$ is oscillatory. We construct a QDF that is positive and conserved along $\mathfrak{B}$ and hence prove that the system is lossless. For $i = \mathtt{w}_1 + 1, \ldots, \mathtt{w}$, let r_i have nonzero roots at $\pm j\omega_{0i}, \pm j\omega_{1i}, \pm j\omega_{2i}, \ldots$ and maximal even polynomial factor equal to s_i. Define $v_{pq}(\xi) := \frac{s_q(\xi)}{\xi^2+\omega_{pq}^2}$. Consider

$$D(\xi) = \begin{bmatrix} 0_{\mathtt{w}_1\times\mathtt{w}_1} & 0 & \ldots & \ldots & \ldots & \\ 0_{\bullet\times\mathtt{w}_1} & D_{\mathtt{w}_1+1} & 0_{\bullet\times 1} & \ldots & \ldots & \ldots \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & D_{\mathtt{w}_1+2} & 0_{\bullet\times 1} & \ldots & \ldots \\ \vdots & \vdots & 0_{\bullet\times 1} & \ddots & \ddots & \vdots \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & \vdots & \ddots & \ddots & 0_{\bullet\times 1} \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & \ldots & \ldots & \ldots & D_{\mathtt{w}}(\xi) \end{bmatrix} \tag{14}$$

where $D_i = \text{col}\,(a_{0i}\xi v_{0i}(\xi), a_{0i}\omega_{0i}v_{0i}(\xi), a_{1i}\xi v_{1i}(\xi), a_{1i}\omega_{1i}v_{1i}(\xi), \ldots)$ if r_i is even and $D_i = \text{col}\,(a_{0i}\xi^2 v_{0i}(\xi), a_{0i}\omega_{0i}\xi v_{0i}(\xi), a_{1i}\xi^2 v_{1i}(\xi), a_{1i}\omega_{1i}\xi v_{1i}(\xi), \ldots)$ if r_i is odd, $a_{ik} \in \mathbb{R}^+$ as in the proof of the sufficiency part of Theorem 12. From the argument used in order to prove the scalar case, it is easy to see that $\phi'(\zeta,\eta) = D(\zeta)^T D(\eta)$ is positive and conserved along $\mathfrak{B}'$, and hence $\phi(\zeta,\eta) = V(\zeta)^T D(\zeta)^T D(\eta) V(\eta)$ is positive and conserved along $\mathfrak{B}$. ■

4. CONCLUSION

In this paper, the main focus has been to give a characterisation for higher order linear autonomous lossless systems as opposed to the characterisation for first order systems using state space method (Weiss et al. [2001] and Weiss and Tucsnak [2003]). Using the material covered in this paper, one can easily implement a computer program wherein the input is a higher order description of a scalar oscillatory system and the outputs are its energy functions and the kinetic and potential energy components of a given energy function for the system. Given a multivariable oscillatory system, using the material in this paper, one can implement a program to compute an energy function for the system.

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