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Identification of First-Order Time-Delay Systems using Two Different Pulse Inputs ⋆

B.A. León de la Barra *, **Lihua Jin** **, **Y.C. Kim** **,[1], and **M. Mossberg** ***

* *University of Tasmania, Australia (e-mail: b.a.leondelabarra@ieee.org)*
** *Chungbuk National University, South Korea (e-mail: jinlihua@chungbuk.ac.kr, yckim@cbu.ac.kr)*
*** *Karlstad University, Sweden (e-mail: magnus.mossberg@kau.se)*

Abstract: This paper provides exact analytical expressions for the DC gain, time constant, and time delay of first-order plus time-delay (FOTD) systems from knowledge of two relative extrema in the transient response to two different finite-duration pulse inputs. The availability of these formulae leads to simple identification methods for process control settings that do not require any prior knowledge about the process DC gain and/or time delay. Variance analysis is used to study the quality of the estimates when the pulse response measurements are corrupted by noise. The proposed identification techniques compare favorably with existing FOTD identification methods both in terms of overall simplicity and user friendliness.

1. INTRODUCTION

It is commonly accepted that identification for control in industrial settings most often means that a simple process model with at most three parameters is fitted to real time data (Ljung [2002]). In fact, FOTD models are very common in process control where systems with essentially monotone step responses are frequently found (Åström and Hägglund [2006]). The identification of FOTD models has been the subject of continued attention for a few decades. Relay-based identification of FOTD systems has received significant attention over the last two decades. Some recent efforts in this area include (Srinivasan and Chidambaram [2003]) that provided analytical expressions for the parameters describing an FOTD system from quantities derived from a single asymmetrical relay test. A relay based estimation method where knowledge of the slope of the output signal at a zero crossing was needed has also been reported (Majhi and Litz [2003]). This latter method also required solving a nonlinear equation for each unknown parameter. More recently, Padhy and Majhi [2006] introduced a relay-based identification method for FOTD processes where the steady-state gain was assumed to be known a priori. Last year, Majhi [2007] introduced relay-based identification of a class of non-minimum phase SISO processes that included the systems to be considered in this paper. However, this approach required the use of a nonlinear algebraic equation solver for parameter estimation raising convergence issues if the initial estimates were not that close to the true parameter values. The same is true for the method previously presented in Majhi [2005]. Under certain conditions, this latter method also requires previous knowledge of the DC gain of the process. Note also that relay-based identification methods need to exclude the initial transient oscillating behavior so that limit cycle data are estimated after a few (typically two or three) stabilized cycles have taken place (Majhi [2007], Srinivasan and Chidambaram [2004]). More recently, a general integral identification method applicable to n^{th} order time-delay systems where the test input is expressible as a sequence of step signals was presented (Liu et al. [2007]). This method assumed that lower and upper bounds on the time delay were known. Recent work by León de la Barra and Mossberg [2007] used finite-duration rectangular pulse inputs to identify the parameters of a prototype second-order transfer function. Exact analytical expressions for the damping ratio and the undamped natural frequency were obtained from knowledge of the peak time and peak value in the rectangular pulse response. The sensitivity of these expressions to uncertainties in the measurements was also studied. The work reported in this paper complements the results in León de la Barra and Mossberg [2007] by tackling the identification of FOTD models. Our approach uses only two relative extrema in the transient response to finite-duration pulses of different shape, cf. Fig. 1, and does not require any prior knowledge about the process DC gain and/or time delay. The proposed identification techniques also compare favorably with existing FOTD identification methods both in terms of overall simplicity and user friendliness. In particular, we contend that our approach is likely to be more time efficient than relay-based FOTD identification as it will use precisely the initial transient response data neglected by relay-based FOTD identification techniques.

2. PROBLEM STATEMENT

Consider an FOTD system characterized by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{1+Ts}e^{-Ls} \tag{1}$$

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[1] Corresponding author. Tel.: +82-43-261-2475.

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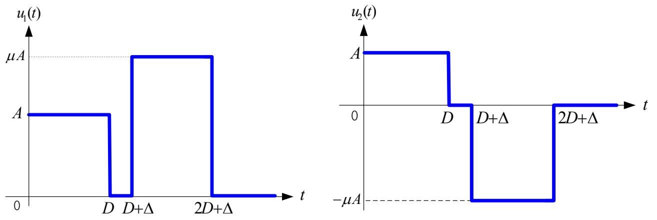
where $K \neq 0$, $L \geq 0$, and $T > 0$ respectively. This paper deals with the problem of estimating the three parameters that characterize system (1). For this purpose, one may certainly use some of the ubiquitous step response based FOTD identification methods (Åström and Hägglund [2006]). A merit of pulse testing is, compared with step (and ramp) testing, that the process input and output will return to their pre-testing steady-state values soon after the end of the identification experiment. Minimizing the experiment's impact on the overall process operation is also an important issue that any identification procedure should take into account (Hwang and Huang [2006]). Thus, this paper considers the problem of obtaining explicit analytical expressions for K, L, and T from knowledge of two points in the transient response of system (1) to simple finite-duration pulse inputs. Two different pulse inputs, as shown in Fig. 1, will be considered here. Note that Hwang and Huang [2006] has also argued that different pulse shapes could be used to better excite specific frequency ranges for different systems.



(a) A double rectangular pulse (b) A delayed doublet pulse

Fig. 1. Finite-duration pulse inputs with the same energy.

In Fig. 1, A and μ represent the pulse amplitude and a multiplier, D is the pulse width, and Δ denotes the separation time between two rectangular pulses.

In this paper we also consider pulses having the same total signal energy that is given by

$$\int_0^\infty u_i^2(t)dt = (\mu^2+1)A^2D, \quad i = 1,2. \tag{2}$$

3. MAIN RESULTS

In this section, we present simple methods to exactly identify the parameters of an FOTD system from knowledge of two points in its transient response to two pulse inputs as illustrated in Fig. 1.

3.1 Double Rectangular Pulse Response

A double rectangular pulse input can be expressed as

$$\begin{aligned} u_1(t) = &A\,[H(t) - H(t-D)] \\ &+\mu A\,[H(t-D-\Delta) - H(t-2D-\Delta)] \end{aligned} \tag{3}$$

where $A \neq 0$, $D, \Delta, \mu > 0$, and

$$H(t) = \begin{cases} 0, \ t < 0 \\ 1, \ t \geq 0 \end{cases}$$

is the Heaviside unit function. Since the Laplace transform of $u_1(t)$ is given by

$$U_1(s) = \frac{A}{s}\left[\left(1-e^{-Ds}\right) + \mu\left(e^{-(D+\Delta)s} - e^{-(2D+\Delta)s}\right)\right] \tag{4}$$

it is easy to show that the double rectangular pulse response can be derived as

$$\begin{aligned} Y_1(s) = \frac{AK}{s\,(1+Ts)}\ &e^{-Ls}\left[\left(1-e^{-Ds}\right)\right. \\ &\left.+\mu\left(e^{-(D+\Delta)s} - e^{-(2D+\Delta)s}\right)\right]. \end{aligned} \tag{5}$$

If we define

$$t_a = L + D, \quad t_b = L + D + \Delta, \quad t_c = L + 2D + \Delta, \tag{6}$$

then the response in the time domain is described by

$$y_1(t) = \begin{cases} 0, & t < L \\ AK\left(1 - e^{-\frac{t-L}{T}}\right), & L \leq t < t_a \\ AK\left(e^{-\frac{t-t_a}{T}} - e^{-\frac{t-L}{T}}\right), & t_a \leq t < t_b \\ AK\left(e^{-\frac{t-t_a}{T}} - e^{-\frac{t-L}{T}}\right. & t_b \leq t < t_c \\ \qquad \left. -\mu e^{-\frac{t-t_b}{T}} + \mu\right), & \\ AK\left(e^{-\frac{t-t_a}{T}} - e^{-\frac{t-L}{T}}\right. & t \geq t_c \\ \qquad \left. -\mu e^{-\frac{t-t_b}{T}} + \mu e^{-\frac{t-t_c}{T}}\right). & \end{cases} \tag{7}$$

A typical double rectangular pulse response for $AK > 0$ is illustrated in Fig. 2.

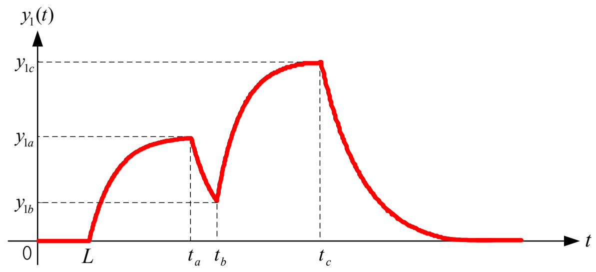


Fig. 2. Double rectangular pulse response for an FOTD system.

It can be easily seen that each expression in (7) is either monotonically increasing or decreasing for the corresponding interval. Therefore, it is straightforward to verify that the extreme values of the double rectangular pulse response are given by

$$y_{1a} = y\,(t_a) = AK\left(1 - e^{-\frac{D}{T}}\right), \tag{8}$$

$$y_{1b} = y\,(t_b) = AKe^{-\frac{\Delta}{T}}\left(1 - e^{-\frac{D}{T}}\right), \tag{9}$$

$$y_{1c} = y\,(t_c) = AK\left(1 - e^{-\frac{D}{T}}\right)\left(\mu + e^{-\frac{D+\Delta}{T}}\right). \tag{10}$$

The following result characterizes how K, L, and T can be identified by using the above extreme values.

Result 1: From the double rectangular pulse response of an FOTD system given by (1) it follows that

$$(i) \quad T = \frac{\Delta}{\ln\left(y_{1a}\right) - \ln\left(y_{1b}\right)}, \quad \text{if} \ \ \Delta \neq 0 \ \ \text{or} \tag{11}$$

$$T = \frac{D+\Delta}{\ln(y_{1a}) - \ln(y_{1c} - \mu y_{1a})}, \tag{12}$$

$$(ii)\quad K = \frac{y_{1a}}{A\left[1 - \left(\frac{y_{1b}}{y_{1a}}\right)^{\frac{D}{\Delta}}\right]}, \quad \text{if} \quad \Delta \neq 0 \quad \text{or} \tag{13}$$

$$K = \frac{y_{1a}}{A\left[1 - \left(\frac{y_{1c} - \mu y_{1a}}{y_{1a}}\right)^{\frac{D}{D+\Delta}}\right]}, \tag{14}$$

$$(iii)\quad L = t_a - D = t_b - D - \Delta = t_c - 2D - \Delta. \tag{15}$$

Derivation: See Appendix A.

Remark 1: Note that in a practical setting having two alternative expressions to evaluate the steady-state gain K and time constant T, and three different expressions to estimate the time delay L, respectively, could be used to make a preliminary assessment about whether the underlying process generating the data should be characterized by an FOTD model. Note also that these additional expressions originate from the obvious fact that a double rectangular pulse has two additional vertical edges when compared to a single rectangular pulse.

3.2 Delayed Doublet Pulse Response

A delayed doublet pulse input can be described by

$$u_2(t) = A\left[H(t) - H(t-D)\right] \\ -\mu A\left[H(t-D-\Delta) - H(t-2D-\Delta)\right]. \tag{16}$$

The Laplace transform of $u_2(t)$ is expressed by

$$U_2(s) = \frac{A}{s}\left[\left(1 - e^{-Ds}\right) \right. \\ \left. -\mu\left(e^{-(D+\Delta)s} - e^{-(2D+\Delta)s}\right)\right] \tag{17}$$

and the delayed doublet pulse response is described by

$$Y_2(s) = \frac{AK}{s(1+Ts)}\; e^{-Ls}\left[\left(1 - e^{-Ds}\right) \right. \\ \left. -\mu\left(e^{-(D+\Delta)s} - e^{-(2D+\Delta)s}\right)\right]. \tag{18}$$

The delayed doublet pulse response in the time domain can be derived as

$$y_2(t) = \begin{cases} 0, & t < L \\ AK\left(1 - e^{-\frac{t-L}{T}}\right), & L \le t < t_a \\ AK\left(e^{-\frac{t-t_a}{T}} - e^{-\frac{t-L}{T}}\right), & t_a \le t < t_b \\ AK\left(e^{-\frac{t-t_a}{T}} - e^{-\frac{t-L}{T}} \right. & t_b \le t < t_c \\ \quad \left. +\mu e^{-\frac{t-t_b}{T}} - \mu\right), & \\ AK\left(e^{-\frac{t-t_a}{T}} - e^{-\frac{t-L}{T}} \right. & t \ge t_c \\ \quad \left. +\mu e^{-\frac{t-t_b}{T}} - \mu e^{-\frac{t-t_c}{T}}\right). & \end{cases} \tag{19}$$

Fig. 3 depicts a typical delayed doublet pulse response for $AK > 0$.

It is obvious that each expression in (19) is either monotonically increasing or decreasing for the corresponding interval. Therefore, it is straightforward to verify that the extreme values of the delayed doublet pulse response are given by

$$y_{2a} = y(t_a) = AK\left(1 - e^{-\frac{D}{T}}\right), \tag{20}$$

$$y_{2b} = y(t_b) = AKe^{-\frac{\Delta}{T}}\left(1 - e^{-\frac{D}{T}}\right), \tag{21}$$

$$y_{2c} = y(t_c) = -AK\left(1 - e^{-\frac{D}{T}}\right)\left(\mu - e^{-\frac{D+\Delta}{T}}\right). \tag{22}$$

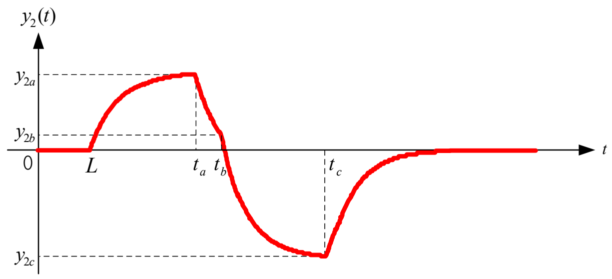


Fig. 3. Delayed doublet pulse response for an FOTD system.

The following result specifies how K, L, and T can be estimated by using the above extreme values.

Result 2: From the delayed doublet pulse response of an FOTD system given by (1) it follows that

$$(i)\quad T = \frac{\Delta}{\ln(y_{2a}) - \ln(y_{2b})}, \quad \text{if} \quad \Delta \neq 0 \quad \text{or} \tag{23}$$

$$T = \frac{D+\Delta}{\ln(y_{2a}) - \ln(y_{2c} + \mu y_{2a})}, \tag{24}$$

$$(ii)\quad K = \frac{y_{2a}}{A\left[1 - \left(\frac{y_{2b}}{y_{2a}}\right)^{\frac{D}{\Delta}}\right]}, \quad \text{if} \quad \Delta \neq 0 \quad \text{or} \tag{25}$$

$$K = \frac{y_{2a}}{A\left[1 - \left(\frac{y_{2c} + \mu y_{2a}}{y_{2a}}\right)^{\frac{D}{D+\Delta}}\right]}, \tag{26}$$

$$(iii)\quad L = t_a - D = t_b - D - \Delta = t_c - 2D - \Delta. \tag{27}$$

Derivation: See Appendix B.

Remark 2: A non-delayed rectangular doublet pulse has been used in Åström and Hägglund [2006] to estimate the parameters of an FOTD system. As expected, the results in Åström and Hägglund [2006] constitute a special case of Result 2 and can be obtained by making $\mu = 1$ and $\Delta = 0$ in equations (24), (26). Note also that the expression for the time constant T given by Åström and Hägglund [2006] is incorrect as it would generate negative time constants from stable responses.

4. ROBUSTNESS TO MODEL MISMATCH

From the viewpoint of practical implementation, the proposed simple identification methods may have significant estimation errors when the response data is corrupted with measurement noise and the actual process is of higher order. In this section, the robustness of the estimated model against mismatched model order will be investigated. The assessment of accuracy is performed in both the time domain and the frequency domain. For the time domain accuracy, the integral of the squared error (ISE) bwtween the impulse responses of the estimated model and the real process is used.

$$J = \int_0^{t_s} e^2(t)dt. \tag{28}$$

Wherein t_s is the settling time of the impulse response.

It is well known that a good fitness of an identified model in the time domain does not necessarily imply a good matching of the frequency response. Let the estimated model of (1) be $\hat{G}(s)$. The frequency domain estimation error can be measured by the following worst case error (Wang and Zhang [2001]).

$$E = \max_{\omega\in[0,\omega_c]} \left\{ \left| \frac{\hat{G}(j\omega) - G(j\omega)}{G(j\omega)} \right| \times 100\% \right\} \tag{29}$$

where ω_c is the phase crossover frequency, i.e., $\angle G(j\omega_c) = -\pi$.

Now let us examine the robustness of the identification methods based on Results 1 and 2, and an existing step response method (Smith and Corripio [1985]) through an example. In Smith and Corripio [1985], we have

$$K = y_{ss}, \quad T = \frac{3}{2}(t_2 - t_1), \quad L = t_2 - T \tag{30}$$

where y_{ss} is the steady-state value of the unit step response, t_1 and t_2 are the times at which $y(t_1) = 0.283y_{ss}$ and $y(t_2) = 0.632y_{ss}$, respectively.

Example 1: Consider a real system of third-order plus time-delay given by,

$$G(s) = \frac{100}{(s+2)(s+5)(s+10)} e^{-0.2s}. \tag{31}$$

The parameters specifying the pulse inputs are assumed to be $A = 1$, $\mu = 2.25$, $D = 0.44$, and $\Delta = 0.27$. In addition, a zero mean white noise with variance $\lambda^2 = 5 \times 10^{-3}$ is added at the process output.

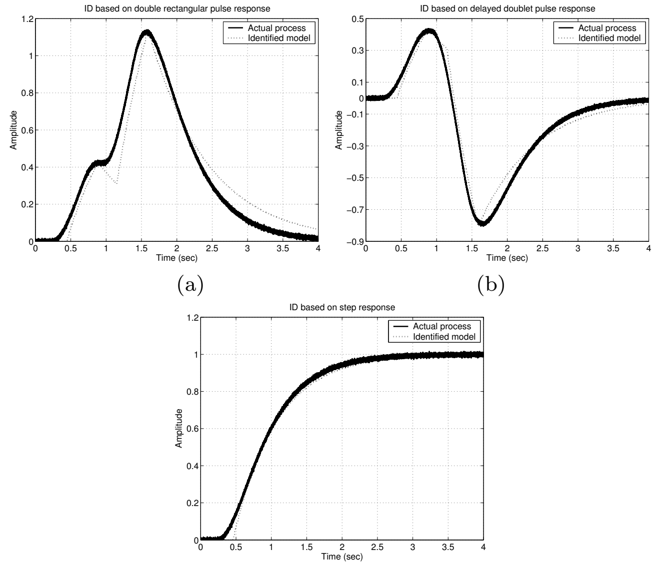
Each identified parameter and each robustness measure are given in Table 1.

Table 1. The identified model parameters and estimation errors for Example 1.

	Double Rectangular	Delayed Doublet	Step
$\hat{K}$	1.0363	1.0182	1.0008
$\hat{T}$	0.8385	0.7904	0.6077
$\hat{L}$	0.4407	0.4371	0.4577
J	2.0006	1.5696	2.6714
E	0.0008	0.0363	0.0182

The double rectangular pulse, delayed doublet pulse, and step responses of the real process with noise and the identified model to different inputs are shown in Fig. 4 (a)–(c). Due to the difference in the order of the actual plant and estimated models, the estimation errors always exist except for the extrema as in Fig. 4(a) and (b).

The impulse responses and Bode diagrams of the actual plant and identified models are comparatively shown in Figs. 5 and 6.



(a) (b)

(c)

Fig. 4. Pulse and step responses of the actual process and identified models.

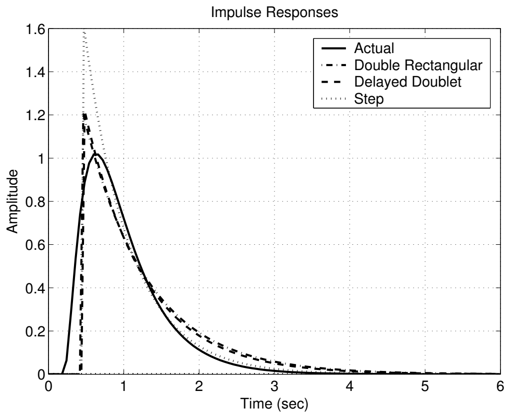


Fig. 5. Impulse responses of the actual plant and identified models.

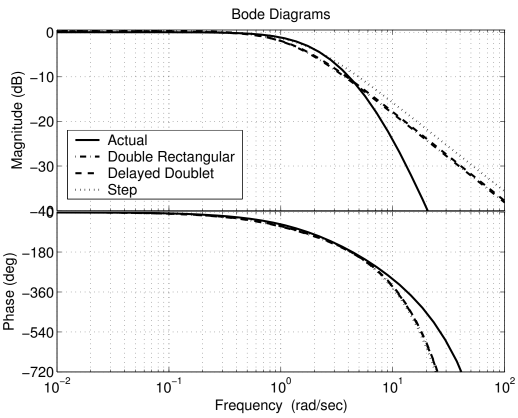


Fig. 6. Bode diagrams of the actual plant and identified models.

In this example, the three identification methods give similar accuracy in the time domain, the double rectangular pulse results in the best accuracy in the frequency domain, whereas the step response method provides the best estimation of the steady-state gain. However, most identification methods based on extremal values may have generally significant estimation errors for the case of model mismatch.

5. VARIANCE ANALYSIS

Let e_i and τ_i denote a pulse response extreme value and its time location, respectively. Let us also assume that inexact measurements of these quantities

$$\hat{e}_i = e_i + \varepsilon, \tag{32}$$

$$\hat{\tau}_i = \tau_i + \varepsilon \tag{33}$$

are available, where ε represents zero mean white noise of variance σ^2.

The estimators of the system parameters K and T given in Results 1 and 2, and step response based identification are functions of the known quantities A, μ, D, and Δ (that characterize the different pulse inputs), and of two measured pulse response extreme values, say $\hat{e}_1$ and $\hat{e}_2$, respectively. On the other hand, each estimator of L only depends on a measured extreme value time location, say τ_1, corresponding to the pulse response extreme value e_1.

The errors in the measurements, as characterized by (32)–(33), will affect the accuracy of the estimated parameters in the following way.

Consider the vector

$$\hat{m} = [\hat{e}_1 \quad \hat{e}_2 \quad \hat{\tau}_1]^T \tag{34}$$

with mean value

$$m = [e_1 \quad e_2 \quad \tau_1]^T \tag{35}$$

and covariance matrix $\sigma^2 \cdot I$, where I denotes the identity matrix. We have that

$$\hat{\vartheta} = \begin{bmatrix} \hat{K} \\ \hat{T} \\ \hat{L} \end{bmatrix} = \begin{bmatrix} f_1(\hat{m}) \\ f_2(\hat{m}) \\ f_3(\hat{m}) \end{bmatrix} = f(\hat{m}), \tag{36}$$

where $f(\hat{m})$ and $\hat{m}$ are different for Results 1 and 2, and existing step response method, but where the same framework can be used for describing all possible estimators. Note also that if for a given pulse shape there are alternative expressions for the estimators, $\hat{m}$ can have more than one value for that specific pulse response. The results to be presented in this section only consider the first (or leftmost) expression for each estimator.

When $\hat{m}$ is sufficiently close to m, the approximation

$$f(\hat{m}) \approx f(m) + f'(m) \cdot (\hat{m} - m) \tag{37}$$

should be accurate enough, where f' is the 3 by 3 derivative of f. This means that

$$\begin{aligned} cov\left(\hat{\vartheta}\right) &\approx E\left\{[f(\hat{m}) - f(m)] \cdot [f(\hat{m}) - f(m)]^T\right\} \\ &\approx f(m) \cdot E\left[(\hat{m} - m) \cdot (\hat{m} - m)^T\right] \cdot [f(m)]^T \\ &= \sigma^2 \cdot f(\hat{m}) \cdot [f(\hat{m})]^T \end{aligned} \tag{38}$$

The following example explores how the variances (38) of the estimates $\hat{K}$ and $\hat{T}$, introduced in Results 1 and 2, and of the step response method are affected when the pulse response measurements are corrupted by noise.

Example 2: The parameters defining the different pulse inputs are given by $A = 1$, $\mu = 2$, $D = 1$, and $\Delta = 0.2$, respectively.

The system parameters are chosen as $K = 1$ and $L = T = 0.5$, and the variances of $\hat{K}$ and $\hat{T}$ are shown in Figs. 7 and 8 as a function of the noise variance.

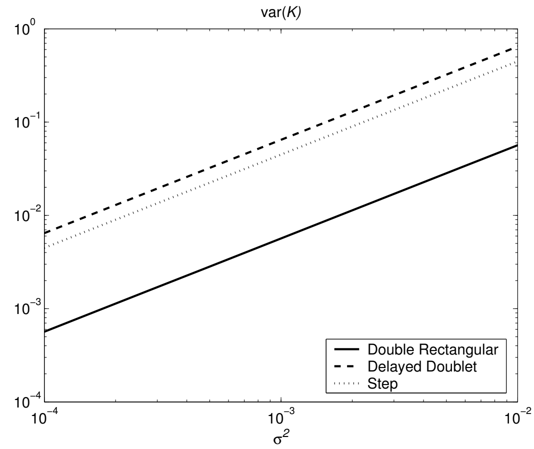


Fig. 7. Variance of $\hat{K}$ versus noise variance.

From Fig. 7 it is clear that the $\hat{K}$ estimates given by the double rectangular pulse have the lowest variance and that the $\hat{K}$ estimates given by the delayed doublet pulse have the highest variance, respectively. It is worth pointing out that a delayed doublet pulse with the same parameters than a double rectangular pulse will always have less energy at zero frequency and this could partly explain why the delayed doublet pulse exhibits the highest variance in estimating the static process gain.

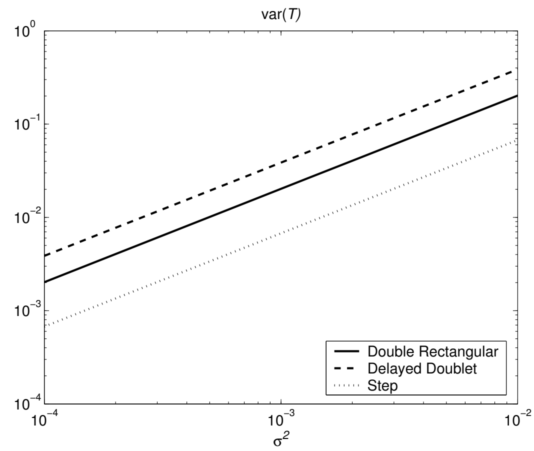


Fig. 8. Variance of $\hat{T}$ versus noise variance.

From Fig. 8 it is seen that the $\hat{T}$ estimates given by step input have the lowest variance and that the $\hat{T}$ estimates given by the delayed doublet pulse have the highest variance, respectively.

6. CONCLUSION

This paper has provided exact analytical expressions for the DC gain, time constant, and time delay of FOTD systems from knowledge of two points in the transient response to different finite-duration pulses. Our approach did not require any prior knowledge about the process DC gain and/or time delay. Variance analysis was used to study the quality of the estimates when the pulse response measurements were corrupted by noise. The robustness of the estimated model against a mismatch in the model order has also been investigated. The estimation accuracy was evaluated in both the time domain and the frequency domain. Comparing the proposed methods with step response based identification, the pulse response methods generally result in slightly larger estimation errors than the step response method unless the original plant can be well described by an FOTD system. It is also noted that the proposed methods depend on the input parameters A, μ, D, and Δ for high-order processes.

REFERENCES

L. Ljung, Identification for control: simple process models, *Proc. of the 41st IEEE Conference on Decision and Control*, pages 4652–4657, 2002.

K.J. Åström and T. Hägglund, *Advanced PID Control*, ISA, pages 49–51, 2006.

K. Srinivasan and M. Chidambaram, Modified relay feedback method for improved system identification, *Computers and Chemical Engineering*, volume 27, pages 727–732, 2003.

S. Majhi and L. Litz, Relay based estimation of process model parameters, *Proc. of the American Control Conference*, pages 2949–2953, 2003.

P.K. Padhy and S. Majhi, Relay based PI-PD design for stable and unstable FOPDT processes, *Computers and Chemical Engineering*, volume 30, pages 790–796, 2006.

S. Majhi, Relay-based identification of a class of nonminimum phase SISO processes, *IEEE Trans. on Automatic Control*, volume 52, number 1, pages 134–139, 2007.

S. Majhi, On-line PI control of stable processes, *Journal of Process Control*, volume 15, pages 859–867, 2005.

K. Srinivasan and M. Chidambaram, An improved autotune identification method, *Chemical and Biochemical Engineering Quarterly*, volume 18, pages 249–256, 2004.

M. Liu, Q.G. Wang, B. Huang and C.C. Hang, Improved identification of continuous-time delay processes from piecewise step tests, *Journal of Process Control*, volume 17, pages 51–57, 2007.

B.A. León de la Barra and M. Mossberg, Identification of under-damped second-order systems using finite-duration rectangular pulse inputs, *Proc. of the American Control Conference*, 2007.

S.H. Hwang and Y.C. Huang, Identification of continuous-time delayed systems using an input pulse of arbitrary shape, *Journal of the Chinese Institute of Engineers*, volume 29, number 2, pages 301–309, 2006.

Q.G. Wang and Y. Zhang, Robust identification of continuous systems with dead-time from step responses, *Automatica*, volume 37, pages 377–390, 2001.

C.A. Smith and A.B. Corripio, *Principles and practice of automatic process control*, John Wiley & Sons, pages 216–220, 1985.

Appendix A. DERIVATION OF RESULT 1

We first show the derivation of (12). If we substitute (8) into (10) we have

$$y_{1c} = y_{1a}\left(\mu + e^{-\frac{D+\Delta}{T}}\right). \tag{A.1}$$

Rewriting

$$\frac{y_{1c}}{y_{1a}} - \mu = e^{-\frac{D+\Delta}{T}} \tag{A.2}$$

and taking logarithm on both sides of (A.2)

$$\ln\left(\frac{y_{1c}}{y_{1a}} - \mu\right) = -\frac{D+\Delta}{T}. \tag{A.3}$$

Rearranging (A.3) leads to (12), i.e.,

$$T = \frac{D+\Delta}{\ln(y_{1a}) - \ln(y_{1c} - \mu y_{1a})}. \tag{A.4}$$

Alternatively, combining (8) and (9)

$$y_{1b} = y_{1a} e^{-\frac{\Delta}{T}} \tag{A.5}$$

leads to

$$T = \frac{\Delta}{\ln(y_{1a}) - \ln(y_{1b})}. \tag{A.6}$$

Now, if we rewrite (8) using (A.6) we have

$$y_{1a} = AK\left[1 - \left(\frac{y_{1b}}{y_{1a}}\right)^{\frac{D}{\Delta}}\right] \tag{A.7}$$

leading to (13). And (14) is obtained by using (A.4) in (8). The derivation of (15) is trivial. $\nabla\ \nabla\ \nabla$

Appendix B. DERIVATION OF RESULT 2

Combining (20) and (22) we have

$$y_{2c} = -y_{2a}\left(\mu - e^{-\frac{D+\Delta}{T}}\right), \tag{B.1}$$

and rearranging (B.1)

$$\frac{y_{2c}}{y_{2a}} + \mu = e^{-\frac{D+\Delta}{T}}. \tag{B.2}$$

Therefore, (24) is obtained by taking logarithm on both sides of (B.2). It is also clear from (24) that

$$e^{-\frac{D}{T}} = \left(\frac{y_{2c}}{y_{2a}} + \mu\right)^{\frac{D}{D+\Delta}} \tag{B.3}$$

and (26) follows by substituting (B.3) into (20). (23) and (25) follow by using (21) in (20). The derivation of (27) is trivial. $\nabla\ \nabla\ \nabla$