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# Improved convergence rate for a recursive procedure in a production and storage problem

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**Abstract:** We study a stochastic optimal impulse control problem which arises in a production and storage system involving identical items and stochastic demand. The problem is posed in the class of piecewise deterministic Markov process and it can be solved by two successive approximation methods. The first uses an uniformly contraction operator that has identical end costs at demand arrival time and at completion of an item time. The other method proposed in this work uses an extension of this operator with different end costs at these times. We show that the second method yields a recursive procedure with faster convergence rate.

Keywords: Markov Decision Process;Production Systems;Stochastic Control;Jump Process; Inventory Control;

## 1. INTRODUCTION

In the production and storage problem studied in this work the state space is composed of a discrete variable, defined by the stock level, and a continuous variable, defined by the progression on production of an item. The trajectory of the state space is characterized by random discontinuities (jumps), at demand arrival time and at the time of completion of an item. The decision-maker should define whether or not the production of an item should be initiated or interrupted. The production of an item can eventually be interrupted before the total completion is attained, allowing a preemptive decision action. An instantaneous set up cost is incurred when the production is interrupted or initialized, and the imbalance between production and demand of items is taken into account with a penalty function which is associated to the operating value of stock or back order for these items and the progression on the production. This problem is well posed in the framework of piecewise deterministic Markov process (PMDP).
The PMDP has been used to model Manufacturing flow control models with failure prone machines and random jumps associated to machine breakdowns, as we can see in Boukas and Yan (1996) and Yan and Zhang (1997). Continuous control problems are studied in Boukas and Yan (1996) whereas a combination of continuous and interventions control problems is addressed in Yan and Zhang (1997). The model in Boukas and Yan (1996) considers preventive maintenance in order to reduce failure rates and to delay the aging of machines. Multi-item single machine problems with piecewise deterministic demand are studied in Jean-Marie and Tidball (1999), Mancinelli and Gonzalez (1997).In do Val and Salles (1999) and Salles and do Val (2001), the Production and Storage models have the state variables represented by the stock level and progression on production, as the model studied in this work. However, in do Val and Salles (1999) it is not considered the set up cost, and the decision maker takes the actions continuously along the time. In Arruda et al. (2004) and Arruda et al. (2005) the progression on production variable is discretized in stages of production and the production and storage model is viewed as a discrete time process embedded in a PMDP process.
It is a well known fact in many situations of practical interest, that general methods yielded by dynamic programming often lead to computationally complex problems. So it is important to explore the structural properties of the problem in order to get efficient computational procedures. For example, in Aragonel and Gonzalez (1997) it is studied a computational procedure for the multi item single machine which allow to compute the solution in a short time; in Arruda et al. (2005) it is proposed an algorithm based on approximation schemes in order to reduce the computational dimensionality of the multi-product problem. This paper continues along this line, by showing an improvement of the recursive procedure proposed by Salles and do Val (2001), for the production and storage problem that is formulated in section 2. In Proposition 3 of section 3 it is compared the convergence rate of the method proposed in Salles and do Val (2001) (see Theorem 2) with the one we propose here (See Theorem 4). Finally, in section 4 we show the numerical results and we make the final comments.

## 2. THE PRODUCTION AND STORAGE PROBLEM

Consider a production and storage system subject to random demand. The policy maker in such a system should decide on appropriate intervention epochs, which define the time to re-start the production when the production is in the idle state and the number of items in stock is low.

Or conversely, the intervention should determine the time to halt the production, when the stock level is too high. Individual items are considered and the associated demand arrives at random in time and in size. Assume that it forms a compound Poisson process with rate $\lambda$, with the lot size requested by the $i$-th customer denoted by $\omega_i$. The lot sizes form a sequence of *i.i.d.* random variables with a distribution $p_k, k = 1, 2, \ldots, \ell$, such that $P(\omega_i = k) = p_k$ ($\ell$ is the maximum lot size to consider).

Let $\theta_i$ and $\sigma_i$ $i = 1, 2, \ldots$ denote respectively, the sequences of completion times of each item and the arrival times of the demand. Thus, the number of item in stock or in back order is given by the process

$$n_t := \sum_i \mathbb{1}_{\{\theta_i \le t\}} - \sum_i \omega_i \mathbb{1}_{\{\sigma_i \le t\}}, \quad t \ge 0.$$

where $\mathbb{1}$ is the indicator function. We consider that $n_t$ takes its values in a finite integer set $\bar{N} = \{N^-, \ldots, N^+\}$, where $N^- < 0$ is the lower bound for the back order level and $N^+ > 0$ is the physical limit for storage. The production of each item evolves at a constant rate $u$ and the investment to complete an item is indicated by $\Gamma$. We adopt a state variable $\xi \in [0, \Gamma]$ to indicate the progression on production (or how much we have invested in the production) of an item being manufactured, and set the state process as $z_t := (n_t, \xi_t)$. Whenever an item is on production, $t \to z_t$ evolves in a subset $S' := \bar{N} \times [0, \Gamma)$ and when the production is idle, $t \to z_t$ evolves in another copy of set $\bar{N} \times [0, \Gamma)$ which we denote by $S''$.

The trajectory of $z_t$ is described as follows. At arrival time $\sigma_i$ the process jumps from $\lim_{t \uparrow \sigma_i}(n_t, \xi_t) := (n_{\sigma_i^-}, \xi_{\sigma_i^-})$ to $(n_{\sigma_i}, \xi_{\sigma_i})$ with $n_{\sigma_i} = n_{\sigma_i^-} - \omega_i$ and $\xi_{\sigma_i^-} = \xi_{\sigma_i}$. When an item is completed at a time $\theta_j$, the jump occurs from $(n_{\theta_j^-}, \Gamma)$ to $(n_{\theta_j^-} + 1, 0)$. Let $T_i, i = 1, 2, \ldots$ the jump times due to the demand arrival or completion of an item. For $z = (n, \xi) = (n_{T_{i-1}}, \xi_{T_{i-1}})$ with $T_0 = 0$ and $i = 1, \ldots,$ the process $z_t$, for $T_{i-1} \le t < T_i$ follows a deterministic trajectory defined by the drift function:

$$\varphi(s, z) = \begin{cases} (n, u \cdot s + \xi), & 0 \le s \le t^*(z), \quad \text{for } z \in S', , \\ (n, \xi), & s \ge 0, \quad \text{for } z \in S''. \end{cases}$$

where $t^*(z) := \inf\{s : \varphi(s, z) \in \bar{N} \times \{\Gamma\}\}$. Observe that $t^*(z)$ is the time to produce an item assuming no stoppage along the production interval, when we start the production at $z_0 = z$. If $z \in S'$, $t^*(z) \le \Gamma/u$ , and if $z \in S''$, $t^*(z) = \infty$, since the production is idle. The inter jump time $(T_i - T_{i-1})$ is characterized by the following distribution:

$$P_z(T_i - T_{i-1} \le s) = \begin{cases} 1 - e^{-\lambda(z)s} & 0 \le s < t^*(z) \\ 1 & s \ge t^*(z) \end{cases} \tag{2.1}$$

where $z = (n_{T_{i-1}}, \xi_{T_{i-1}})$.

The process $z_t = (n_t, \xi_t)$ belongs to the class of PDMP, as defined in Davis (1993) with state space $S \cup \partial S$, where $S = S' \cup S''$ and $\partial S = \bar{N} \times \{\Gamma\}$ is the boundary of $S$. Let $F_t := \sigma(z_s : s \le t)$ be the filtration of $z_t$; thus the transition probability of $z_t$ ($P_z(z_{T_i} \in A | F_{T_i^-})$ for $z \in S \cup \partial S$ and any Borel measurable set $A \in S$) is defined by the function $\mu(z_{T_i}, z_{T_i^-}) := p_k \mathbb{1}_{\{T_i = \sigma_i\}} + \mathbb{1}_{\{T_i = \theta_i\}}$.

The sequence of intervention times that transfer the process $z_t$ from the production subset ($S'$) to the non production subset ($S''$) (or conversely) is denoted by $\pi = (\tau_1, \tau_2, \tau_3, \ldots)$ and let $\Pi$ be the class of all admissible intervention policies $\pi$ with respect to the filtration $\mathcal{F}_t$. If an intervention occurs at time $\tau_i$, the process is transferred from $z_{\tau_i^-} = (n_{\tau_i^-}, \xi_{\tau_i^-}) \in S'$ ($\in S''$) to $\bar{z}_{\tau_i} = (n_{\tau_i}, \xi_{\tau_i}) \in S''$ ($\in S'$) with $n_{\tau_i^-} = n_{\tau_i}$ and $\xi_{\tau_i^-} = \xi_{\tau_i}$.

Let us denote by $\beta > 0$ the cost per unit of time for running the production, $\alpha > 0$ a discount rate, and let $n \to L(n)$ be a convex function that represents the stock ($n > 0$ ) or the shortage ($n \le 0$) costs. We represent:

$$f(z) := L(n) + \beta \, \mathbb{1}_{\{z \in S'\}} \tag{2.2}$$

for all $z = (n, \xi) \in S$. When an intervention occurs at time $\tau_i$ an instantaneous cost $g(z_{\tau_i^-})$ is paid, and between these intervention epochs a cost is incurred at rate $t \to f(z_t)$. With these definitions, the expect cost for the production and storage problem with respect to a intervention strategy $\pi \in \Pi$, $V^\pi : \bar{N} \times [0, \Gamma] \to \mathbb{R}$, is defined by

$$V^\pi(z) := E_z^\pi \Big\{ \int_0^\infty e^{-\alpha s} f(z_s) ds + \sum_{i=1}^\infty e^{-\alpha \tau_i} g(z_{\tau_i^-}) \mathbb{1}_{\{\tau_i < \tau\}} \Big\} \tag{2.3}$$

whenever $z_0 = z = (n, \xi)$. The cost associated to the optimal policy is the *value function* for the production and storage problem that is given by:

$$V(z) := \inf_{\pi \in \Pi} V^\pi(z) \tag{2.4}$$

The problem defined above is a kind of impulse control problem of PDMP for which general optimization methods were proposed by Davis (1993).

*Remark 1.* The existence of stabilizing policies for the problem (2.4) is guaranteed if we assume that the demand will always be supplied by the production capacity, i.e. $\Gamma < 1/(\lambda \sum_{k=1}^{\ell} k p_k)$. In addition, the process $z_t$ will always have jumps into the finite state space $S$, if we assume zero demand arrival rate ($\lambda(z_t) = 0$) in the lower back order level ($z_t \in N^- \times [0, \Gamma)$) and zero production rate (u=0) in the higher stock level ($z_t \in N^+ \times [0, \Gamma)$).

## 3. SOLUTIONS FOR THE PRODUCTION AND STORAGE PROBLEM

The solution is determined from successive approximation methods. The first, proposed in Salles and do Val (2001), define a contraction operator that uses the same end costs at demand arrival time and at completion of an item time. The second method, proposed in this work, uses a similar operator that disconnect these end costs. These operators are defined in the following sequence.

We denoted $a \wedge b$ for $\min\{a, b\}$, and $\partial^* S$ for the subset of $\partial S$ that is really reached by the production process $z_t$; thus, for a given production strategy $\pi$, we can not stop the production when the process $z_t$ is in the neighborhood of $\partial^* S$. Set $C_b(S \cup \partial^* S)$ as the space of real continuous and bounded functions on $S \cup \partial^* S$. For any functions $\phi, \phi', \psi \in C_b(S \cup \partial^* S)$, let us define:

$$\mathcal{R}_t[\phi,\phi',\psi](z) := E_z\Big\{\int_0^{t\wedge T_1} e^{-\alpha s} f(z_s)ds + e^{-\alpha T_1}[\phi(z_{T_1})1\!\!1_{\{T_1=\sigma_1\}} + \phi'(z_{T_1})1\!\!1_{\{T_1=\theta_1\}})]1\!\!1_{\{t\geq T_1\}} + e^{-\alpha t}(\psi(\bar z_t)+g(z_t))1\!\!1_{\{t<T_1\}}\Big\}, \quad (3.1)$$

$$\mathcal{R}[\phi,\phi',\psi](z) := \inf_{0\leq t\leq t^*(z)} \mathcal{R}_t[\phi,\phi',\psi](z) \quad (3.2)$$

In definition of operator $\mathcal{R}_t$ we associated a penalty cost $\phi$ at demand arrival time ($\sigma_1$), a penalty cost $\phi'$ at completion time of an item ($\theta_1$) and a penalty cost $\psi + g$ at intervention time $t$. We assume that

($H_1$) $g(z) \geq g_0 > 0$, for each $z \in S$.
($H_2$) $f, g, \lambda \in C_b(S \cup \partial^* S)$ and $\mathcal{Q}[\phi](z) := \int_S \phi(dy)\mu(dy,z), \in C_b(S)$.
($H_3$) The total production time of an item without stoppage ($\Gamma/u$) is not zero.

For $\phi$ and $\phi'$ in $C_b(S \cup \partial^* S)$, let us consider the following operators:

$$\mathcal{N}[\phi](z) := \frac{1}{\alpha+\lambda(z)}(f(z) + \lambda(z)\sum p_k\phi(z_k)), \quad (3.3)$$

$$\text{for } z = (n,\xi) \text{ and } z_k = (n-k,\xi) \quad (3.4)$$

$$\tilde{\mathcal{P}}[\phi,\phi'](z) := \begin{cases} \mathcal{R}[\phi,\phi',\mathcal{N}[\phi]](z), \text{for } z \in S' \\ (\mathcal{R}[\phi,\phi',\mathcal{N}[\phi]](\bar z) + g(z)) \wedge \mathcal{N}[\phi](z), \\ \text{for } z \in S'', \end{cases} \quad (3.5)$$

$$\mathcal{P}[\phi](z) := \tilde{\mathcal{P}}[\phi,\phi](z) \quad (3.6)$$

Observe that $\mathcal{P}$ has the same end cost $\phi$ at demand arrival time and at completion of an item time, whereas in operator $\tilde{\mathcal{P}}$ these end costs can be different ($\phi$ and $\phi'$).

*Theorem 2.* Suppose that $H_1$, $H_2$ and $H_3$ hold. For $W_0 \in C_b(S \cup \partial^* S)$, the sequence of functions

$$W_i(z) := \mathcal{P}[W_{i-1}](z), \quad \forall z \in S, \quad (3.7)$$

converges to $V$ uniformly, as $i \to \infty$, and $V \in C_b(S\cup\partial^* S)$ is the unique solution of $V = \mathcal{P}[V]$.

Proof:See Theorem 3 in Salles and do Val (2001).

### 3.1 *The Proposed Method*

Since the process $z_t$ has an integer variable $n_t \in \bar N$ we may divide the state space $S$ in $N = N^+ + 1 - N^-$ subsets $S^j, j = 1,\dots N$, defined by $S^j = (j-1+N^-) \times [0,\Gamma]$. Therefore $S = S^1 \cup \dots \cup S^N$. Let us define the part of any function $\phi(z)$ on subspace $S^j$ by $\phi^j(z)$, i.e. $\phi^j(z) = \phi(z)$ for $z \in S^j$. Given an initial function $U_0 \in C_b(S \cup \partial^* S)$, let the sequence of functions $U_i$, $i = 1,\dots$ defined recursively by:

$$U_i^1(z) = \tilde{\mathcal{P}}[U_{i-1}, U_{i-1}](z) \text{ and}$$
$$U_i^j(z) = \tilde{\mathcal{P}}[\tilde U_i^{j-1}, U_{i-1}](z), \quad \forall z \in S^j,\ j = 2,\dots,N \quad (3.8)$$

where $\tilde U_i^{j-1}(z) = U_i^1(z)1\!\!1_{\{z\in S^1\}} + U_i^2 1\!\!1_{\{z\in S^2\}} + \dots + U_i^{j-1}(z)1\!\!1_{\{z\in S^{j-1}\}}$.

Let us define the constants $\rho_1 := \sup_z |E_z[e^{-\alpha T_2}]|$ and $\rho_2 := \sup_z |E_z[e^{-\alpha\theta_2}]|$. It is clear that $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$, since $0 < T_2 \leq \theta_2 < \infty$. We compare the convergence rate of the sequences $W_i$ and $U_i$ in the following result:

*Proposition 3.* Suppose that $H_1$ and $H_3$ hold and $\lambda(z) = 0$ for $z \in S^1$. For each $i = 2,3,\dots$ we have the following statements:

(i) $\|W_{i+1} - W_i\| \leq \rho_1\|W_{i-1} - W_{i-2}\|$ and $\|U_{i+1} - U_i\| \leq \rho_2\|U_{i-1} - U_{i-2}\|$;
(ii) $\rho_2/\rho_1 = (\alpha+\lambda)/(\alpha + \lambda e^{(\alpha+\lambda)\Gamma/u}) < 1$.

Proof of item (i): It follows directly from expressions (A.1) and (A.2) in Appendix A.
Proof of item (ii): Since $T_2 - T_1$ has probability distribution given by (2.1)we have that

$$\rho_1 = \sup_z |E_z[e^{-\alpha T_2}]| = \sup_z |E_z = [e^{-\alpha T_1}E_{z_{T_1}}[e^{-\alpha(T_2-T_1)}]]| = \sup_z |E_z[\frac{(\alpha e^{-(\alpha+\lambda)t^*(z_{T_1})} + \lambda)}{\alpha+\lambda}] \quad (3.9)$$

The sup in expression (3.9) occurs when $T_1 = 0$, i.e. when $z \in \partial^* S$, since the process $z_t$ jumps when an item is completed. In view of assumption ($H_3$) we have $t^*(z_{T_1}) = \Gamma/u > 0$ and consequently $0 < \rho_1 = \frac{(\alpha e^{-(\alpha+\lambda)\Gamma/u}+\lambda)}{\alpha+\lambda} < 1$. Now, let us analyse the expression:

$$\rho_2 = \sup_z |E_z[e^{-\alpha\theta_2}]| = \sup_z |E_z[e^{-\alpha\theta_1}E_{z_{\theta_1}}[e^{-\alpha(\theta_2-\theta_1)}]]| \quad (3.10)$$

Observe that $\theta_2 \geq T_2$ and that $\theta_2 = T_2$ if and only if $T_1 = \theta_1 = t^*(z)$ and $T_2 - T_1 = t^*(z_{T_1}) = \Gamma/u > 0$, i.e. if there isnt any demand arrival before finishing two items. Therefore, from (2.1) we conclude that:

$$E_{z_{\theta_1}}[e^{-\alpha(\theta_2-\theta_1)}] = \\ = E_{z_{T_1}}[e^{-\alpha(T_2-T_1)}1\!\!1_{\{T_1=t^*(z), T_2-T_1=t^*(z_{T_1})\}})] = \\ = e^{-(\alpha+\lambda)t^*(z_{T_1})} = \rho_2 < 1$$

where the last equality is got from the fact that the sup in expression (3.10) occurs when $\theta_1 = 0$, i.e. when $z \in \partial^* S$. Thus $\rho_2/\rho_1$ is given by the expression in part (ii) of Proposition 2.1. □

In view of Proposition 2.1, the convergence rate in the sup norm of sequence $W_i$ is larger than the convergence rate of sequence $U_i$. The next Theorem shows that $U_i$ converges to the solution of the Production and Storage problem (2.4).

*Theorem 4.* Suppose that $H_1$, $H_2$ and $H_3$ hold, and $\lambda(z) = 0$ for $z \in S^1$. Then $U_i(z) \in C_b(S\cup\partial^* S), i,\ i = 1,\dots$ converges to $V$ uniformly as $i \to \infty$.

Proof:First we have to show that $U_{i+1}^j(z) \in C_b(S \cup \partial^* S)$, supposing that $U_i(z) \in C_b(S \cup \partial^* S)$ and that $U_{i+1}^k \in C_b(S\cup\partial^* S)$ for $k = 1,\dots,j-1,\ j \geq 2$. From (3.3) we can write for $z = (n,\xi) \in S^j$ that $\mathcal{N}[\tilde U_{i+1}^{j-1}](z) = \frac{1}{\alpha+\lambda(z)}(f(z) + \lambda(z)\sum_{k=1}^{\ell} p_k U_{i+1}^{j-k}(z_k))$. Thus $\mathcal{N}[\tilde U_{i+1}^{j-1}](z) \in C_b(S \cup \partial^* S)$ for $z \in S^j$. In view of this and the fact that $U_{i+1}^j(z) = \tilde{\mathcal{P}}[\tilde U_{i+1}^{j-1}, U_{i-1}](z) = \mathcal{R}[\tilde U_{i+1}^{j-1}, U_i, \mathcal{N}[\tilde U_{i+1}^{j-1}]](z)$ we conclude from Lemma (53.38) p. 224 in Davis (1993) that $U_{i+1}^j(z) \in C_b(S' \cup \partial^* S')$. Consequently, from (3.5), we have that $U_{i+1}^j(z) \in C_b(S'' \cup \partial^* S'')$. Thus $U_{i+1}^j(z) \in C_b(S\cup\partial^* S)$ for $j \geq 2$. Now, we need to show that $U_{i+1}^1(z) \in C_b(S\cup\partial^* S)$. From Remark 2.1, $\mathcal{N}[U_i](z) = f(z)/\alpha$, for $z \in S^1$, since we assume that $\lambda(z) = 0,\ z \in S^1$. Thus, from $H_2$, $\mathcal{N}[U_i](z) \in$

$C_b(S \cup \partial^* S)$. Using the same arguments above, we have that $U^1_{i+1}(z) \in C_b(S \cup \partial^* S)$. From Proposition 2.1 the sequence $U_i$ is cauchy in the sup norm, therefore $U_i$ converges to some function $U \in C_b(S \cup \partial^* S)$ uniformly as $i \to \infty$, that satisfies $U = \tilde{\mathcal{P}}[U,U](z) = \mathcal{P}[U](z)$. Thus,from Theorem 2.1 $V = U$. □

## 4. NUMERICAL EXAMPLES

Table 1. Numerical Examples.

| Parameters | Study Cases A | Study Cases B |
|---|---|---|
| $(\alpha)$ | 0.25 | 0.05 |
| $(\beta)$ | 0 | 0 |
| $(\lambda)$ | 0.3 | 0.3 |
| $(L, n \geq 0)$ | $10\,n$ | $2.5\,n$ |
| $(L, n < 0)$ | $-10\,n$ | $-2.5\,n$ |
| Set Up Cost $(g)$ | $2.5\exp\{\alpha\xi\}$ | $2.5\exp\{\alpha\xi\}$ |
| $(\Gamma/u)$ | 2 | |
| $(p_k)$ | 2 | |

Table 2. Normalized Iteration for convergence of Methods 1 and 2

| Case | Method 1 | Method 2 |
|---|---|---|
| A (I=39) | 1 | 0.48 |
| B (I=244) | 1 | 0.55 |

The result in Proposition 2.1 is illustrated in two numerical examples with parameters given in Table 1, with $\ell = 3$. The state space is $S = [0,2) \times \{-146,\ldots,54\}$, and we initialized the recursive procedures in Theorems 2.1 and 2.2 (denoted here by Method 1 and 2, respectively) with the same initial function given by $\|L(n)/\alpha\|$. We show in table 2 the normalized iteration for the convergence error of each sequence in the sup norm to be lower than 0.02. We take as reference the total iteration $(I)$ for convergence of Method 1. We observe that the normalized iteration for convergence of Method 2 is lower than the Method 1 and is approximately given by $\rho_2/\rho_1$, where $\rho_1$ and $\rho_2$ are the convergence rate of Methods 1 and 2 respectively, defined in Proposition 2.1. We also observe that the number of iteration I to solve the case A is lower than the case B, because the constants $\rho_1$ and $\rho_2$ increase when the discount factor $(\alpha)$ decreases.

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## Appendix A. AUXILAR RESULT

In this section we present the Lemma that is used in the proof of item (i) of Proposition 2.1.

*Lemma 5.* For each $z \in S$ and $i = 1, 2, \ldots$

$$W_{i+1}(z) = \min_{\pi\in\Pi} E^\pi_z\Big\{\int_0^{T_2} e^{-\alpha s} f(z_s)\,dt \tag{A.1}$$

$$+ \sum_{j=1}^{\iota_i} e^{-\alpha\tau_j} g(z_{\tau_j}) \mathbb{1}_{\{\tau_j < T_2\}} + e^{-\alpha T_2} W_{i-1}(z_{T_2})\Big\},$$

$$U_{i+1}(z) := \min_{\pi\in\Pi} E^\pi_z\Big\{\int_0^{\theta_2} e^{-\alpha s} f(z_s)\,ds$$

$$+ \sum_{j=1}^{\iota_i} e^{-\alpha\tau_j} g(z_{\tau_j}) \mathbb{1}_{\{\tau_j < \theta_2\}} + e^{-\alpha\theta_2} U_{i-1}(z_{\theta_2})\Big\}. \tag{A.2}$$

where $\iota_i$ is an integer such that $\iota_i \geq i$

Proof: Suppose that

$$W_{i+1}(z) = \min_{\pi\in\Pi} E^\pi_z\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,dt$$

$$+ \sum_{j=1}^{\iota_w} e^{-\alpha\tau_j} g(z_{\tau_j}) \mathbb{1}_{\{\tau_j < T_1\}} + e^{-\alpha T_1} W_i(z_{T_1})\Big\}, \tag{A.3}$$

$$U^j_{i+1}(z) = \min_{\pi\in\Pi} E^\pi_z\Big\{\int_0^{\theta_1} e^{-\alpha s} f(z_s)\,dt$$

$$+ \sum_{j=1}^{\iota_u} e^{-\alpha\tau_j} g(z_{\tau_j}) \mathbb{1}_{\{\tau_j < \theta_1\}} + e^{-\alpha\theta_1} U_i(z_{\theta_1})\Big\}, \tag{A.4}$$

where $\iota_w$ and $\iota_u$ are integers

for each $z \in S$, $i = 2, 3, \ldots$ and $j = 1, \ldots, N$. Since $\{z_t : t \geq 0\}$ is a Strong Markov Process, we conclude from (A.3) and the dynamic programming principle that:

$$
\begin{aligned}
\mathrm{W}_{i+1}(z) &= \min_{\tau_1\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+ e^{-\alpha\tau_1} g(z_{\tau_1})\mathbb{1}_{\{\tau_1<T_1\}} + e^{-\alpha T_1} W_i(z_{T_1})\Big\} \\
&= \min_{\tau_1\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+ e^{-\alpha\tau_1} g(z_{\tau_1})\mathbb{1}_{\{\tau_1<T_1\}} \\
&+ e^{-\alpha T_1} \min_{\pi\in\Pi} E^\pi_{z_{T_1}}\Big\{\int_{T_1}^{T_2} e^{-\alpha(s-T_1)} f(z_s)\,ds \\
&+ \sum_{j=1}^{i+1} e^{-\alpha(\tau_j-T_1)} g(z_{\tau_j})\mathbb{1}_{\{T_1<\tau_j<T_2\}} \\
&+ e^{-\alpha(T_2-T_1)} W_{i-1}(z_{T_2})\Big\}|\mathcal{F}_{T_1}\Big\} \\
&= \min_{\pi\in\Pi} E_z^\pi\Big\{\int_0^{T_2} e^{-\alpha s} f(z_s)\,ds \\
&+ \sum_{j=1}^{i+1} e^{-\alpha\tau_j} g(z_{\tau_j})\mathbb{1}_{\{\tau_j<T_2\}} + e^{-\alpha T_2} W_{i-1}(z_{T_2})\Big\},
\end{aligned}
$$

showing expression (A.1). From (A.4) and using the same arguments above we show (A.2). Now, to finish this prove we need to show the expressions (A.3) and (A.4).

Defining $\tau_1 := t \wedge T_1$ we conclude from (3.1), (3.2) and (3.6) for $z \in S'$ that:

$$
\begin{aligned}
&W_{i+1}(z) = \mathcal{P}[W_i](z) = \mathcal{R}[W_i, W_i, \mathcal{N}[W_i]](z) = \\
&\min_{\tau_1\in\Pi} E_z^\pi\Big\{\int_0^{\tau_1} e^{-\alpha s} f(z_s)\,ds \\
&+ e^{-\alpha\tau_1}(g(z_{\tau_1}) + \mathcal{N}[W_i](\overline{z}_{\tau_1}))\mathbb{1}_{\{\tau_1<T_1\}} \\
&+ e^{-\alpha T_1} W_i(z_{T_1})\mathbb{1}_{\{\tau_1\geq T_1\}}\Big\}.
\end{aligned}
$$

In addition, we observe for $\overline{z}_{\tau_1} \in S''$ that

$$
\begin{aligned}
\mathcal{N}[W_i](\overline{z}_{\tau_1}) &= E_{\overline{z}_{\tau_1}}\Big\{\int_{\tau_1}^{T_1} e^{-\alpha(s-\tau_1)} f(z_s)\,ds \\
&+ e^{-\alpha(T_1-\tau_1)} W_i(z_{T_1})\Big\}
\end{aligned}
$$

and from (A.5) and (A.5), we conclude, for $z \in S'$, that

$$
\begin{aligned}
W_{i+1}(z) &= \min_{\tau_1\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+e^{-\alpha\tau_1} g(z_{\tau_1})\mathbb{1}_{\{\tau_1<T_1\}} + e^{-\alpha T_1} W_i(z_{T_1})\Big\}. \qquad \text{(A.5)}
\end{aligned}
$$

For $z \in S''$ it follows from Lemma 21 in Salles and do Val (2001) and from (A.5) that:

$$
\begin{aligned}
&W_{i+1}(z) = \mathcal{P}[W_i](z) = \mathcal{R}[W_i, W_i, W_{i+1}](z) = \\
&= \min_{\tau_1\in\Pi} E_z^\pi\Big\{\int_0^{\tau_1} e^{-\alpha s} f(z_s)\,ds + e^{-\alpha\tau_1} g(z_{\tau_1})\mathbb{1}_{\{\tau_1<T_1\}} \\
&+ e^{-\alpha\tau_1} \min_{\tau_2\in\Pi} E^\pi\Big\{\int_{\tau_1}^{T_1} e^{-\alpha(s-\tau_1)} f(\bar{z}_s)\,ds \\
&+ e^{-\alpha(\tau_2-\tau_1)} g(\bar{z}_{\tau_2})\mathbb{1}_{\{\tau_2<T_1\}} \\
&+ e^{-\alpha(T_1-\varsigma_1)} W_i(\bar{z}_{T_1})|\mathcal{F}_{\tau_1}\Big\}\mathbb{1}_{\{\tau_1<T_1\}} \\
&+ e^{-\alpha T_1} W_i(z_{T_1})\mathbb{1}_{\{\tau_1\geq T_1\}}\Big\} \\
&= \min_{\pi\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+ \sum_{j=1}^{2} e^{-\alpha\tau_j} g(z_{\tau_j})\mathbb{1}_{\{\tau_j<T_1\}} + e^{-\alpha T_1} W_i(z_{T_1})\Big\}.
\end{aligned}
$$

where the last equality is obtained using the fact that $\{z_t : t \geq 0\}$ is a Strong Markov Process, showing the expression (A.3). Now we start the proof of expression (A.4). In view of

$$
U_{i+1}^j(z) = \tilde{\mathcal{P}}[\tilde{U}_{i+1}^{j-1}, U_i](z) = \mathcal{R}[\tilde{U}_{i+1}^{j-1}, U_i, \mathcal{N}[\tilde{U}_{i+1}^{j-1}]](z)
$$

we can apply the same arguments used to obtain (A.5) and (A.6), more the fact that $\theta_1 = \infty$ for $z \in S''$, to conclude that:

$$
\begin{aligned}
U_{i+1}^j(z) &= \min_{\tau\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+ \sum_{j=1}^{\iota_w} e^{-\alpha\tau_j} g(z_{\tau_j})\mathbb{1}_{\{\tau_j<T_1\}} \\
&+ e^{-\alpha T_1}(U_i(z_{T_1})\mathbb{1}_{\{T_1=\theta_1\}} + \tilde{U}_{i+1}^{j-1}(z_{T_1})\mathbb{1}_{\{T_1=\sigma_1\}})\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{U}_{i+1}^{j-1}(z_{T_1})\mathbb{1}_{\{T_1=\sigma_1\}} = U_{i+1}^1(z_{T_1})\mathbb{1}_{\{T_1=\sigma_1, z_{T_1}\in S^1\}} \\
&+ U_{i+1}^2(z_{T_1})\mathbb{1}_{\{T_1=\sigma_1, z_{T_1}\in S^2\}} + \dots \\
&+ U_{i+1}^{j-1}(z_{T_1})\mathbb{1}_{\{T_1=\sigma_1, z_{T_1}\in S^{j-1}\}}
\end{aligned}
$$

Since $\lambda(z) = 0$ for $z \in S^1$ we conclude that $\mathbb{1}_{\{T_1=\sigma_1\}} = 0$ given $z \in S^1$. According to this fact and from (A.6) we have that $U_{i+1}^1(z)$ satisfies (A.4) for $j = 1$. Therefore, from (A.6) and (A.6) with $j = 2$ we have for $z \in S^2$ that

$$
\begin{aligned}
&U_{i+1}^2(z) = \min_{\tau\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+ \sum_{j=1}^{\iota_w} e^{-\alpha\tau_j} g(z_{\tau_j})\mathbb{1}_{\{\tau_j<T_1\}} + e^{-\alpha T_1}(U_i\mathbb{1}_{\{T_1=\theta_1\}} \\
&+ U_{i+1}^1(z_{T_1})\mathbb{1}_{\{T_1=\sigma_1\}})\Big\} = \min_{\tau\in\Pi} E_z^\pi\Big\{\int_0^{T_1} e^{-\alpha s} f(z_s)\,ds \\
&+ \sum_{j=1}^{\iota_w} e^{-\alpha\tau_j} g(z_{\tau_j})\mathbb{1}_{\{\tau_j<\sigma_1\}} + e^{-\alpha T_1}(U_i\mathbb{1}_{\{T_1=\theta_1\}} \\
&+ \min_{\tau\in\Pi} E^\pi_{z_{\sigma_1}}\Big\{\int_{\sigma_1}^{\theta_1} e^{-\alpha(s-\sigma_1)} f(z_s)\,ds \\
&+ \sum_{j=1}^{\iota_w} e^{-\alpha\tau_j} g(z_{\tau_j})\mathbb{1}_{\{\tau_j<\theta_1\}} \\
&+ e^{-\alpha(\theta_1-\sigma_1)} U_i(z_{\theta_1})|\mathcal{F}_{\sigma_1}\Big\}\mathbb{1}_{\{T_1=\sigma_1, z_{T_1}\in S^1\}}\Big\}
\end{aligned}
$$

Therefore, using the Strong Markov property we show that $U^2_{i+1}(z)$ also satisfies (A.4). By induction, we show that $U^j_{i+1}(z)$ $j = 3, \dots N$ satisfies (A.4). □