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# Calculation of All $H_\infty$ Robust Stabilizing Gains for SISO LTI Systems

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**Abstract:** Robust stabilization of continuous time single-input single-output (SISO) linear time invariant (LTI) systems with multiplicative uncertainties is considered in this paper. In particular, it has been shown that all gains that robustly stabilize a given uncertain SISO LTI system can be found by utilizing a generalization of the Nyquist theorem. The method proposed involves calculation of roots of two real polynomials and does not require any search or gridding over a parameter, and as a result offers computational advantages over existing methods in literature.

## 1. INTRODUCTION

Most of the controllers used in the practical world are low order controllers, which usually come in a form of PID controller (Åström and Hägglund 1996). Recently, a large amount of research has focused on finding the set of all stabilizing fixed-order controllers as a result of this practical motivation. To this extent, Ho et al (1997) have demonstrated that all stabilizing P, PI and PID controllers can be found by the help of a generalized Hermite-Biehler Theorem. Munro (1999), Munro et al (1999) and Söylemez et al (2003) employed a generalization of the Nyquist theorem to find analytical descriptions of stabilizing low-order controllers. Meanwhile, Ackermann and Kaesbauer (2001) and Bajcinca (2006) have used the idea of singular frequencies for the same purpose. Extensions of these results have been given to cover stabilization of systems with parameter uncertainties (see Ho et al (2001), Munro and Söylemez (1999)).

Sometimes it is not possible (or practical) to represent uncertainties in a system model with parametric uncertainties. Such uncertainties are usually encapsulated in a norm bounded system block that acts on a nominal system in an additive or multiplicative manner (Skogestad and Postlethwaite 2005). Although it is possible to find robust controllers that can stabilize systems with such uncertainties by the help of $H_\infty$ control theory, the resulting compensators are usually of high order (at least as high as the order of the plant) and therefore impractical in many cases. Several attempts exist to put constraints on the order of $H_\infty$ controllers (see Iwasaki and Skelton (1995) for example). However, as stated in Ho (2001) many of these approaches suffer from computational intractability.

Ho (2001) has provided a method for finding the set of PID controllers that satisfy given $H_\infty$ performance criteria for the first time (see also Ho and Lin (2003)). The method proposed by Ho involves a gridding on the proportional term ( $K_p$ ) and calculation of intersection of stabilizing regions on the $K_i - K_d$ plane for a set of polynomials with complex coefficients using an extension of the Hermite-Biehler theorem. Blanchini et al (2004) give an alternative method based on intuitive graphical considerations to determine second order controllers that satisfy given $H_\infty$ specifications. This method also requires a gridding on the third parameter ( $K_p$ in the case of PID control). Nevertheless, in some practical cases direct determination of the set of proportional controllers that provide robust stability is required. To the best knowledge of authors, there is no such direct methods available in the literature for this purpose. The main aim of this paper is to provide such a method.

Nyquist theorem and its generalization is considered in the following section. Calculation of robust stabilizing gains and the main results of the paper are given in Section 3. Section 4 includes conclusions and possible future work.

## 2. NYQUIST THEOREM AND STABILIZING GAINS

Consider the simple feedback connection shown in Fig. 1. For a given value of the constant controller *K*, Nyquist theorem provides a way to determine the stability of the closed-loop system by examining the frequency response of the open-loop system.

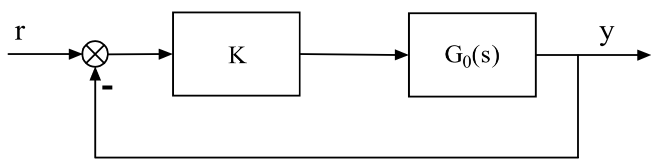



Fig. 1. Closed-loop control system with proportional control

Nyquist theorem can be summarized as follows:

**Theorem 2.1 (Nyquist 1932):** Let N denote the number of encirclements of the polar (Nyquist) plot of $G_0(s)$ around the critical point $-1$ in the complex plane. If $u_0$ and $u_c$ are the number of unstable poles of the open-loop system $G_0(s)$ , and closed-loop system (for $K = 1$), respectively, the following equation holds

$$N = u_c - u_0 \tag{1}$$

**Remark 2.1:** For any given gain $K$, Nyquist theorem can be modified to find the number of unstable closed-loop system poles by finding the encirclements of the Nyquist plot of $G_0(s)$ around the critical point $\frac{-1}{K}$.

Actually, it is possible to exploit the idea presented in Remark 2.1 so as to determine the set of all stabilizing gains ($K$) by finding the location and direction of the crossings of the Nyquist plot of the real axis (without actually drawing the plot).

To this extend, consider the single-input single-output control system of Fig. 1 where

$$G_0(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \tag{2}$$

is the plant to be controlled (with $a_i, b_i \in \mathbb{R}$).

Decomposing the numerator and denominator polynomials of (2) into their even and odd parts and substituting $s = jw$ gives

$$G_0(jw) = \frac{N_e(-w^2) + jwN_o(-w^2)}{D_e(-w^2) + jwD_o(-w^2)} \tag{3}$$

where $N_e$, $N_o$, $D_e$, and $D_o$ are polynomials corresponding to even and odd parts of numerator and denominator polynomials, respectively. It is then possible to write

$$G_0(jw) = \frac{D_e N_e + D_o N_o w^2}{D_e^2 + D_o^2 w^2} + jw\left[\frac{D_e N_o - D_o N_e}{D_e^2 + D_o^2 w^2}\right] \tag{4}$$

and

$$G_0(jw) = \frac{X_0(w^2)}{Z_0(w^2)} + jw\frac{Y_0(w^2)}{Z_0(w^2)} \tag{5}$$

where

$$X_0(w^2) \triangleq D_e N_e + D_o N_o w^2 \tag{6}$$

$$Y_0(w^2) \triangleq D_e N_o - D_o N_e \tag{7}$$

$$Z_0(w^2) \triangleq D_e^2 + D_o^2 w^2 \tag{8}$$

and where for notational simplification $D_e$, $D_o$, $N_e$ and $N_o$ are used instead of $D_e(-w^2)$, $D_o(-w^2)$, $N_e(-w^2)$ and $N_o(-w^2)$, respectively. Note that the imaginary part of $G_0(jw)$ is given by

$$\mathrm{Im}[G_0(jw)] = w\frac{Y_0(w^2)}{Z_0(w^2)} \tag{9}$$

By denoting $v \triangleq w^2$ and the positive real roots of *Y(v)* as $v_1^*, v_2^*, ..., v_\gamma^*$ it is possible to show that the Nyquist plot of $G_0(jw)$ crosses the real axis only if $w = 0$, $w = \infty$, or $w = \pm\sqrt{v_i^*}$ for $i = 1, 2, ..., \gamma$. Denoting $v_{\gamma+1}^* = 0$ and $v_{\gamma+2}^* = \infty$, the real axis crossing points are found as $x_i = X_0(v_i^*)/Z_0(v_i^*)$ for $i = 1, 2, ..., \gamma + 2$. Relabeling the pairs $(x_i, v_i^*)$ such that $x_i \le x_{i+1}$, it is possible to state the following theorem.

**Theorem 2.2: (Munro et al (1999))**
Consider a linear time-invariant system given by a proper rational transfer function $G_0(s) = N(s)/D(s)$ given as in (2), and assume that D(s) has no roots on the imaginary axis. Let $X_0(w^2)$, $Y_0(w^2)$ and $Z_0(w^2)$ be polynomials as defined (6)-(8), and the pairs $(x_i, v_i^*)$ $(i = 1, 2, ..., \gamma + 2)$ be as defined above. Furthermore, denote the first coefficient of $Y(v)$ as $y_1$, and the last nonzero coefficient of $Y(v)$ as $y_0$. Then, for a given gain $k \in \mathbb{K}_i \triangleq (-1/x_{i-1}, -1/x_i)$, the number of unstable poles of the closed-loop system ($u_i$) is given by

$$u_i = u_0 + \sum_{t=1}^{i-1} r_t \tag{10}$$

where $u_0$ is the number of unstable poles of $G_0(s)$, and $r_i$ denotes number of pole crossings from left half plane to right half plane (rhp) defined as

$$r_i \triangleq \begin{cases} (1-(-1)^l)Sgn(Y_0^{(l)}(v_i^*)) & if\ \ 0 < v_i^* < \infty \\ Sgn(y_0) & if\ \ v_i^* = 0 \\ -Sgn(y_1) & if\ \ v_i^* = \infty \end{cases} \tag{11}$$

in which $Sgn()$ is the standard signum function, which takes values from the set $\{-1 \quad 0 \quad 1\}$ depending on the sign of its argument, and $Y_0^{(l)}(v_i^*)$ is the first nonzero derivative of $Y(v)$ at the point $v_i^*$. The stabilizing intervals are then those of $\mathbb{K}_i$ for which $u_i = 0$. Note that if $\mathrm{Sgn}(x_{i-1}) \ne \mathrm{Sgn}(x_i)$ the corresponding gain interval is divided into two parts: $\mathbb{K}_i = [-1/x_{i-1} \quad \infty] \cup [-\infty \quad -1/x_i]$.

**Example 2.1:** Consider a system described by the transfer function

$$G_0(s) = \frac{0.109693s^4 + 0.0728112s^3 + 0.7354s^2 + 0.318s + 1}{1.41834s^5 + 3.39206s^4 + 7.62744s^3 + 5.9986s^2 + 3.33s + 1}$$

Nyquist plot of the system is plotted at different scales in Fig. 2. Noting that the open-loop system is stable, closed-loop system becomes stable only when the critical point $-1/K$ is on one of the regions with $N = 0$ (see Fig. 2). Furthermore, it is possible to find these regions, and hence stabilizing gains (*without actually drawing the Nyquist diagram*) with the help of Theorem 2.2. From (6)-(8),

$$X(w^2) = 0.268815w^8 - 2.14613w^6 + 5.24513w^4 - 5.67506w^2 + 1$$
$$Y(w^2) = -0.155583w^8 + 1.63275w^6 - 5.8774w^4 + 8.09596w^2 - 3.012$$
$$Z(w^2) = 2.0117w^{10} - 10.1306w^8 + 26.9288w^6 - 8.03144w^4 - 0.9083w^2 + 1$$

Positive real roots of $Y(w^2)$ are 0.759835, 1.53309, 1.88847, 2.0001. Hence, the following table is constructed using Theorem 2.2. It is then possible to state that the nominal system is stable under closed-loop if, and only if, $K \in L$, where

$$L = [-1 \quad 2.17752] \cup [46.0416 \quad 99.9875] \cup [167.803 \quad \infty]$$

**Table 1:** Calculation of stabilizing gain intervals

| $i$ | $w_i$ | $x_i$ | $r_i$ | $u_i$ | $\mathbb{K}_i$ |
|---|---|---|---|---|---|
| 1 | 0.759835 | -0.459237 | 2 | 0 | $[0 \quad 2.17752]$ |
| 2 | 1.53309 | -0.021719 | -2 | 2 | $[2.17752 \quad 46.0416]$ |
| 3 | 1.88847 | -0.010001 | 2 | 0 | $[46.0416 \quad 99.9875]$ |
| 4 | 2.0001 | -0.005959 | -2 | 2 | $[99.9875 \quad 167.803]$ |
| 5 | $\infty$ | 0 | 1 | 0 | $[167.803 \quad \infty]$ |
| 6 | 0 | 1.0 | -1 | 1 | $[-\infty \quad -1]$ |
| 7 | - | $\infty$ | - | 0 | $[-1 \quad 0]$ |

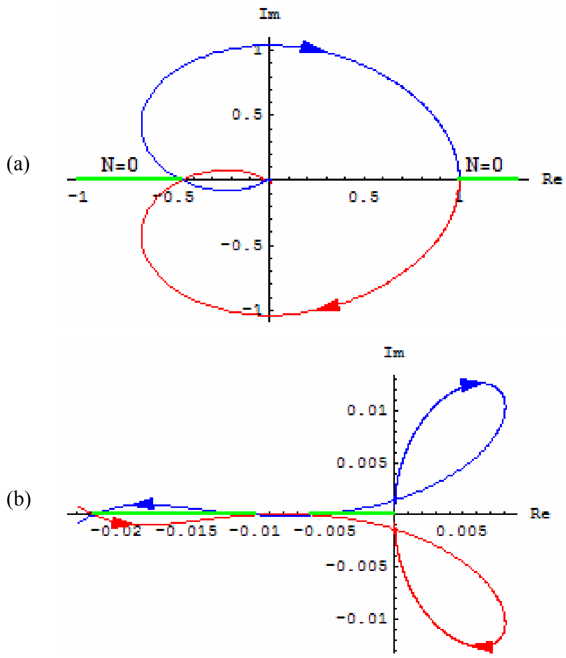



**Fig. 2.** Nyquist Plot for Example 2.1 at different scales. Red curves correspond to the part of the plot for $w > 0$, whereas blue curves correspond to $w < 0$.

## 3. ROBUST STABILIZATION

Usually, finding exact models of systems is very difficult, if not impossible, in practice. Therefore it is a common practice to determine the uncertainty in a model for a given system. There are several ways to represent the uncertainties for a given LTI system. One such way is representing the uncertainty as an unstructured uncertainty as shown in Fig. 3.

In Fig. 3, $G_0(s)$ represents the nominal system, $\Delta$ is the uncertainty block that represents all systems with $H_\infty$ norm less than 1 ($\|\Delta\|_\infty \le 1$), and $W(s)$ is a weight transfer function that determines the " relative size" of uncertainty at different frequencies. The system to be controlled is denoted by G in Fig. 3, and is given as

$$G(s) = G_0(s)(1 + W(s)\Delta) \tag{12}$$

The aim of control here is to find the set of gains $K$ that stabilize the closed-loop system under all possible uncertainties. Note that this problem can also be posed as that of finding gains that satisfy an $H_\infty$ constraint on the weighted closed-loop system transfer function $T(s) = \dfrac{KG_0(s)}{1 + KG_0(s)}$ such that $\|W(s)T(s)\|_\infty < 1$.

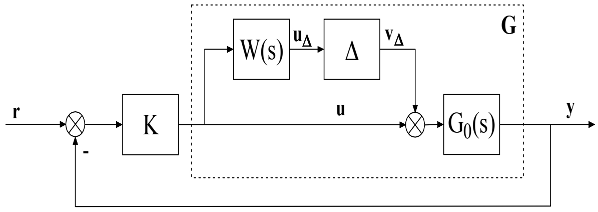



**Fig. 3.** Closed-loop control of an uncertain system.

We remark that equation (12) actually defines a family of systems, and hence, Nyquist plot of G is a family of curves rather than a single curve (see Fig. 5). As a result, Nyquist plot of G crosses the real axis in segments of the real axis instead of at single points.

Actually, it is possible to argue that frequency response of a system ($G(jw)$) with multiplicative uncertainty at a given frequency ($w^*$) is a disk centered at $G_0(jw^*)$ and radius $r = |G_0(jw)W(jw)|$ (see Fig. 4) (Skogestad and Postlethwaite 2005). When examined carefully, Fig. 4 reveals that Nyquist plot of $G(jw)$ crosses the real axis for $p_1 \le \mathrm{Re}(s) \le p_2$ at a given frequency $w = w^*$, where

$$\begin{aligned} p_{1,2}(w) &= \mathrm{Re}(G_0(jw)) \mp \alpha \\ &= \underbrace{\frac{X_0(w)}{Z_0(w)}}_{\triangleq a(w)} \mp \sqrt{\underbrace{r^2 - \beta^2}_{\triangleq b(w)}} \\ &= a(w) \mp \sqrt{b(w)} \end{aligned} \tag{13}$$

in which

$$\beta = \mathrm{Im}(G_0(jw)) = \frac{wY_0(w)}{Z_0(w)} \tag{14}$$

and

$$\begin{aligned} r^2 &= |G_0(jw)|^2 |W(jw)|^2 \\ &= \frac{(X_0^2 + w^2Y_0^2)(X_W^2 + w^2Y_W^2)}{Z_0^2 Z_W^2} \end{aligned} \tag{15}$$

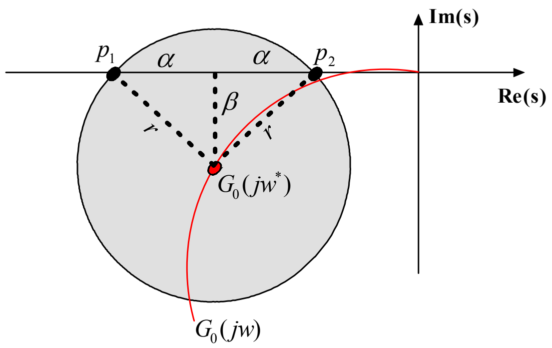



**Fig. 4.** Frequency response of $G(jw)$ at a given frequency ( $w^*$ ).

In (15), $X_W$, $Y_W$, and $Z_W$ are functions of $w^2$ and can be calculated like $X_0$, $Y_0$, and $Z_0$, respectively (see (6)-(8)) by using $W(s)$ in place of $G_0(s)$ in definitions (3)-(8). Note again that for notational simplicity dependency on the frequency variable $w$ is not shown in (15). This convention will also be used in the following, when appropriate.

It is possible to show that

$$b(w) = \frac{N_b(w)}{D_b(w)} \tag{16}$$

where

$$N_b \triangleq (X_0^2 + w^2 Y_0^2)(X_W^2 + w^2 Y_W^2) - w^2 Y_0^2 Z_W^2 \tag{17}$$

$$D_b \triangleq Z_0^2 Z_W^2 \tag{18}$$

Now, in order to be able to find the minimum and the maximum values that can be assumed by $p_1$ and $p_2$ for each real axis crossing of the Nyquist plot, let us define the polynomial

$$R(w^2) \triangleq Z_0^4 (N_b D_b^{'} - D_b N_b^{'})^2 - 4 N_b D_b^3 (X_0 Z_0^{'} - Z_0 X_0^{'})^2 \tag{19}$$

where $D_b^{'}$, $N_b^{'}$, $Z_0^{'}$, and $X_0^{'}$ denotes the first derivative of $D_b$, $N_b$, $Z_0$ and $X_0$ with respect to $w$, respectively. Denote the ordered nonnegative real roots of $R(w^2) = 0$ as $w_1 = 0, w_2 = 0, w_3, w_4, \ldots, w_{2t-1}, w_{2t}$ such that $w_k \le w_{k+1}$. For each pair of roots $w_{2k-1}$ and $w_{2k}$ (for $k = 1, 2, \cdots, t$), define

$$p_{k\min} \triangleq \min(p_1(w_{2k-1}), p_1(w_{2k})) \tag{20}$$

$$p_{k\max} \triangleq \max(p_2(w_{2k-1}), p_2(w_{2k})) \tag{21}$$

If $G_0(s)W(s)$ is biproper we also define

$$p_{(t+1)\min} \triangleq a_n(1 - W_\infty) \tag{22}$$

$$p_{(t+1)\max} \triangleq a_n(1 + W_\infty) \tag{23}$$

where $a_n = \operatorname*{Lim}_{s\to\infty} G_0(s)$ and $W_\infty = \operatorname*{Lim}_{s\to\infty} W(s)$, and take $t = t+1$.

Intervals on the real axis are then defined as

$$X_k \triangleq [p_{k\min} \quad p_{k\max}] \tag{24}$$

with the corresponding gain intervals

$$K_{uk} \triangleq \begin{cases} \left[\dfrac{-1}{p_{k\min}} \quad \dfrac{-1}{p_{k\max}}\right] & \text{if } Sgn(p_{k\min}) = Sgn(p_{k\max}) \\ \left[-\infty \quad \dfrac{-1}{p_{k\max}}\right] \cup \left[\dfrac{-1}{p_{k\min}} \quad \infty\right] & \text{otherwise} \end{cases} \tag{25}$$

The union of these gain intervals is called as U:

$$U = \bigcup_{k\in[1,2,\cdots,t]} K_{uk} \tag{26}$$

Now, it is possible to state the following lemma

**Lemma 3.1:**

Nyquist plot of the uncertain system $G(s)$ defined in (12) crosses the real axis (only) at the intervals $X_k$, and hence the number of rhp poles of the closed-loop system shown in Fig. 3 is uncertain (changes with model uncertainty) if, and only if, $K \in U$ .

**Proof of Lemma 3.1:**

Proof of the lemma depends on the observation that real axis crossings occur if, and only if, real points $p_{1,2}(w) \in \mathbb{R}$ can be found through (13). We note that crossing intervals take their extreme values at frequencies, when

$$\frac{\partial}{\partial w} p_{1,2}(w) = 0 \tag{27}$$

Using (13) in (27) we have,

$$\frac{\partial}{\partial w}(a(w) \mp \sqrt{b(w)}) = \frac{\partial}{\partial w} a(w) \mp \frac{1}{2\sqrt{b(w)}} \frac{\partial}{\partial w} b(w) = 0 \tag{28}$$

It is then possible to show that (27) is equivalent to

$$\left(\frac{\partial}{\partial w} b(w)\right)^2 = 4 b(w) \left(\frac{\partial}{\partial w} a(w)\right)^2 \tag{29}$$

Noting that $a(w) = \dfrac{X_0(w)}{Z_0(w)}$ and $b(w) = \dfrac{N_b(w)}{D_b(w)}$, (29) can be rearranged as

$$\left(\frac{N_b D_b^{'} - D_b N_b^{'}}{D_b^2}\right)^2 = 4 \frac{N_b}{D_b} \left(\frac{X_0 Z_0^{'} - Z_0 X_0^{'}}{Z_0^2}\right) \tag{30}$$

Noting that $D_b > 0$ and $Z_0 > 0$ (30) can be shown to be equal to $R(w^2) = 0$, where $R(w^2)$ is as defined in (19). Therefore for a nonnegative real root of $R(w^2)$, Nyquist plot of the uncertain system crosses the real axis at an extreme point (boundary of Nyquist plot band). Real axis crossing for $w \to \infty$ needs special consideration, when $G_0(s)W(s)$ is biproper. For such systems, $\operatorname*{Lim}_{w\to\infty}(G(jw))$ is a disk centered at $a_n$ with radius $a_n W_\infty$, and hence, a real axis crossing occurs between $p_{(t+1)\min}$ and $p_{(t+1)\min}$, which are defined in (22) and (23), respectively. Lemma 3.1 then immediately follows. ∎

We can now state the main result of the paper.

**Theorem 3.1 (main result):**
For a given uncertain system $G(s)$ as defined in (12), let the whole set of gain intervals that stabilize the nominal system $G_0(s)$ found using Theorem 2.2 be $L$. Then, the set of gains that robustly stabilize the system with respect to multiplicative uncertainty weight $W(s)$ is given by

$$J \triangleq L \setminus U \tag{31}$$

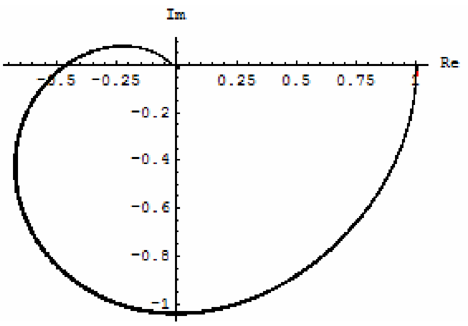
where $U$ is found from (26), and \ sign in (31) denotes set subtraction.

Proof of the theorem is straightforward following the discussions made above, and is not given here.

**Example 3.1:** Consider the system given in Example 2.1, and assume that there is a multiplicative uncertainty in the system model described by the weight transfer function $W(s)=\dfrac{5s+0.5}{s+500}$. Note that this weight function indicates that there is around 0.1% uncertainty in the model for low frequencies, and 500% uncertainty for high frequencies. We also remark that using $H_\infty$ optimal control theory it is possible to describe all robustly stabilizing controllers for this system using a parameterization around a central controller of order 6. In this example, it is demonstrated that the system is robustly stabilizable using constant compensators.

Nyquist plot for the uncertain system is given in different scales in Fig. 5. It is possible to determine the sections of the real axis that is crossed by the Nyquist plot (*without drawing the Nyquist plot*) using Lemma 3.1.

(a)




(b)

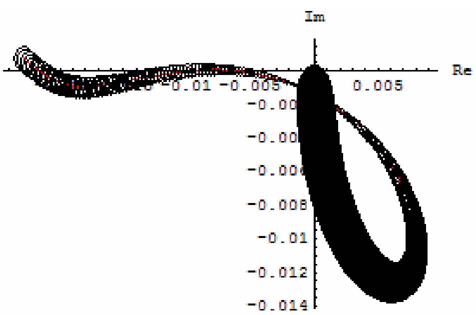



**Fig. 5:** Nyquist Plot for the uncertain system given in Example 3.1. Only the part of the plot for $w>0$ is drawn to reduce complexity.

From (17) and (18),

$$N_b(w^2)=0.5809\,10^{-9}w^{22}+1.392\,10^{-4}w^{20}-1.515\,w^{18}+31.77\,w^{16}-281w^{14}$$
$$1357w^{12}-3870w^{10}+6563w^{8}-6309w^{6}+3048w^{4}-$$
$$567\,w^{2}+6.25\,10^{-5}$$

$$D_b(w^2)=4.0469\,10^{-9}w^{24}+2.023\,10^{-3}w^{22}+252.9w^{20}-2547\,w^{18}+$$
$$13185.6\,w^{16}-36119.8w^{14}+55264w^{12}-25633w^{10}-292w^{8}+$$
$$4278w^{6}-952.3w^{4}-113.5\,w^{2}+62.5$$

Nonnegative real roots of $R(w^2)$ found from (19) are then found as

$w_1=0$, $w_2=0$, $w_3=0.758098$, $w_4=0.76161$, $w_5=1.51906$, $w_6=1.55064$, $w_7=1.83135$, $w_8=2.0280$, $w_9=247.856$, $w_{10}=252.175$

Then, intervals of $K$ for which the number of closed-loop system unstable poles are uncertain can be found as in Table 2 using Lemma 3.1. It is then possible to state that the closed-loop system is robustly stable for $K\in[-0.999\quad 2.1488]\cup[47.5813\quad 82.584]\cup[200.91\quad 1621.9]$ by the help of Theorem 3.1.

**Table 2:** Calculation of uncertainty intervals

| $k$ | $w_{2k-1}$ and $w_{2k}$ | $p_{k\,\min}$ | $p_{k\,\max}$ | $K_{uk}$ |
|---|---|---|---|---|
| 1 | $w_1=0$, $w_2=0$ | 0.999 | 1.001 | [-1.001 -0.999] |
| 2 | $w_3=0.7581$<br>$w_4=0.7616$ | -0.46537 | -0.45309 | [2.1488 2.2071] |
| 3 | $w_5=1.5191$<br>$w_6=1.5506$ | -0.02236 | -0.02101 | [44.718 47.581] |
| 4 | $w_7=1.8313$<br>$w_8=2.0280$ | -0.01211 | -0.00498 | [82.584 200.91] |
| 5 | $w_9=247.85$<br>$w_{10}=252.17$ | -6.16 $10^{-4}$ | -6.21 $10^{-4}$ | [-∞ -1610.7]<br>[1621.9 ∞] |

A closer examination of Nyquist plot crossing between $w=0.757708$ and $w=0.761995$ is shown in Fig. 6. It is possible to see from this figure that Nyquist plot of the uncertain system starts crossing the real axis at $w=0.757708$ (purple circle), reaches a minimum value real axis crossing at $w=w_3=0.7581$ (blue circle), and a maximum value real axis crossing at $w=w_4=0.7616$ (red circle). The real axis crossing finishes at $w=0.761995$ (green circle).

**Example 3.2:** Consider the same system in the previous example with the difference of the weight transfer function, which is given in this case as $W(s)=\dfrac{10s+1}{s+10}$. Note that this indicates a relatively larger uncertainty on the system model. Intervals of $K$ for which the number of closed-loop system poles is uncertain are found from Lemma 3.1 as $U=[-\infty\quad -16.307]\cup[-1.1111\quad -0.9091]\cup[0.9589\quad \infty]$. Hence, the system becomes robustly stable if, and only if, $K\in[-0.9091\quad 0.9589]$.

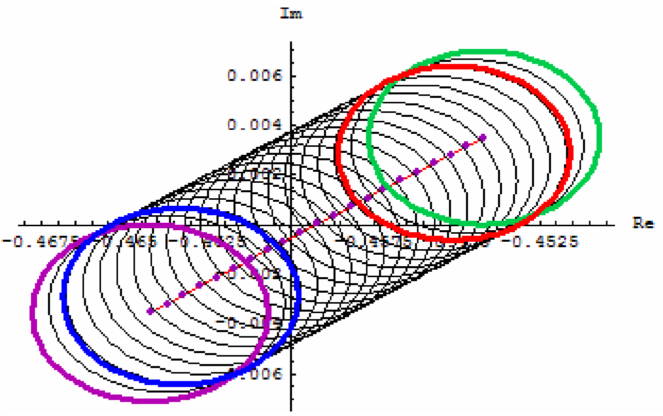



**Fig. 6:** Nyquist plot of $G(s)$ between $w = 0.757708$ and $w = 0.761995$. Note that these frequencies correspond to two roots of $N_b(w^2)$. Red curve (line) is the Nyquist plot of $G_0(s)$, and circles indicate the uncertainty around this Nyquist plot.

**Example 3.3:** Consider an uncertain system described by the nominal system transfer function

$$G_0(s) = \frac{5s^3 + 14.75s^2 + 49.5s + 72}{s^4 - 11s^3 + 48s^2 - 104s + 96}$$

and multiplicative uncertainty weight transfer function $W(s) = \dfrac{1.2s^2 + 2s + 1}{s^2 + 40s + 100}$. Nyquist plot for the uncertain system $G(s)$ is shown in Fig. 7. Note that the nominal system transfer function is unstable. The closed-loop system is stable under nominal working conditions ($W(s) = 0$) for $K \in L = [2.3145 \quad \infty]$ due to Theorem 2.2. Application of Lemma 3.1 reveals that the number of RHP poles of the closed-loop system is uncertain for $K \in U$, where

$$U = [-\infty \quad -51.048] \cup [-4.208 \quad -3.563] \cup [-1.346 \quad -1.320] \cup [1.741 \quad 3.211] \cup [22.75 \quad \infty]$$

Therefore, the system is robustly stable for $K \in J = L \setminus U = [3.211 \quad 22.75]$.

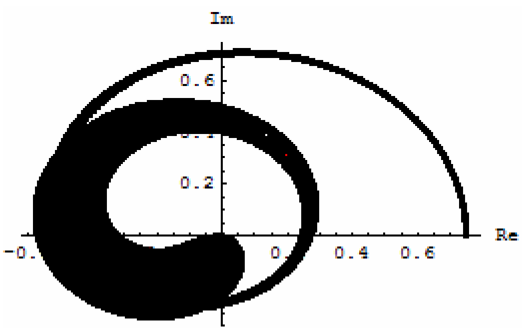



**Fig. 7:** Nyquist plot of $G(s)$ for Example 3.3.

## 6. CONCLUSIONS

A method is proposed to determine all stabilizing proportional controllers for a given system with multiplicative uncertainty. The method is applicable to systems with unstable or nonminimum phase transfer functions. Although it is assumed that the nominal system does not have any poles on the imaginary axis in derivations of formulations, it is actually possible to extend the results to cover such cases rather easily. Since many design problems are reduced to finding suitable gains in lower levels of the design we believe that the proposed method will have immediate application areas.

Future research will focus on extending the results to higher order controllers and/or time delay systems.

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