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Stabilizing Sampled-data Linear Systems with Markovian Packet Losses and Random Sampling ⋆

Li Xie * **Lihua Xie** **

* *Department of Automatic Control, Beijing Institute of Technology, Beijing 100081, China. xieli.lixie@gmail.com*
** *School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, 639798, Singapore. elhxie@ntu.edu.sg*

Abstract: We consider the stability properties and stabilizing problem of sampled-data networked controlled linear systems with Markovian packet losses. A binary Markov chain is used to characterize the packet loss phenomenon of the network. Then with the Markovian packet loss assumption, we obtain a discrete-time augmented Markov jump linear system which describes the continuous-time linear system evolving in deterministic discrete time. Furthermore, we show that the sampled-data system under consideration can also be considered as a randomly sampled system with an i.i.d. random sampling period. A number of necessary and sufficient conditions for the stochastic stability properties are established by using the known results of Markov jump linear systems and randomly sampled systems. Those conditions are based on the relationships of stability properties between the systems evolving in deterministic continuous time, deterministic discrete time, and random discrete time. In addition, the asymptotic stability of the system is also studied by using Lyapunov exponent method. Numerical examples are used to illustrate the main results of the paper.

Keywords: Sampled-data linear systems, networked control, Markovian packet losses, i.i.d. random sampling

1. INTRODUCTION

Networked control systems have been a very hot research area over the past decade. Networked control systems are feedback control systems using networked message to achieve closed-loop stability and desired performance. In networked control systems, there are wireless communication channels or networks between sensors, actuators, and controllers, which make tele-control and sensor-network-based control applicable. While we enjoy a lot of benefits from networked control systems, we are also facing some new interesting and challenging problems raised by networked control systems. For example, due to congestion and fading in communication channels, data losses may occur. Then data packet losses may result in system performance degradation or even instability.

A sampled-data system is a networked control system if discrete-time signals of the sampled-data system are transmitted to the discrete-time controller via a digital communication channel, such as telerobot and distributed sensor-network-based control system. Traditionally, the communication link is assumed to be an ideal one which has infinite bandwidth and data packet dropout does not occur. In this paper, we address the packet loss issue for the control problems of sampled-data networked linear systems.

Recent work has advanced the control research for networked control systems with packet losses; see Part B in Section III and Part B in Section IV of Hespanha et al. [2007]. Most of existing papers are concerned with the control problem of discrete-time networked control systems with packet losses. For example, Sinopoli et al. [2004], Schenato et al. [2007] considered Kalman filtering and LQG control for discrete-time linear systems with randomly intermittent observations, and the packet loss process is assumed to be an i.i.d. Bernoulli binary random sequence. Smith and Seiler [2003] considered estimation with lossy measurements in which the random packet loss is assumed to be governed by a two-state Markov chain. More recently, Huang and Dey [2007] studied the stability of Kalman filtering with Markovian packet losses by introducing stopping times to describe the transmission time or update time of measurements; see also Xie and Xie [2007]. Imer et al. [2006] considered an optimal control problem of LTI systems over unreliable communication links in which the packet loss process is an i.i.d. Bernoulli binary random sequence and may occur both from the sensor to the controller and from the controller to the actuator. Hu and Yan [2007] carried out a robust stability analysis of discrete-time linear systems with static state feedback controller with respect to the distribution of the i.i.d. Bernoulli packet loss process.

For sampled-data networked linear systems, Yu et al. [2004] considered the stabilizing problem with the assump-

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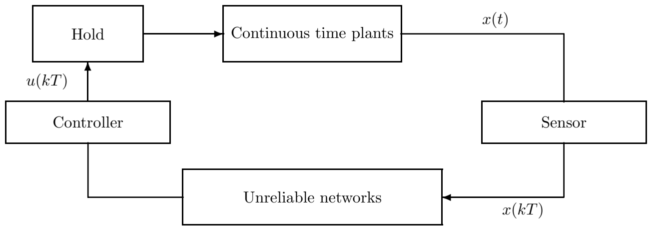

Fig. 2.1. Sampled-data linear systems via an unreliable network

tions that the period during which the measurements are lost is bounded and the successive update instant is known to the controller. Compared with Smith and Seiler [2003] for discrete-time systems with Markovian dropouts, Yu et al. [2004] is referred to using a deterministic dropout rate method by Hespanha et al. [2007]. Montestruque and Antsaklis [2004] considered a stability problem of model-based networked sampled-data systems, where update time is defined as the time between two successful consecutive transmissions and considered as a bounded time-varying variable, an i.i.d., and a finite state Markov chain, respectively.

The aim of the present paper is to consider the stability analysis and stabilizing problem of sampled-data networked linear systems with Markovian packet losses *in stochastic framework*. We will show that the i.i.d. assumption of the update time in Montestruque and Antsaklis [2004] is reasonable if we assume that the packet loss process has the Markov property. Traditionally, the i.i.d. assumption for random sampling period (i.e., update time) was also made for randomly sampled linear or nonlinear systems; see Bergen [1960], Bharucha [1961], Kushner and Tobias [1969], Agniel and Jury [1971]. The assumption of the boundedness of the packet loss period may not be reasonable since the packet loss may occur randomly, which implies that the update time may assume any values.

Notation: We use $\|x\|$ to denote the Euclidean norm of x and $\|A\|$ the induced matrix norm of matrix A. A' denotes the transpose of real matrix A. Let the spectral radius of the matrix $A_{n\times n}$ be $\rho(A) \triangleq \max\{|\lambda_i|, i = 1,\dots,n\}$, where λ_i is the i-th eigenvalue of A and the symbol $|\cdot|$ denotes the magnitude of a (complex) number. Let the triple $(\Omega, \mathcal{F}, \mathbf{P})$ be the underlying probability space. Also we let $\mathbf{E}$ denote the mathematical expectation with respect to the probability measure $\mathbf{P}$ and $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. The symbol $\otimes$ is the Kronecker product and vec the vector valued function of matrices. Let T denote the sample period and I the identity matrix with a compatible dimension.

2. LINEAR SYSTEMS WITH MARKOVIAN PACKET LOSSES

Consider a networked sampled-data control system:

$$\dot{x}(t) = Ax(t) + Bu(kT), \quad t \in [kT, (k+1)T) \tag{2.1}$$

$$u(kT) = \gamma_k Kx(kT) + (1-\gamma_k)u(kT - T); \tag{2.2}$$

see Figure 2.1. Here and in the sequel, we assume that all real matrices have compatible dimensions. In Figure 2.1, the continuous-time plant is described by a continuous-time linear time-invariant differential equation (2.1) driven by a discrete-time static feedback controller. Meanwhile the state $x(t)$ is fully measured by sensors and then sampled, quantized, encoded, and further transmitted by an unreliable network or communication channel. The controller receives the discrete-time state $x(kT)$ after decoded and generates the control signal $u(kT)$ based on the control law $Kx(kT)$. Before the discrete-time control signal $u(kT)$ is fed to the plant, $u(kT)$ is first passed through a zero-order hold and converted into a continuous-time signal. That is, the system under consideration is a sampled-data linear time-invariant control system. Also, we consider that $u(kT)$ is a static state feedback with a constant gain K for simplicity.

We describe data losses in unreliable channels by using a discrete-time binary Markov model. More specifically, in (2.2), if $\gamma_k = 1$, then the state $x(t)$ is successfully transmitted to the controller without any error; otherwise we consider the transmission fails and in this situation the controller (actually, the hold) holds the previous control value as its current output. Here we assume $u(-T) = 0$. In this paper, we do not address other issues such as the effects of quantization error and time delays.

We assume that the packet loss process $\{\gamma_k, k \geq 0\}$ is a time-homogeneous Markov chain under the probability measure $\mathbf{P}$ with the range set $\mathcal{S} = \{0, 1\}$. That is, we assume that $\{\gamma_k, k \geq 0\}$ satisfies the Markov property:

$$\mathbf{P}(\gamma_{k+1} = i|\gamma_0, \gamma_1, \dots, \gamma_k) = \mathbf{P}(\gamma_{k+1} = i|\gamma_k), \quad i \in \mathcal{S} \tag{2.3}$$

and the transition probability is independent of the time step k. We now make an assumption for the transition probability of the Markov chain $\{\gamma_k, k \geq 0\}$.

Assumption 2.1. Let the transition probability matrix of γ_k be

$$(p_{ij})_{i,j\in\mathcal{S}} = (\mathbf{P}(\gamma_{k+1} = j|\gamma_k = i))_{i,j\in\mathcal{S}} := \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}. \tag{2.4}$$

We assume that the failure and recovery rates p and q are such that the transition probability matrix $(p_{ij})_{i,j\in\mathcal{S}}$ is irreducible. Also we assume that the initial state x_0 is independent of the Markov chain $\{\gamma_k, k \geq 0\}$.

The irreducible assumption excludes two trivial cases, i.e., $p = 0$ or/and $q = 0$. By using the Markov property (2.3) and (2.4), we have the finite dimensional distribution of $\{\gamma_k, k \geq 0\}$:

$$\begin{aligned} \mathbf{P}(\gamma_0 = s_0, &\gamma_1 = s_1, \dots, \gamma_k = s_k) \\ &= p_{s_{k-1}k_n} p_{s_{k-2}s_{k-1}} \dots p_{s_0 s_1} p_{s_0} \end{aligned} \tag{2.5}$$

where $p_{s_{k-1}s_k} = \mathbf{P}(\gamma_k = s_k|\gamma_{k-1} = s_{k-1})$ and $s_k \in \{0, 1\}$ for $k \geq 1$; for example, $p_{10} = \mathbf{P}(\gamma_k = 0|\gamma_{k-1} = 1)$. Here $p_{s_0} = \mathbf{P}(\gamma_0 = s_0)$ is the initial distribution of $\{\gamma_k, k \geq 0\}$.

3. A DISCRETE-TIME MARKOVIAN JUMP LINEAR SYSTEM

Let the initial time $t_0 = 0$, the initial state $x_0 = x(0)$, and $k \in \mathbb{N}$. It follows from (2.1) that we have for any $t \in [kT, (k+1)T)$,

$$x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\tau)} \mathrm{d}\tau Bu(kT). \tag{3.1}$$

Furthermore, for $t = (k+1)T$, (3.1) and (2.2) yield that

$$x((k+1)T) = e^{AT}x(kT) + \int_0^T e^{A(T-\tau)}\mathrm{d}\tau Bu(kT) \quad (3.2)$$

$$u((k+1)T) = \gamma_{k+1}Ke^{AT}x(kT) + \big((1-\gamma_{k+1})I + \gamma_{k+1}K\int_0^T e^{A(T-\tau)}\mathrm{d}\tau B\big)u(kT). \quad (3.3)$$

We can write (3.2)–(3.3) in a more compact form:

$$y((k+1)T) := \begin{bmatrix} x((k+1)T) \\ u((k+1)T) \end{bmatrix} := \mathcal{M}_c(\gamma_{k+1}) \begin{bmatrix} x(kT) \\ u(kT) \end{bmatrix} = \mathcal{M}_c(\gamma_{k+1})y(kT). \quad (3.4)$$

where

$$\mathcal{M}_0 := \begin{bmatrix} e^{AT} & \int_0^T e^{A(T-\tau)}\mathrm{d}\tau B \\ 0 & I \end{bmatrix} \quad (3.5)$$

$$\mathcal{M}_1 := \begin{bmatrix} e^{AT} & \int_0^T e^{A(T-\tau)}\mathrm{d}\tau B \\ Ke^{AT} & K\int_0^T e^{A(T-\tau)}\mathrm{d}\tau B \end{bmatrix} \quad (3.6)$$

Then the system (3.4) is a Markov jump linear system with two operation modes. Obviously, the mode $\mathcal{M}_0$ is unstable since 1 is its eigenvalue. The stability of the mode $\mathcal{M}_1$ depends on the controller gain K.

3.1 Stability properties via the Markovian jump linear system

In this section, we study the relationship between the stability properties at deterministic continuous time and deterministic discrete time. Then the stability properties of the system (2.1)–(2.2) can be established via the Markovian jump linear system (3.4).

Definition 3.1. The system (2.1)–(2.2) is (almost surely) asymptotically stable, mean square stable, and stochastic stable (at continuous time) if for any initiate state x_0 and γ_0,

$$\lim_{t\to\infty} \|x(t)\| = 0, \lim_{t\to\infty} \mathbf{E}[\|x(t)\|^2] = 0,$$

and $\mathbf{E}[\int_0^\infty \|x(t)\|^2\mathrm{d}t] < \infty$, respectively.

Definition 3.2. The system (3.2)–(3.3) is (almost surely) asymptotically stable, mean square stable, and stochastic stable if for any initiate state x_0 and γ_0,

$$\lim_{k\to\infty} \|y(kT)\| = 0, \lim_{k\to\infty} \mathbf{E}[\|y(kT)\|^2] = 0$$

and $\mathbf{E}[\sum_{k=1}^\infty \|y(kT)\|^2] < \infty$, respectively.

We first investigate the stochastic stability properties which are referred to the stochastic stability property and the mean square stability property. Here, we call the stability properties of the system (3.2)–(3.3) in discrete time and the stability properties of the system (2.1)–(2.2) in continuous time, for simplicity, the stability properties in discrete time and the stability properties in continuous time.

Theorem 3.1. With the above definitions, we have

(S1) The mean square stability in continuous time is equivalent to the mean square stability in discrete time.
(S2) The stochastic stability in discrete time implies the stochastic stability in continuous time.

For the converse statement of (S2), we refer to Theorem 5.1. We now consider asymptotic stability properties. It is obvious that the asymptotic stability in continuous time implies that the discrete-time state $x(kT)$ is asymptotically stable since the later is an infinite sub-sequence of the continuous-time state $x(t)$.

Corollary 3.1. The asymptotic stability of the system (3.2)–(3.3) implies the asymptotic stability in continuous time.

3.2 Stability properties and stabilization in discrete time

Since the system (3.2)–(3.3) is actually a discrete time Markovian jump linear system with two operation modes, we know that the mean square stability is equivalent to the stochastic stability for the system (3.2)–(3.3); see Theorem 1 in Ji and Chizeck [1990]. By *directly using* the results of Markov jump linear systems, we next give a necessary and sufficient condition for the mean square stability. Then it follows from Theorem 3.1 and Corollary 3.1 that these results can be used to check the stability properties of the system (2.1)–(2.2).

Theorem 3.2. The system (3.2)–(3.3) is mean square stable if and only if there exists two positive definite matrices P_0 and P_1 such that

$$\sum_{j=0}^{1} p_{ij}\mathcal{M}_i' P_j \mathcal{M}_i - P_i < 0 \quad (3.7)$$

holds for all $i \in \mathcal{S}$. Furthermore, there exists a state-feedback controller (2.2) such that the system (3.2)–(3.3) is mean square stable if there exist matrices $P_0 > 0, P_1 > 0, \bar{K}, Y$ satisfying the coupled LMIs:

$$\mathcal{M}_0'\big((1-q)P_0 + qP_1\big)\mathcal{M}_0 - P_0 < 0 \quad (3.8)$$

$$\begin{bmatrix} P_1 & \bar{A}'\bar{K}' \\ \bar{K}\bar{A} & Y + Y' - \big(pP_0 + (1-p)P_1\big) \end{bmatrix} > 0 \quad (3.9)$$

where $\bar{K} = Y\mathrm{diag}(I_{n\times n}, K_{m\times n})$ and $\bar{A}$ is defined by

$$\bar{A} = \begin{bmatrix} e^{AT} & \int_0^T e^{A(T-\tau)}\mathrm{d}\tau B \\ e^{AT} & \int_0^T e^{A(T-\tau)}\mathrm{d}\tau B \end{bmatrix}.$$

Then the controller gain K exists if the following matrix equations are congruent:

$$Y_{12}K = \bar{K}_{12},\ Y_{22}K = \bar{K}_{22} \quad (3.10)$$

where

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}_{(n+m)\times(n+m)},\ \bar{K} = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{bmatrix}_{(n+m)\times 2n}.$$

By (3.7), we next obtain necessary conditions for the mean square stability by following Ji and Chizeck [1991].

Corollary 3.2. The necessary conditions for the mean square stability of the system (3.2)–(3.3) are

$$(1-q)\rho^2(e^{AT}) < 1$$

$$(1-p)\rho^2(e^{AT} + \int_0^T e^{A(T-\tau)}\mathrm{d}\tau BK) < 1. \quad (3.11)$$

It can be easily observed that if the feedback control $u(kT) = Kx(kT)$ stabilizes the systems under no packet losses, i.e. $\rho(e^{AT} + \int_0^T e^{A(T-\tau)}\mathrm{d}\tau BK) < 1$, the second inequality of (3.11) is trivially satisfied. The first inequality (3.11) says that for a less stable plant, the recovery rate q needs to be higher. The next corollary follows from Theorem 2.2 in Fang et al. [1994].

Corollary 3.3. Let $\mu = [\mu_0\ \mu_1]'$ be the stationary distribution of the Markov chain $\{\gamma_k, k \geq 0\}$. Then the system (3.2)–(3.3) is asymptotically stable in discrete time if there exists a matrix norm $\|\cdot\|$ such that

$$\sum_{i=0}^{1} \mu_i \log \|\mathcal{M}_i\| < 0. \tag{3.12}$$

4. AN EQUIVALENT LINEAR SYSTEM WITH RANDOM SAMPLING

Consider the system (2.1)–(2.2) again. We now introduce the following random time for the Markov chain $\{\gamma_k, k \in \mathbb{N}\}$:

$$\begin{aligned} t_1 &= \inf\{k : k \geq 1, \gamma_k = 1\} \\ t_2 &= \inf\{k : k > t_1, \gamma_k = 1\}, \\ &\vdots \\ t_k &= \inf\{k : k > t_{k-1}, \gamma_k = 1\}. \end{aligned} \tag{4.1}$$

Hence these time instants $t_k \geq 1$ are random variables and also stopping times of the Markov chain $\{\gamma_1, \ldots, \gamma_k\}$. In particular, t_k is called the k-th passage time to state 1. The random time $\{t_k, k \in \mathbb{N}\}$ is a strictly monotonously increasing time sequence and $t_k \to \infty$ as $k \to \infty$. Here as before, we let the initial time $t_0 = 0$ and the initial state $x_0 = x(0)$. Note that by the definition, the stoping times $\{t_k, k \in \mathbb{N}\}$ are different from the stoping times $\{\alpha_k, \beta_k, k \geq 1\}$ considered first in Huang and Dey [2007] and then in Xie and Xie [2007].

We now define a continuous time system with randomly sampled states (observations):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(t) &= x(t_kT), \quad t = t_kT, k = 0, 1, \ldots, \end{aligned} \tag{4.2}$$

where $u(t) = Kx(t_kT), \forall t \in [t_kT, t_{k+1}T)$, and the gain K is a constant design parameter. For the systems (4.2) under consideration, the state $x(t)$ is only observed at the random time instants $\{t_kT, k \geq 0\}$. Also the control $u(t)$ is a static state feedback with a constant gain and equals $Kx(t_kT)$ at the random sampling time instant t_kT. Moreover, $u(t)$ is held during the random interval $(t_kT, t_{k+1}T)$ during which no state signal $x(t)$ is available. Then the closed-loop system becomes

$$\dot{x}(t) = Ax(t) + BKx(t_kT), \quad \forall t \in [t_kT, t_{k+1}T). \tag{4.3}$$

The system (4.2) is shown in Figure 4.1. It is obvious that

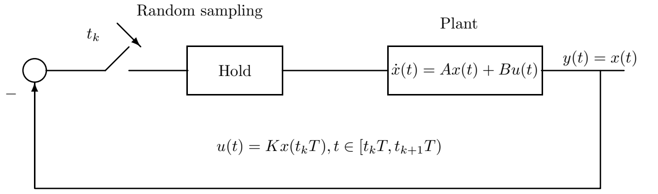

Fig. 4.1. An equivalent system: randomly sampled system

the system (2.1)–(2.2) is equivalent to the system (4.2). It follows from (4.3) that we have for $t \in [t_kT, t_{k+1}T)$,

$$\begin{aligned} x(t) &= e^{A(t-t_kT)}x(t_kT) + \int_{t_kT}^{t} e^{A(t-\tau)}BKx(t_kT)\mathrm{d}\tau \\ &= \left[e^{A(t-t_kT)} + \int_0^{t-t_kT} e^{A(t-t_kT-\tau)}\mathrm{d}\tau BK\right]x(t_kT) \end{aligned} \tag{4.4}$$

and for $t = t_{k+1}T$,

$$\begin{aligned} x(t_{k+1}T) = &\Big[e^{A(t_{k+1}-t_k)T} \\ &+ \int_0^{(t_{k+1}-t_k)T} e^{A((t_{k+1}-t_k)T-\tau)}\mathrm{d}\tau BK\Big]x(t_kT) \\ := &\mathcal{M}_s(t_{k+1}-t_k)x(t_kT). \end{aligned} \tag{4.5}$$

The system (4.5) is a random recursion system evolving in the random sampling time instants $\{t_k, k \in \mathbb{N}\}$. We next study the stability properties of the system (4.5). By establishing the relationship between the stability properties in discrete time and random time, in Section 5, we will show that the stability conditions of the system (2.1)–(2.2) can be obtained via the random recursion system (4.5).

4.1 Statistical properties of the random sampling period.

By using the Markov property of the packet loss process $\{\gamma_k, k \in \mathbb{N}\}$, we now characterize the statistical properties of the random sampling period. The following lemma follows from the irreducibility of $\{\gamma_k, k \in \mathbb{N}\}$ since an irreducible Markov chain having a finite state space is positive recurrent. For its proof, we refer to Lemma 1 of Huang and Dey [2007].

Lemma 4.1. Under Assumption 2.1, the stopping times $\{t_k, k \geq 1\}$ are finite with probability one (almost surely).

Hence we can define the following sojourn times:

$$t_k^* := t_k - t_{k-1} \tag{4.6}$$

with $k \in \mathbb{N} - \{0\}$. Then the increment process t_k^* takes values in the set $\mathbb{N} - \{0\}$. In Montestruque and Antsaklis [2004], t_k^* is defined as the time between the $(k-1)$-th and k-th successful transmissions and called update time. Without loss of generality, we next let $\gamma_0 = 1$. That is, the initial state x_0 is known by both of the continuous-time plant and the discrete-time networked controller. If γ_0 is random, the following distribution has a complicated analytical expression.

Lemma 4.2. Under Assumption 2.1, the sojourn times $\{t_k^*, k \geq 1\}$ are i.i.d. Furthermore the distribution of t_k^* is given by

$$\mathbf{P}(t_k^* = i) = \begin{cases} 1-p, & \text{if } i = 1; \\ pq(1-q)^{i-2}, & \text{otherwise.} \end{cases} \tag{4.7}$$

4.2 Mean square stability in random sampling time

In this section, we consider the mean square stability properties of the random recursive system (4.5). The system (4.5) is also called the system (4.2) in the random sampling time $\{t_k, k \geq 0\}$.

Definition 4.1. The system (4.2) is mean square stable in random sampling time, that is, the random recursive system (4.5) is mean square stable, if for any x_0 and γ_0,

$$\lim_{t\to\infty} \mathbf{E}[\|x(t_kT)\|^2] = 0 \tag{4.8}$$

where t_k is defined by (4.1).

The following necessary and sufficient condition for the mean square stability of the system (4.5) directly follows from Theorem 2 in Kushner and Tobias [1969].

Theorem 4.1. The system (4.5) is mean square stable if and only if there exists a matrix $P > 0$ such that

$$\mathbf{E}[\mathcal{M}_s'(t_k^*)P\mathcal{M}_s(t_k^*)] - P < 0. \tag{4.9}$$

In the next theorem, we suppose that A is non-singular. Then an explicit expression for the inequality (4.9) can be obtained. For arbitrary state matrices A, we can use the Jordan form of A to obtain an expression for the inequality (4.9).

Theorem 4.2. The system (4.5) is mean square stable if and only if the following conditions hold:

(i) $(1-q)\rho^2(e^{AT}) < 1$,
(ii) there exists a matrix $S > 0$ satisfying the inequalities:

$$1. \quad S - (1-q)e^{A'T}Se^{AT} > 0; \tag{4.10}$$

$$\begin{aligned} 2. \quad & (1-p)C'PC + qpD'SD - qpD'F'PE \\ & - qpE'PFD + pE'PE - P < 0 \end{aligned} \tag{4.11}$$

where $P = S - (1-q)e^{A'T}Se^{AT}$ and

$$E = A^{-1}BK, \quad C = e^{AT}(I + A^{-1}BK) - A^{-1}BK,$$
$$F = [I - (1-q)e^{AT}]^{-1}, \quad D = e^{2AT}(I + A^{-1}BK).$$

5. STABILITY PROPERTIES VIA THE RANDOMLY SAMPLED SYSTEM

Definition 5.1. The system (2.1)–(2.2) is stochastically stable in random sampling time if for any initiate state x_0 and γ_0,

$$\mathbf{E}[\sum_{k=0}^{\infty} \|x(t_kT)\|^2] < \infty \tag{5.1}$$

where t_k is defined by (4.1).

The next theorem establishes the equivalence of stochastic stability properties for random sampling time. When $\{t_k^*, k \geq 1\}$ is a finite state Markov chain, the similar results have been obtained for Markov jump linear systems evolving in deterministic discrete time in Ji and Chizeck [1990], Feng et al. [1992]. Here we consider $\{t_k^*, k \geq 1\}$ is an i.i.d. with the countable range set $\{1, \ldots, n, \ldots\}$.

Theorem 5.1. The mean square stability of the system (4.5) with the random sampling time $\{t_kT, k \geq 1\}$ is equivalent to the stochastic stability, i.e., the equality (4.8) and the inequality (5.1) are equivalent.

Remark 5.1. A necessary and sufficient condition for the mean square stability of the system (4.5) has been obtained by extending the results of Bergen [1960], as pointed out in Agniel and Jury [1971], for randomly time-varying systems with zero input in Bharucha [1961]. That is, the system (4.5) is mean stable if and only if

$$\lim_{k\to\infty} (\mathbf{E}[(M_s'(t_k^*) \otimes M_s'(t_k^*))])^k = 0.$$

This condition is equivalent to

$$\rho(\mathbf{E}[(M_s(t_k^*) \otimes M_s(t_k^*))]) < 1. \tag{5.2}$$

The condition (5.2) is also equivalent to (4.9); see Theorem 2 in Kushner and Tobias [1969] and Appendix B in Agniel and Jury [1971].

We now establish the equivalence of the mean square stabilities in random sampling time and discrete time.

Theorem 5.2. The mean square stability of the system (4.5) in random time is equivalent to the mean square stability of the system (3.4) in discrete time.

The next proposition is concerned with the converse statement of (S2) in Theorem 3.1 which is true providing that an extra condition holds.

Proposition 5.1. Let $\mathcal{A}(t) = e^{At} + \int_0^t e^{A(t-\tau)}\mathrm{d}\tau BK$. If the system (2.1)–(2.2) in continuous time is stochastic stable and the matrix $\mathbf{E}[\int_0^{t_1^*T} \mathcal{A}(t) \otimes \mathcal{A}(t)\mathrm{d}t]$ has not any zero column, then the system (4.5) in random sampling time is mean square stable.

By using the statement (S1) of Theorem 3.1, Theorem 5.2, and (5.2) in Remark 5.1, we give a sufficient and necessary condition to guarantee the mean square stability of the system (2.1)–(2.2) as well as stochastic and asymptotic stability.

Theorem 5.3. The necessary and sufficient condition for the mean square stability of the system (2.1)–(2.2) is

$$\rho(\mathbf{E}[(M_s(t_k^*) \otimes M_s(t_k^*))]) < 1. \tag{5.3}$$

Meanwhile, the mean square stability of the system (2.1)–(2.2) also implies the stochastic stability of the system (2.1)–(2.2). Furthermore, the system (2.1)–(2.2) is asymptotically stable as well.

We are now in the position to establish a sufficient condition for the asymptotic stability in continuous time. We first make an assumption on the system parameters.

Assumption 5.1. The recovery rate q, the sample period T, and the state matrix A satisfy

$$(1-q)\rho(e^{AT}) < 1. \tag{5.4}$$

Assumption 5.1 ensures a related function of the i.i.d. process $\{t_k^*, k \geq 1\}$ has a finite expectation such that we can use the strong law of large numbers to study the asymptotic stability.

Theorem 5.4. Under Assumption 5.1, the system (2.1) and (2.2) is asymptotically stable if there exists a matrix norm $\|\cdot\|$ such that

$$\mathbf{E}[\log \|\mathcal{M}_s(t_1^*)\|] < 0 \tag{5.5}$$

where t_1^* and $\mathcal{M}_s(t_1^*)$ are defined by (4.6) and (4.5).

Corollary 5.1. Consider scalar systems (2.1)–(2.2). If $(1-q)\rho^2(AT) < 1$, then

$$\mathbf{E}[\log \|\mathcal{M}_s(t_1^*)\|] < 0 \tag{5.6}$$

is also a necessary condition for the asymptotic stability of the system (2.1) and (2.2).

6. ILLUSTRATIVE EXAMPLES

Example 6.1. Consider an unstable scalar plant with $A = 0.4, B = 1$. Let the failure rate $p = 0.3$ and the recovery rate $q = 0.6$. In Figure 6.1, we give the admissible range of the gain K for the mean square stability and the asymptotic stability, respectively, by solving the inequalities (5.3) or (4.9) and (5.6). That is, if $K \in (-1.641, -0.4)$, the system is asymptotically stable, and if $K \in (-0.5257, -0.4)$, the system is mean square stable. Here $T = 1, x_0 = 0.8, 1 \leq k \leq 50$. In order to guarantee those stabilities, K must be less than -0.4.

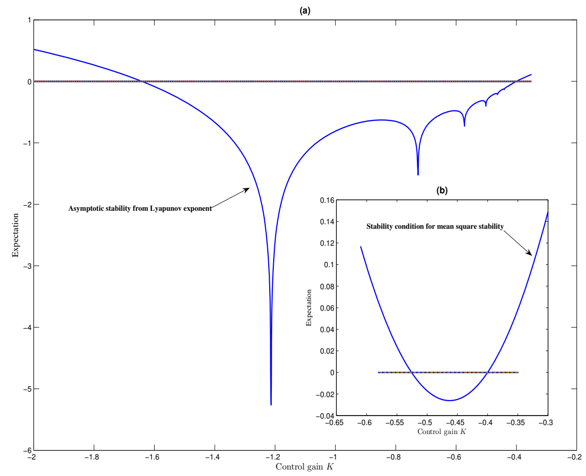

Fig. 6.1. Control gain vs. expectation values

7. CONCLUSION

In this paper, we have considered the stability properties and stabilizing problem of sampled-data networked controlled linear systems with Markovian packet losses. With the Markovian packet loss assumption, we first obtain a discrete-time augmented Markov jump linear system which describes the continuous-time linear system evolving in deterministic discrete time. Furthermore, we show that the sampled-data system under consideration can also be considered as a randomly sampled system with an i.i.d. random sampling period.

It is noted that for the sampled-data systems under consideration, there are three associated time instant sequences, i.e., deterministic continuous time sequence, deterministic discrete time sequence, and random sampling time sequence. We have established the relationships of the stability properties for the systems evolving in continuous time, discrete time, and random time. The involved systems in discrete time and random time are a Markov jump linear system and a randomly sampled linear system. By using the known results of Markov jump linear systems and randomly sampled systems, we obtain a number of necessary and sufficient conditions for the stochastic stability properties of the systems in discrete time and random time. Then based on the relationships of stability properties between the systems evolving in deterministic continuous time, deterministic discrete time, and random time, which are established here, we have shown that those conditions are valid for the sampled-data system. In addition, the asymptotic stability of the system is also studied by using Lyapunov exponent method. Numerical examples are used to illustrate the main results of the paper. In order to design stabilizing controllers, we found that numerical tools including LMIs need to be developed in future for solving related inequalities.

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