Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

The Transmission Zero at s Radius and the Minimum Phase Radius of LTI Systems ⋆

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Abstract: In this paper, the transmission zero at s radius and the minimum phase radius of a linear time-invariant (LTI) system are introduced. The former radius gives a measure of how "near" a LTI system is from having a transmission zero at a specified point $s \in \mathbb{C}$ in the complex plane, and the latter radius measures how "near" a minimum phase system is to a nonminimum phase system. Formulas for computing both radii are presented, along with the procedures for constructing the minimum norm perturbations that achieve the respective radius. Some properties of the two radii and numerical examples are also given.

1. INTRODUCTION

Consider the following LTI multivariable system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^r$ are respectively the state, input, and output vectors, and A, B, C, and D are constant matrices with the appropriate dimensions for $n \geq 1$, $m \geq 1$, $r \geq 1$, and $\max(r, m) \leq n$. The transmission zeros of such a system play an important role in various areas of control theory. In the robust servomechanism problem (RSP), for instance, one of the necessary and sufficient conditions for the existence of a solution is related to the absence of certain transmission zeros in the complex plane (Davison (1976)). For example, there exists a solution to the RSP for tracking constant references if and only if there are no transmission zeros at the origin. Also, minimum phase systems are systems that have no transmission zeros in the closed right-half complex plane, and it is well known that minimum phase systems have certain advantages over nonminimum phase systems. For example, minimum phase systems can achieve perfect regulation (Scherzinger and Davison (1985)) and perfect tracking/disturbance rejection (Davison and Scherzinger (1987)). However, when a system (1) is subject to parametric perturbations (i.e. $A \rightarrow A+\Delta_A$, $B \rightarrow B+\Delta_B$, $C \rightarrow C+\Delta_C$, and $D \rightarrow D+\Delta_D$), a minimum phase system may be very "close" to a system that is nonminimum phase. Similarly, a system with no transmission zeros at, say, the origin, may be very "close" to having one. Therefore in both cases, a continuous measure is more informative and sometimes more desirable than a binary "yes/no" metric.

In the current literature, there has been recent work carried out related to measuring the robustness of a system's *pole* properties (i.e. controllability/observability and stability) with respect to parametric perturbations. In particular, various continuous measures have been developed that measure how close a controllable (observable) system is to being uncontrollable (unobservable) (Eising (1984); Hu and Davison (2004)), how close a stable system is to an unstable one (Qiu et al. (1995)), and how close a decentralized system with no decentralized fixed modes (DFM) is to having a DFM (Vaz and Davison (1988); Lam and Davison (2007)). However, there has been no work done on measuring the robustness of a system's transmission zero properties such as: i) how close a system is to having a particular transmission zero at $s \in \mathbb{C}$ in the complex plane; and ii) how close a minimum phase is to a nonminimum phase system. Hence, the main result of this paper is to define and introduce continuous measures for these two problems, which will be called the *transmission zero at s radius* and *minimum phase radius* respectively.

This paper is organized as follows. Section 2 provides a brief review of transmission zeros and minimum phase systems, and also defines the transmission zero at s radius and the minimum phase radius. Section 3 provides readily computable formulas for computing these two radii, and also discusses some properties of these two radii. Section 4 then provides a procedure for computing the minimum norm system perturbations that achieve the two radii, followed by numerical examples which are presented in Section 5.

2. PRELIMINARIES

The notation used in this paper is standard. The field of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively, and $\mathbb{C}_+$ denotes the closed right half complex plane. The i-th singular value of a matrix $M \in \mathbb{C}^{p\times m}$ is denoted by $\sigma_i(M)$ where $\sigma_1(M) \geq \sigma_2(M) \geq \cdots$. $\|A\|$ denotes the spectral norm of a matrix M and is equal to $\sigma_1(M)$. Also, $\overline{M}$, M^T, M^*, and M^+ denote respectively the complex conjugate, transpose, complex conjugate transpose, and Moore-Penrose pseudoinverse of M. The real and imaginary components of the matrix M are given by $\Re M$ and $\Im M$ respectively. The set of eigenvalues of a square matrix $A \in \mathbb{C}^{n\times n}$ is given by $\lambda(A)$. Finally, system (1) is sometimes denoted by (C, A, B, D).

⋆ This work has been supported by NSERC under grant No. A4396.

The following definition is made in Davison and Wang (1974).

Definition 1. (Transmission zeros). Given a LTI system (1), the *transmission zeros* (TZ) are defined to be the set of complex numbers $s \in \mathbb{C}$ that satisfy the following inequality:

$$\text{rank}\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} < n + \min(r, m) \tag{2}$$

Furthermore, system (1) is called *degenerate* if the set of transmission zeros include the whole complex plane. For the reminder of the paper, it will in general be assumed that (C, A, B, D) is *non-degenerate*.

Definition 2. A LTI system (1) is said to be *nonminimum phase* (nonMP) if at least one of its transmission zeros is contained in $\mathbb{C}_+$; otherwise the system is said to be *minimum phase* (MP).

The transmission zero at s radius measures how close a system is to having a transmission zero at s, and is defined as follows.

Definition 3. (Transmission Zero at s Radius). Given the LTI system (1), and given $s \in \mathbb{C}$, the *transmission zero (TZ) at s radius*, r^{TZ}, is defined to be:

$$r_{\mathbb{F}}^{TZ}(C, A, B, D, s) = \inf\left\{ \left\| \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} \right\| \;\middle|\; \begin{array}{l} \Delta_A \in \mathbb{F}^{n\times n}, \Delta_B \in \mathbb{F}^{n\times m}, \Delta_C \in \mathbb{F}^{r\times n}, \Delta_D \in \mathbb{F}^{r\times m}, \\ (C + \Delta_C, A + \Delta_A, B + \Delta_B, D + \Delta_D) \text{ has a TZ at } s \end{array} \right\} \tag{3}$$

where $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ [1].

Similarly, the minimum phase radius measures how near a minimum phase system is to a nonminimum phase system, and is defined as follows.

Definition 4. (Minimum Phase Radius). Given a LTI system (1) that is minimum phase, the *minimum phase (MP) radius*, r^{MP}, is defined to be:

$$r_{\mathbb{F}}^{MP}(C, A, B, D) = \inf\left\{ \left\| \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} \right\| \;\middle|\; \begin{array}{l} \Delta_A \in \mathbb{F}^{n\times n}, \Delta_B \in \mathbb{F}^{n\times m}, \Delta_C \in \mathbb{F}^{r\times n}, \Delta_D \in \mathbb{F}^{r\times m}, \\ (C + \Delta_C, A + \Delta_A, B + \Delta_B, D + \Delta_D) \text{ is nonMP} \end{array} \right\} \tag{4}$$

where $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$.

3. MAIN RESULTS

The main result of this paper is given by the following two theorems.

Theorem 5. Given a LTI system (1) and given $s \in \mathbb{C}$, then

$$r_{\mathbb{C}}^{TZ}(C, A, B, D, s) = \sigma_{n+\min(r,m)}\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \right) \tag{5}$$

and

$$r_{\mathbb{R}}^{TZ}(C, A, B, D, s) = \sup_{\gamma \in (0,1]} \sigma_{2(n+\min(r,m))-1}\left(\begin{bmatrix} \Re W & -\gamma \Im W \\ \gamma^{-1}\Im W & \Re W \end{bmatrix} \right) \tag{6}$$

where $W = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$.

Proof. The proof is provided in Appendix A.

Theorem 6.

$$r_{\mathbb{C}}^{MP}(C, A, B, D) = \min_{s \in \mathbb{C}+} r_{\mathbb{C}}^{TZ}(C, A, B, D, s) \tag{7}$$

$$r_{\mathbb{R}}^{MP}(C, A, B, D) = \min_{s \in \mathbb{C}+} r_{\mathbb{R}}^{TZ}(C, A, B, D, s) \tag{8}$$

Proof. Theorem 6 follows directly from the minimization of Theorem 5 over the closed right-half of the complex plane.

Lemma 7. (Properties of $r_{\mathbb{C}}^{TZ}$ and $r_{\mathbb{R}}^{TZ}$).

(1) $r_{\mathbb{F}}^{TZ}(C, A, B, D, s) = r_{\mathbb{F}}^{TZ}(C, A, B, D, \bar{s})$
(2) $r_{\mathbb{F}}^{TZ}(C, A, B, D, s) = r_{\mathbb{F}}^{TZ}\big(CT^{-1}, TAT^{-1}, TB, D, s\big)$
(3) $r_{\mathbb{C}}^{TZ}(C, A, B, D, s) \leq r_{\mathbb{R}}^{TZ}(C, A, B, D, s)$

where $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, T is a real orthogonal matrix, and equality in Property (3) is achieved for $s \in \mathbb{R}$.

Proof. The proof is provided in Appendix A.

Lemma 8. Given a LTI system (1) with $D = 0$, the system is arbitrarily close to a large real nonminimum phase zero; i.e.

$$r_{\mathbb{F}}^{TZ}(C, A, B, 0, s) \to 0 \text{ as } s \to \infty \tag{9}$$

where $s \in \mathbb{R}$ and $F \in \{\mathbb{C}, \mathbb{R}\}$. Hence given a LTI minimum phase system (1) with $D = 0$, the system is arbitrarily close to a nonminimum phase system; i.e.

$$r_{\mathbb{F}}^{MP}(C, A, B, 0) = 0 \tag{10}$$

where $F \in \{\mathbb{C}, \mathbb{R}\}$.

Proof. The proof is provided in Appendix B.

4. CONSTRUCTING THE MINIMUM NORM PERTURBATIONS

Given the system (1) and $s \in \mathbb{C}$, this section presents a procedure that computes the minimum norm perturbation matrices $\Delta_A \in \mathbb{F}^{n\times n}$, $\Delta_B \in \mathbb{F}^{n\times m}$, $\Delta_C \in \mathbb{F}^{r\times n}$, and $\Delta_D \in \mathbb{F}^{r\times m}$ that achieve $r_{\mathbb{F}}^{TZ}$; i.e.

$$\text{rank}\left(\begin{bmatrix} A + \Delta_A - sI & B + \Delta_B \\ C + \Delta_C & D + \Delta_D \end{bmatrix} \right) < n + \min(r, m) \tag{11}$$

where $\left\| \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} \right\| = r_{\mathbb{F}}^{TZ}$ and $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. To compute the perturbations that achieve $r_{\mathbb{C}}^{TZ}$ (i.e. $\mathbb{F} = \mathbb{C}$), the singular value decomposition of $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ in used. For computing the perturbations that achieve $r_{\mathbb{R}}^{TZ}$, a result by Karow (2003) is used. In particular, Theorem 10 in Section 4.2.

4.1 Perturbations that achieve $r_{\mathbb{C}}^{TZ}$

Consider the following general result on the singular value decomposition (e.g. see Karow (2003)).

Theorem 9. Given $M \in \mathbb{C}^{q\times l}$ and $k \in \mathbb{N}$, let the singular value decomposition of M be given as follows:

$$M = \sum_{i=1}^{l} \sigma_i(M)\, u_i v_i^*$$

where $U = [u_1, \cdots, u_q]$ and $V = [v_1, \cdots, v_l]$ are both unitary matrices. Let $\mathcal{X} = \text{span}\{v_k, \ldots, v_l\}$ and let $X =$

[1] In terms of nomenclature, $r_{\mathbb{C}}^{TZ}$ (i.e. $\mathbb{F} = \mathbb{C}$) is called the *complex* transmission zero at s radius and $r_{\mathbb{R}}^{TZ}$ is called the *real* TZ at s radius.

$[v_k, \ldots, v_l]$ be a matrix whose columns form a basis of $\mathcal{X}$, then

$$x^*((\sigma_k(M))^2 I_l - M^* M)x \geq 0 \quad \text{for all } x \in \mathcal{X}$$

Let

$$\Delta := -\sum_{i=k}^{l} \sigma_i(M)\, u_i v_i^* = -MXX^* \tag{12}$$

then

$$\text{rank}(M + \Delta) < k \quad \text{and} \quad \|\Delta\| = \sigma_k(M)$$

Therefore by Theorem 9, the perturbations that achieve $r_{\mathbb{C}}^{TZ}$ can be obtained from the singular value decomposition of $T := \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$; i.e. let

$$T = \sum_{i=1}^{n+\min(r,m)} \sigma_i(T)\, u_i v_i^* \tag{13}$$

be the singular value decomposition of T, where $U = [u_1, \cdots, u_{n+r}]$ and $V = [v_1, \cdots, v_{n+m}]$ are both unitary matrices, and let $X_C = \left[v_{n+\min(r,m)}, \cdots, v_{n+m}\right]$. Then the system perturbations are given by:

$$\begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} = -TX_C X_C^* \tag{14}$$

4.2 Perturbations that achieves $r_{\mathbb{R}}^{TZ}$

Consider the following result on real perturbation values, which is analogous to Theorem 9 for singular values.

Theorem 10. (Karow (2003)). Given $M \in \mathbb{C}^{q \times l}$ and $k \in \mathbb{N}$. If $\tau_k(M) = \infty$, then there exists no $\Delta \in \mathbb{R}^{q \times l}$ such that $\text{rank}(M + \Delta) < k$. Assume now $\tau_k(M) < \infty$, and let $\mathcal{X}$ be any $(l - k + 1)$-dimensional subspace of $\mathbb{C}^l$ satisfying the hermitian-symmetric inequality

$$x^* \left(\tau_k(M)^2 I_l - M^* M\right) x \geq \left| x^T \left(\tau_k(M)^2 I_l - M^T M\right) x \right|$$

for all $x \in \mathcal{X}$, and let $X \in \mathbb{C}^{l \times (l-k+1)}$ be any matrix whose columns form a basis of $\mathcal{X}$. Set

$$\Delta := -\left[\,\Re(MX) \;\; \Im(MX)\,\right]\left[\,\Re(X) \;\; \Im(X)\,\right]^+ \in \mathbb{R}^{q \times l} \tag{15}$$

Then

$$\text{rank}(M + \Delta) < k \quad \text{with} \quad \|\Delta\| = \tau_k(M)$$

Hence from Theorem 10, we immediately see that the perturbations can be constructed by:

$$\begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} = -\left[\,\Re(TX_s) \;\; \Im(TX_s)\,\right]\left[\,\Re(X_s) \;\; \Im(X_s)\,\right]^+ \tag{16}$$

where $T := \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$, and where X_s is a complex matrix whose columns form the basis for a subspace that satisfies

$$x^* \hat{H} x \geq \left| x^T \hat{S} x \right| \tag{17}$$

where $\hat{H} := (r_{\mathbb{R}}^{TZ})^2 I - T^* T$ and $\hat{S} := (r_{\mathbb{R}}^{TZ})^2 I - T^T T$. Such a basis (i.e. X_s) can be obtained by performing a simultaneous block diagonalization of $\hat{H}$ and $\hat{S}$. For more information, please see Karow (2003) and Lam and Davison (2008).

5. NUMERICAL EXAMPLES

5.1 Example 1

Consider the following minimum phase system:

$$\dot{x} = \begin{bmatrix} 0.74 & -0.69 & -2.08 \\ -0.12 & 1.62 & 0.63 \\ -0.38 & -0.21 & 0.14 \end{bmatrix} x + \begin{bmatrix} -1.23 & -0.26 \\ 1.02 & 2.51 \\ -0.66 & 1.13 \end{bmatrix} u$$
$$y = \left[\, 1.06 \;\; 0.71 \;\; 0.61 \,\right] x + \left[\, 1.33 \;\; -2.89 \,\right] u \tag{18}$$

By (7), the complex minimum phase radius is found to be $r_{\mathbb{C}}^{MP} = 8.902_{10^{-2}}$, which is achieved at $s = 0.8158_{10^{-1}} \pm j2.517_{10^{-1}}$. Using the procedure presented in Section 4, the corresponding system perturbation is computed to be:

$$\Delta := \left[\begin{array}{c|c} \Delta_A & \Delta_B \\ \hline \Delta_C & \Delta_D \end{array}\right] = \left[\begin{array}{ccc|cc} -1.07_{10^{-2}} + j8.59_{10^{-3}} & 1.61_{10^{-2}} - j1.16_{10^{-2}} & -7.98_{10^{-4}} + j9.81_{10^{-4}} & -7.43_{10^{-3}} + j1.80_{10^{-3}} & -3.74_{10^{-3}} + j1.51_{10^{-3}} \\ -4.58_{10^{-3}} + j5.43_{10^{-3}} & 6.96_{10^{-3}} - j7.49_{10^{-3}} & -3.06_{10^{-4}} + j5.77_{10^{-4}} & -3.58_{10^{-3}} + j1.66_{10^{-3}} & -1.74_{10^{-3}} + j1.14_{10^{-3}} \\ 4.28_{10^{-2}} - j2.90_{10^{-3}} & -6.17_{10^{-2}} + j1.21_{10^{-3}} & 3.78_{10^{-3}} - j1.10_{10^{-3}} & 2.22_{10^{-2}} + j8.57_{10^{-3}} & 1.23_{10^{-2}} + j2.76_{10^{-3}} \\ \hline 1.39_{10^{-2}} + j2.74_{10^{-3}} & -1.99_{10^{-2}} - j4.91_{10^{-3}} & 1.30_{10^{-3}} - j2.57_{10^{-5}} & 6.37_{10^{-3}} + j4.65_{10^{-3}} & 3.68_{10^{-3}} + j1.93_{10^{-3}} \end{array}\right] \tag{19}$$

It can easily be verified that the norm of the perturbation (19) is equal to $r_{\mathbb{C}}^{MP} = 8.902_{10^{-2}}$ and that the perturbed system has a nonminimum phase zero at $s = 0.8158_{10^{-1}} \pm j2.517_{10^{-1}}$.

Likewise by (8), the real minimum phase radius is found to be $r_{\mathbb{R}}^{MP} = 8.983_{10^{-2}}$, which is achieved at $s = 8.209_{10^{-1}} \pm j2.329_{10^{-1}}$. The corresponding perturbation that achieves $r_{\mathbb{R}}^{MP} = 8.983_{10^{-2}}$ is:

$$\Delta = \left[\begin{array}{ccc|cc} 3.72_{10^{-3}} & -5.49_{10^{-3}} & 3.28_{10^{-4}} & -6.38_{10^{-2}} & -2.86_{10^{-2}} \\ 5.20_{10^{-3}} & -7.67_{10^{-3}} & 4.58_{10^{-4}} & -4.20_{10^{-2}} & -1.87_{10^{-2}} \\ 4.49_{10^{-2}} & -6.63_{10^{-2}} & 3.95_{10^{-3}} & 1.98_{10^{-2}} & 1.10_{10^{-2}} \\ \hline 2.18_{10^{-2}} & -3.21_{10^{-2}} & 1.92_{10^{-3}} & -2.20_{10^{-2}} & -8.89_{10^{-3}} \end{array}\right] \tag{20}$$

Again, it can be confirmed that the norm of (20) is equal to $r_{\mathbb{R}}^{MP} = 8.983_{10^{-2}}$ and that the perturbed system has a nonminimum phase zero at $s = 8.209_{10^{-1}} \pm j2.329_{10^{-1}}$.

Now suppose $s = 0$. Then by (5) and (6), the transmission zero at s radius is obtained to be $r_{\mathbb{C}}^{MP} = r_{\mathbb{R}}^{MP} = 0.2882$. The system perturbation that achieves this radius is given by:

$$\Delta = \left[\begin{array}{ccc|cc} -1.86_{10^{-2}} & -9.49_{10^{-3}} & -1.69_{10^{-2}} & 2.38_{10^{-2}} & -3.96_{10^{-4}} \\ -7.30_{10^{-3}} & -3.72_{10^{-3}} & -6.61_{10^{-3}} & 9.32_{10^{-3}} & -1.55_{10^{-4}} \\ -1.36_{10^{-1}} & -6.95_{10^{-2}} & -1.24_{10^{-1}} & 1.74_{10^{-1}} & -2.90_{10^{-3}} \\ \hline -5.79_{10^{-2}} & -2.95_{10^{-2}} & -5.24_{10^{-2}} & 7.39_{10^{-2}} & -1.23_{10^{-3}} \end{array}\right] \tag{21}$$

Remark 11. Given $s = 0$, if the transmission zero at s radius is very small (e.g. $< 10^{-10}$), then the system is very close to having a transmission zero at the origin. In the given example, the radius is relatively not small, and in this case, there exists a solution to the RSP for the system

(18) for all perturbations, if and only if $\left\| \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} \right\| <$ 0.2882.

5.2 Example 2

In this example we shall compute the transmission zero at $s = 0$ radius (denoted by r^{TZ}) for some representative linearized LTI models of industrial systems (e.g. see Lam and Davison (2007)), to show that there can be a large difference in the robustness of the existence of a solution to the robust servomechanism problem for constant tracking/disturbances. For comparison purposes, let r^{TZ} be scaled as follows:

$$r^{TZ}_{scaled} = \frac{r^{TZ}}{\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|} \times 100\%$$

It can be seen from Table 5.2 that there is a large difference in the scaled TZ radius values between the various models.

Table 1. Transmission zero at $s = 0$ radius for various industrial examples

	Plant		r^{TZ}	r^{TZ}_{scaled}
A.	distillation	$n = 11$, S, M	$1.447_{10^{-5}}$	$1.443_{10^{-3}}\%$
B.	gas turbine	$n = 4$, S, M	0.8062	$5.147_{10^{-2}}\%$
C.	turbo	$n = 6$, S, NM	0.1521	0.6397%
D.	helicopter	$n = 4$, US, M	$1.274_{10^{-2}}$	0.1106%
E.	thermal	$n = 9$, S, NM	0	0%
F.	pilot	$n = 6$, US, M	$3.491_{10^{-4}}$	$6.128_{10^{-4}}\%$
G.	boiler	$n = 9$, US, NM	$1.016_{10^{-6}}$	$4.439_{10^{-9}}\%$
H.	mass	$n = 6$, US, M	0.1148	5.483%
I.	2-link	$n = 36$, US, NM	$8.461_{10^{-3}}$	$2.765_{10^{-7}}\%$
J.	2-cart	$n = 8$, US, NM	$8.768_{10^{-2}}$	0.8113%

LEGEND: n is the order of the plant, S denotes a stable plant, US denotes an unstable plant, M denotes a minimum phase plant, and NM denotes a non-minimum phase plant.

5.3 Example 3

This example studies how "close" a system is in obtaining a particular achievable optimal performance index.

It is known that "perfect" tracking and disturbance rejection is achievable for minimum phase systems (e.g. see Davison and Scherzinger (1987)) under certain mild conditions. For nonminimum phase systems, however, it is shown in Qiu and Davison (1993) that there exists a fundamental performance limitation in the possible obtainable tracking and disturbance rejection. Such a limitation can be characterized by the system's nonminimum phase zeros. In particular, let $\lambda_1, \ldots, \lambda_l$ be the nonminimum phase zeros of a given system (1), then the optimal achievable quadratic performance cost for the RSP with constant tracking signals y_{ref} and zero initial conditions, given by:

$$J_\epsilon = \min_u \int_0^\infty \left(e^T e + \epsilon \dot{u}^T \dot{u}\right) dt \tag{22}$$

where $e = y_{ref} - y$ is the error in the system, has the property for constant tracking signals that $\lim_{\epsilon \to 0} J_\epsilon = y^T_{ref} H y_{ref}$, where $H \geq 0$ with the property that $\text{trace}(H) = 2 \sum_{i=1}^{l} \frac{1}{\lambda_i}$.

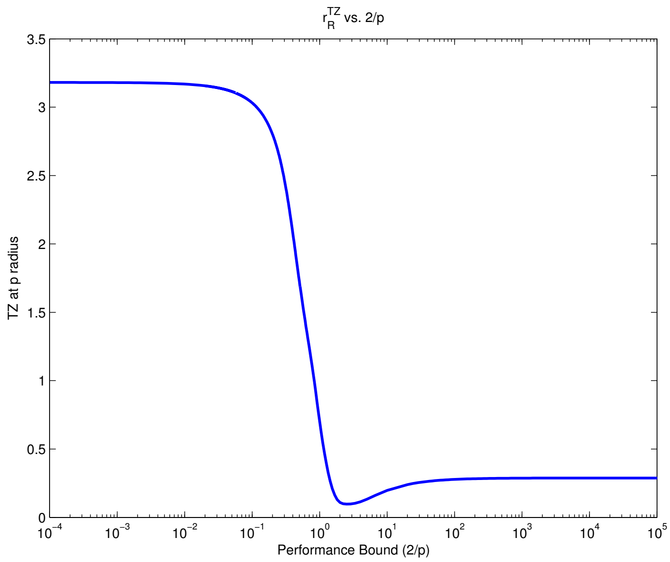

Fig. 1. Distance to performance bound $J = \frac{2}{p}$ for system (18).

This implies for nonminimum phase systems that the optimal limiting performance index that can be achieved is bounded from below by $2 \sum_{i=1}^{l} \frac{1}{\lambda_i}$ (Qiu and Davison (1993)).

Consider again now the same system (18). Given real $p > 0$, Figure 1 plots the system's transmission zero at p radius in relationship to the corresponding achievable performance cost $J = \frac{2}{p}$. It can be seen that system (18) is "closer" to a nonminimum phase system with a large performance limitation than a nonminimum phase system with a small performance bound.

On the other hand, consider again system (18) with $D = 0$. It can be seen from Figure 2 that in this case, the system is "closer" to a nonminimum phase system with a small performance limitation than a system with a large performance limitation. This agrees with Lemma 8 which states that the minimum phase system (18) is arbitrarily close to a nonminimum phase system with a large (real) nonminimum phase zero.

6. CONCLUSIONS

In this paper, two continuous robustness measures related to a LTI system's transmission zeros are introduced, namely the transmission zero at s radius and the minimum phase radius. These two radii respectively measure how "close", with respect to parametric perturbations, the system is to having a transmission zero at a specified $s \in \mathbb{C}$, and how "close" a minimum phase system is to a nonminimum phase system. Formulas for computing both radii and procedures for constructing the corresponding system perturbations that achieve these radii are presented in this paper. Some properties of the two radii are also given, along with numerical examples to illustrate the type of results one may obtain.

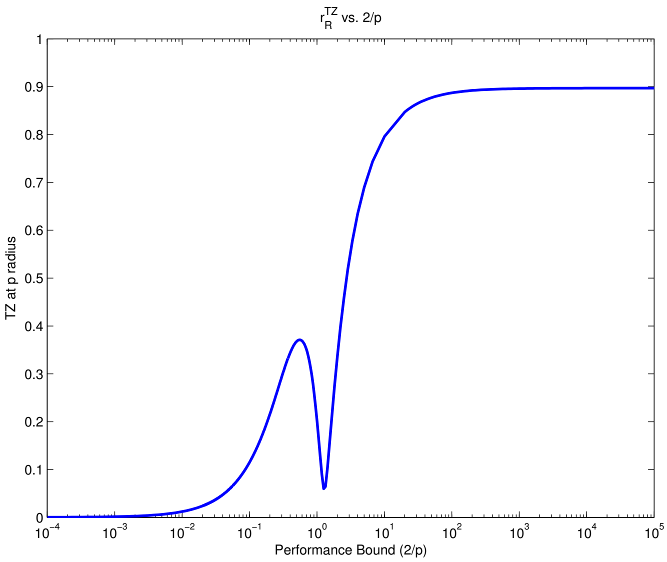

Fig. 2. Distance to performance bound $J = \frac{2}{p}$ for system (18) with $D = 0$.

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Appendix A. PROOF OF THEOREM 5 AND LEMMA 7

A.1 Proof of Theorem 5

Equation (5) follows directly from the property of singular values (e.g. see Golub and Loan (1983)) given by:

Theorem 12. Given $M \in \mathbb{C}^{q\times l}$,

$$\sigma_k(M) = \min\{\|\Delta\| \,|\Delta \in \mathbb{C}^{q\times l}, \text{rank}(M+\Delta) < k\} \quad \text{(A.1)}$$

Definition 13. (Real Perturbation Values). Given a matrix $M \in \mathbb{C}^{q\times l}$, the k-th *real perturbation value* of M, $\tau_k(M)$, is defined as

$$\tau_k(M) := \inf\{\|\Delta\| \,|\Delta \in \mathbb{R}^{q\times l}, \text{rank}(M+\Delta) < k\} \quad \text{(A.2)}$$

where $k \in \mathbb{N}$.

Equation(6) follows directly from the definition of real perturbation values and on the following result found in Bernhardsson et al. (1998). In particular, the k-th real perturbation value of M can be computed by the following result.

Theorem 14. (Bernhardsson et al. (1998)). Given a matrix $M \in \mathbb{C}^{q\times l}$ and $k \in \mathbb{N}$, then

$$\tau_k(M) = \sup_{\gamma\in(0,1]} \sigma_{2k-1}\left(\begin{bmatrix} \Re M & -\gamma\Im M \\ \gamma^{-1}\Im M & \Re M \end{bmatrix}\right) \quad \text{(A.3)}$$

A.2 Proof of Lemma 7

Lemma 7 are direct results obtained from properties of singular values (e.g. see Horn and Johnson (1985)) and from the following properties of real perturbation values (see Bernhardsson et al. (1995)):

Lemma 15. (Properties of $\sigma_k(M)$ and $\tau_k(M)$). Given $M \in \mathbb{C}^{q\times l}$ and $k \in \mathbb{N}$, then:

(1) $\sigma_k(\overline{M}) = \sigma_k(M)$ and $\tau_k(\overline{M}) = \tau_k(M)$
(2) $\sigma_k(Q_1 M Q_2) = \sigma_k(M)$ and $\tau_k(Q_1 M Q_2) = \tau_k(M)$
(3) $\sigma_k(M) \leq \tau_k(M)$

where Q_1 and Q_2 are real orthogonal matrices, and where equality in Property (3) is achieved for $M \in \mathbb{R}^{q\times l}$.

Appendix B. PROOF OF LEMMA 8

The following proof is for $r_{\mathbb{C}}^{TZ}(C, A, B, 0, s)$, but it also applies to $r_{\mathbb{R}}^{TZ}(C, A, B, 0, s)$ by Property (3) of Lemma 7. Given now a minimum phase system (1) with $D = 0$, let $p > 0$ be a given real number, and consider the perturbed system:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + \tilde{D}u
\end{aligned}
\qquad \text{(B.1)}
$$

where $\tilde{D} = \epsilon C(\epsilon A - pI)^{-1}B$ for $\epsilon > 0$. Then it can be verified that the perturbed system (B.1) has m transmission zeros at $\frac{p}{\epsilon} > 0$ since

$$
\begin{aligned}
&\operatorname{rank}\left(\begin{bmatrix} A - \frac{p}{\epsilon}I & B \\ C & \epsilon C(\epsilon A - pI)^{-1}B \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} I_n & (A - \frac{p}{\epsilon}I)^{-1}B \\ C & \epsilon C(\epsilon A - pI)^{-1}B \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} I_n & \epsilon(\epsilon A - pI)^{-1}B \\ 0 & 0 \end{bmatrix}\right) < n + \min(r, m)
\end{aligned}
$$

It can be concluded then that as $\epsilon \to 0$, an arbitrary small perturbation in $\tilde{D}$ gives rise to a perturbed system, which is nonminimum phase, with unstable transmission zeros given by $\frac{p}{\epsilon} > 0$.