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Polynomial Approximation of Closed-form MPC for Piecewise Affine Systems

Michal Kvasnica [*,1], **Frank J. Christophersen** [**], **Martin Herceg** [*], and **Miroslav Fikar** [*]

[*] *Institute of Automation, Information Engineering and Mathematics, Slovak University of Technology, 812 37 Bratislava, Slovakia*
[**] *Automatic Control Laboratory, ETH Zurich, Physikstrasse 3, 8092 Zurich, Switzerland*

Abstract: This paper addresses the issue of the practical implementation of closed-form Model Predictive Controllers (MPC) to processes with very short sampling times. Such questions come in consideration when the solution to MPC problems is expressed in a so-called parametric or closed-form fashion. The underlying idea of this paper is to approximate the optimal control law defined over state space regions by a higher degree polynomial which then guarantees closed-loop stability, constraint satisfaction, and a bounded performance decay. The advantage of the proposed scheme lies in faster controller evaluation and lower storage demand compared to currently available techniques.

Keywords: model predictive control, hybrid systems, Lyapunov stability, approximation

1. INTRODUCTION

In the Model Predictive Control (MPC) framework (Mayne et al., 2000), the process to be controlled is described by a dynamical model, based on which the predicted behavior of the plant can be optimized. If the model of the plant is constrained and linear, and the performance index is based on linear vector norms, it can be shown (Borrelli, 2003) that the underlying optimization problem can be formulated and solved with a *multi-parametric linear program* (mpLP). The resulting closed-form solution, which can be interpreted as a lookup table, is a piecewise affine (PWA) control law defined over polyhedral regions of the state space. Moreover, in (Borrelli, 2003) it was shown that the same type of solutions can be obtained even when the controlled plant is described by a *hybrid* model. Such hybrid systems (Bemporad and Morari, 1999) are systems which combine continuous dynamics with discrete logic, such as on/off switches or finite state machines. These systems are appealing because of their ability to approximate non-linear plants with arbitrary precision.

An advantage of the closed-form solutions is that their on-line application reduces to a simple set-membership test, which can be performed much faster compared to traditional on-line optimization-based techniques. Hence allowing MPC to control processes also with fast sampling rates.

However, the time needed to evaluate the lookup table still limits the minimal admissible sampling time of the controlled system and even in an 'average' case the complexity of the lookup table in the number of defining state space regions tends to be very large and above the storage limit of most control devices (Borrelli, 2003). Therefore, it is often essential for a real-life implementation of the closed-form solution to find an appropriate approximation of the controller or a controller with reduced complexity. Several authors therefore addressed the complexity reduction or approximation issue by either modifying the original MPC problem, retrieving a suboptimal solution, or by post-processing the computed optimal controller, cf. e.g. (Bemporad and Filippi, 2003; Geyer et al., 2004; Tøndel et al., 2003; Ulbig et al., 2007; Lazar et al., 2007). However, a direct guarantee on the reduction of the complexity, closed-loop stability, or performance decay is mostly neglected.

In this paper we propose first to calculate a parametrization of a set of stabilizing feedback laws and then (in the second step) to find a multivariate polynomial contained in such a set. If the polynomial exists and is applied as a state feedback control law to the system, closed-loop stability and constraint satisfaction are guaranteed. The advantage of the polynomial feedback law is that it can be evaluated on-line more efficiently compared to existing approaches. It will be illustrated that the polynomial-based scheme can be evaluated in a constant number of CPU operations, regardless of the complexity of the underlying parametric solution. Similarly, the memory requirement is only a constant function of the degree of the approximation polynomial and does not depend on the complexity of the lookup table. We show that the memory requirements needed to store the approximated control law are reduced by several orders of magnitude compared to existing methods.

2. CONSTRAINED OPTIMAL CONTROL OF HYBRID SYSTEMS

Piecewise affine (PWA) systems (Sontag, 1981) are equivalent to many other hybrid system classes (Heemels et al., 2001) such as mixed logical dynamical systems, linear complementary systems, and max-min-plus-scaling systems and thus form a very general class of linear hybrid systems.

[1] Corresponding author, e-mail: michal.kvasnica@stuba.sk

Moreover, PWA systems can be used to identify or approximate generic nonlinear systems via multiple linearizations at different operating points (Sontag, 1981). Although hybrid systems (and in particular PWA systems) are a special class of nonlinear systems, most of the nonlinear system and control theory does not apply because it usually requires certain smoothness assumptions. For the same reason we also cannot simply use linear control theory in some approximate manner to design controllers for PWA systems.

We consider the class of discrete-time, stabilizable, linear hybrid systems that can be described as constrained *Piecewise affine* (PWA) systems of the following form

$$\begin{aligned} x(t+1) &= f_{\mathrm{PWA}}(x(t), u(t)) && (1) \\ &:= A_d x(t) + B_d u(t) + a_d, \text{ if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}_d, \end{aligned}$$

where $t \geq 0$, the domain $\mathcal{D} := \cup_{d=1}^{N_{\mathcal{D}}} \mathcal{D}_d$ of $f_{\mathrm{PWA}}(\cdot,\cdot)$ is a non-empty compact set in $\mathbb{R}^{n_x+n_u}$ with $N_{\mathcal{D}} < \infty$ the number of system dynamics, and $\{\mathcal{D}_d\}_{d=1}^{N_{\mathcal{D}}}$ denotes a polyhedral partition of the domain $\mathcal{D}$, i.e. the closure of $\mathcal{D}_d$ is $\bar{\mathcal{D}}_d := \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right] \in \mathbb{R}^{n_x+n_u} \mid D_d^x x + D_d^u u \leq D_d^0 \right\}$ and $\mathrm{int}(\mathcal{D}_d) \cap \mathrm{int}(\mathcal{D}_j) = \emptyset$ for all $d \neq j$. Note that linear state and input constraints can be naturally incorporated in the description of $\mathcal{D}_d$. Throughout this work it is assumed that the origin is an equilibrium point of the PWA system (1).

2.1 Constrained Finite Time Optimal Control

We define for the aforementioned PWA system (1) the *constrained finite time optimal control* (CFTOC) problem

$$J_T^*(x(0)) := \min_{U_T} \ \ell_T(x(T)) + \sum_{t=0}^{T-1} \ell(x(t), u(t)) \qquad \text{(2a)}$$

$$\text{s.t.} \ \begin{cases} x(t+1) = f_{\mathrm{PWA}}(x(t), u(t)) \\ x(T) \in \mathcal{X}^f, \end{cases} \qquad \text{(2b)}$$

where $\ell(\cdot,\cdot)$ is the *stage cost*, $\ell_T(\cdot)$ the *final penalty function*, U_T is *the optimization variable* defined as the input sequence $U_T := \{u(t)\}_{t=0}^{T-1}$, $T < \infty$ is *the prediction horizon*, and $\mathcal{X}^f$ is a compact *terminal target set* in $\mathbb{R}^{n_x}$. With a slight abuse of notation, when the CFTOC problem (2a)–(2b) has multiple solutions, i.e. when the optimizer is not unique, $U_T^*(x(0)) := \{u^*(t)\}_{t=0}^{T-1}$ denotes one (arbitrarily chosen) realization from the set of possible optimizers. The CFTOC problem (2) implicitly defines the set of feasible initial states $\mathcal{X}_T \subset \mathbb{R}^{n_x}$ ($x(0) \in \mathcal{X}_T$) and the set of feasible inputs $\mathcal{U}_{T-t} \subset \mathbb{R}^{n_u}$ ($u(t) \in \mathcal{U}_{T-t}$, $t = 0, \ldots, T-1$). In the sequel we will consider linear cost functions of the form

$$\ell(x(t), u(t)) := \|Qx(t)\|_p + \|Ru(t)\|_p, \qquad \text{(3a)}$$

$$\ell_T(x(T)) := \|Px(T)\|_p, \qquad \text{(3b)}$$

where $\|\cdot\|_p$ with $p \in \{1, \infty\}$ denotes the standard vector 1-/∞-norm.

The goal in this section is to give an explicit (closed-form) expression for $u^*(t) : \mathcal{X}_T \to \mathcal{U}_{T-t}$, $t = 0, \ldots, T-1$.

Theorem 2.1. (Solution to CFTOC (Borrelli, 2003)). The solution to the optimal control problem (2a)–(2b) with a linear performance index (3) is a time-varying piecewise affine function of the initial state $x(0)$

$$\mu_{\mathrm{PWA}}(x(0), t) = K_{T-t,i}\, x(0) + L_{T-t,i},$$

if $x(0) \in \mathcal{P}_i$ with $u^*(t) = \mu_{\mathrm{PWA}}(x(0), t)$, where $t = 0, \ldots, T-1$, and $\{\mathcal{P}_i\}_{i=1}^{N_{\mathcal{P}}}$ is a polyhedral partition of the set of feasible states $x(0)$, $\mathcal{X}_T = \cup_{i=1}^{N_{\mathcal{P}}} \mathcal{P}_i$, with the closure of $\mathcal{P}_i$ given by $\bar{\mathcal{P}}_i = \{x \in \mathbb{R}^{n_x} \mid P_i^x x \leq P_i^0\}$. ■

If the resulting feedback law is applied in the *receding horizon* (RH) fashion (Mayne et al., 2000), the control is given as a time-invariant state feedback law of the form

$$\mu_{\mathrm{RH}}(x(t)) := K_{T,i}\, x(t) + L_{T,i}, \qquad (4)$$

if $x(t) \in \mathcal{P}_i$, where $i = 1, \ldots, N_{\mathcal{P}}$ and $u(t) = \mu_{\mathrm{RH}}(x(t))$ for $t \geq 0$.

Definition 2.2. (Feasibility). A CFTOC problem is called *feasible at time t* if there exists a control action at time t for the measured state $x_t := x(0)$, which satisfies the state and input constraints over the considered prediction horizon T. A receding horizon control problem is called *feasible for all time* if it is feasible for all $t \geq 0$. □

Assumption 2.3. (Stability, feasibility). Note that in the following it is assumed that the parameters T, Q, R, P, and $\mathcal{X}^f$ are chosen in such a way that (4) is closed-loop stabilizing, feasible for all time (Christophersen, 2007) and that a polyhedral piecewise affine Lyapunov function of the form

$$V(x) = V_i^x x + V_i^0, \quad \text{if} \ \ x \in \mathcal{P}_i$$

for the *closed-loop system*

$$f^{\mathrm{CL}}(x(t)) := f_{\mathrm{PWA}}(x(t), \mu_{\mathrm{RH}}(x(t))), \qquad (5)$$

$x(t) \in \mathcal{X}_T$, exists and is given. □

This is not a restricting requirement but rather the aim of most (if not all) control strategies. Furthermore, we remark that if the parameters are chosen according to e.g. (Mayne et al., 2000) one can simply take $V(\cdot)$ equal to the optimal cost $J_T^*(\cdot)$.

In the course of this paper our focus lies on the reduction of the complexity of the closed-form control law $\mu_{\mathrm{RH}}(\cdot)$ without losing closed-loop stability nor feasibility for all time.

3. STABILITY TUBES

In order to present the complete result for the new controller approximation approach, the two underlying core ideas need to be explained. The first idea is based on the inherent freedom of the Lyapunov decay inequality (6b) of Theorem 3.1, repeated for completeness in the following and proved e.g. in (Lazar et al., 2008).

Theorem 3.1. (Asymptotic/exponential stability). Let $\mathcal{X}_T$ be a bounded positively invariant set in $\mathbb{R}^{n_x}$ for the autonomous (closed-loop) system $x(t+1) = f^{\mathrm{CL}}(x(t))$ with $x(t) \in \mathcal{X}_T$ and let $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, and $\beta(\cdot)$ be K-class functions (Vidyasagar, 1993). If there exists a non-negative function $V : \mathcal{X}_T \to \mathbb{R}_{\geq 0}$ with $V(\ _{n_x}) = 0$ such that

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|), \qquad \text{(6a)}$$

$$\Delta V(x) := V(f^{\mathrm{CL}}(x)) - V(x) \leq -\beta(\|x\|), \qquad \text{(6b)}$$

for all $x \in \mathcal{X}_T$, then the following results hold:

(a) The equilibrium point $\ _{n_x}$ is *asymptotically stable* (Vidyasagar, 1993) in the Lyapunov sense in $\mathcal{X}_T$.

(b) If $\underline{\alpha}(\|x\|) := \underline{a}\|x\|^\gamma$, $\overline{\alpha}(\|x\|) := \overline{a}\|x\|^\gamma$, and $\beta(\|x\|) := b\|x\|^\gamma$ for some positive constants $\underline{a}, \overline{a}, b, \gamma > 0$ then the equilibrium point $_{n_x}$ is *exponentially stable* (Vidyasagar, 1993) in the Lyapunov sense in $\mathcal{X}_T$. ∎

Simply speaking, if all the prerequisites of Theorem 3.1 are fulfilled with a given controller $\mu_{\rm RH}(\bullet)$, the resulting behavior of the closed-loop system is stabilizing. If, for the given function $V(\bullet)$, $\beta(\bullet)$ is now relaxed, one can (possibly) find a set of controllers that will render the closed-loop system stabilizing and feasible. (Note, that setting $\beta(\bullet)$ close to the zero-function is sufficient for pure asymptotic stability. Naturally, however, the altered closed-loop system is likely to exhibit a modified, possibly detuned, performance and transient behavior.)

These sets of controllers are denoted in the following as *stability tubes*. (The concept and results of stability tubes – along with their computation – are elaborated in further detail in (Christophersen, 2007, Ch. 10).)

Definition 3.2. (Stability tube). Let $V(\bullet)$ be a Lyapunov function for the general nonlinear, closed-loop system $x(t+1) = f(x(t), u(t))$, with $x(t) \in \mathcal{X}_T$, under the stabilizing control $u(t) = \mu(x(t))$ and constraints $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}$ and let the prerequisites of Theorem 3.1 be fulfilled. Furthermore, let $\beta(\bullet)$ be a K-class function. Then the set

$$\mathcal{S}(V, \beta) := \Big\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathbb{R}^{n_x \times n_u} \;\Big|\; f(x, u) \in \mathcal{X}_T, \\ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D},\; V(f(x, u)) - V(x) \le -\beta(\|x\|) \Big\}$$

is called *stability tube*. □

Theorem 3.3. (Stability tube (Christophersen, 2007)). Let the assumptions of Definition 3.2 be fulfilled. Then every control law $u(t) = \widetilde{\mu}(x(t))$, $x(t) \in \mathcal{X}_T$, (also *any* sequence of control samples $u(t)$) fulfilling

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{S}(V, \beta) \tag{7}$$

asymptotically stabilizes the system $x(t+1) = f(x(t), u(t))$, where $x(t) \in \mathcal{X}_T$, to the origin. ∎

Naturally, for general nonlinear systems, the stability tube $\mathcal{S}(V, \beta)$ can basically take any form. Note, however, that for the considered class of PWA systems, PWA control laws, and PWA Lyapunov functions with $\beta(\bullet)$ consisting of a sum of weighted vector 1-/∞-norms, the stability tube can be described by a collection of polytopic sets in the state-input space and can be computed with basic polytopic operations. In the case considered here, the stability tube can be represented and 'easily' be obtained as a collection (or union) of polytopes of the form $\mathcal{S}(V, \beta) := \cup_{j=1}^{N_\mathcal{S}} \mathcal{S}_j$, where the closure of $\mathcal{S}_j$ is $\bar{\mathcal{S}}_j := \big\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathbb{R}^{n_x + n_u} \mid S_j^{xu} \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \le S_j^0 \big\}$. Without going into details, by construction, we have the following properties: *(a)* for some index set $\mathcal{I}_i \subseteq \{1, \ldots, N_\mathcal{S}\}$, the union $\cup_{j \in \mathcal{I}_i} \mathcal{S}_j$ is defined over the controller region $\mathcal{P}_i$, and *(b)*, $\sum_{i=1}^{N_\mathcal{P}} |\mathcal{I}_i| = N_\mathcal{S}$. This means that each $\mathcal{S}_j$ is defined over a single region $\mathcal{P}_i$, i.e. if for some i_1 and j we have $\mathrm{proj}_x(\mathcal{S}_j) \subseteq \mathcal{P}_{i_1}$ then there does not exist a $i_2 \ne i_1$ with $\mathrm{proj}_x(\mathcal{S}_j) \subseteq \mathcal{P}_{i_2}$. (We remark, that simulations seem to indicate that most often $\mathcal{I}_i = 1$ for all i, i.e. only one $\mathcal{S}_j$ is defined over $\mathcal{P}_i$.)

Example 3.4. To demonstrate the idea of the stability tubes, consider the following simple PWA example

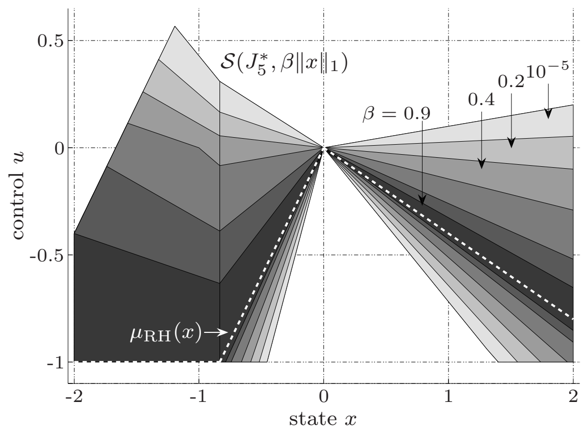

Figure 1. Optimal control law $\mu_{\rm RH}(\bullet)$ (dashed line) for Example 3.4. The shadowed sets are the corresponding stability tubes $\mathcal{S}(J_5^*, \beta\|x\|_1)$ for different values of β.

$$x(t+1) = \begin{cases} 4/5\, x(t) + 2u(t) & \text{if } x > 0, \\ -6/5\, x(t) + u(t) & \text{if } x \le 0, \end{cases} \tag{8}$$

with $u(t) \in [-1,\ 1]$ and $x(t) \in [-2,\ 2]$. When solving the CFTOC Problem (2) with $p = 1$ and $Q = 1$, $R = 1$, $P = 0$, $T = 5$ one obtains the result illustrated in Fig. 1. The control law $\mu_{\rm RH}(\bullet)$ (dashed line) is a PWA function defined over 3 regions, while the stability tube $\mathcal{S}(J_5^*, \beta\|x\|_1)$ for the region is shadowed. Note that $\mathcal{S}(J_5^*, \beta\|x\|_1)$, for a fixed β, is represented by a collection of 3 polytopes. □

Moreover, with the choice of $\beta(\bullet)$ a detuning of the *closed-loop performance* $\sum_{t=0}^{\infty} \ell(x(t), u(t))$, with some control law $u(t) = \widetilde{\mu}(x(t))$, compared to the optimal receding horizon control solution $\mu_{\rm RH}(\bullet)$, can be performed. Thus, one can try to find an approximation $\widetilde{\mu}(\bullet)$ 'inside' the stability tube without losing closed-loop stability, all time feasibility, while still guaranteeing a given, bounded performance decay of $\eta\,\%$. The influence of a different $\beta(\bullet)$ is elaborated in the following.

Corollary 3.5. (Perform. bound (Christophersen, 2007)). Let the assumptions of Def. 3.2 be fulfilled, the stage cost $\ell(\bullet, \bullet)$ be lower bounded by some K-class function, and $\beta > 0$. Then every control law $u(t) = \widetilde{\mu}(x(t))$ with $x(t) \in \mathcal{X}_T$ (also *any* sequence of control samples $u(t)$) fulfilling $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{S}(V, \beta\ell)$ asymptotically stabilizes the system $x(t+1) = f(x(t), u(t))$, where $x(t) \in \mathcal{X}_T$, to the origin and guarantees a level of *closed-loop performance* given by

$$\textstyle\sum_{t=0}^{\infty} \ell(x(t), \widetilde{\mu}(x(t))) \le \frac{1}{\beta} V(x(0)). \tag{9}$$

∎

We remark, that from (9) it follows that the *performance decay* η [in %] with respect to $V(x(0))$ is related to $\beta > 0$ via $\beta(\eta) = \left(1 + \frac{\eta\ [\text{in }\%]}{100}\right)^{-1}$.

Figure 1 illustrates for Example 3.4 the stability tubes for a variety of different β and fixed $V(\bullet)$. Note, that (naturally) the stability tubes are subsets of each other and collapse to the control law $\mu_{\rm RH}(\bullet)$ itself as $\beta \to 1$.

In the next section the set $\mathcal{S}(V, \beta)$ will be used to find a simple polynomial approximation of the optimal control law such that stability and feasibility guarantees are maintained.

4. POLYNOMIAL APPROXIMATION

As outlined in the introduction, we aim at finding a polynomial $\widetilde{\mu}(\cdot)$ of a fixed degree which, when applied as a state feedback control law, guarantees stability of the closed-loop system and approximates the optimal solution $\mu_{\mathrm{RH}}(\cdot)$ to the CFTOC Problem (2) with certain optimality. This is achieved by optimizing the coefficients of a given approximation polynomial such that each point of the polynomial is contained inside the stability tube of the corresponding parametric solution to the CFTOC problem (2). The family of polynomials of interest takes the following form:

$$\widetilde{\mu}(x) := \sum_{l=1,\dots,d} a_l x_1^{l_1} \cdots x_n^{l_n} \tag{10}$$

where d denotes the degree of the approximation polynomial and the matrices $a_l \in \mathbb{R}^{n_u \times n_x}$ are the coefficients to be determined. Note that there is no constant offset a_0 present in (10) since $\widetilde{\mu}(x = {}_{n_x}) = {}_{n_u}$ is required to attain stability in the sense of Theorem 3.1.

From Theorem 3.3 follows, in order to guarantee closed-loop stability, that for all regions $\mathcal{P}_i$, the polynomial $\widetilde{\mu}(\cdot)$ has to satisfy

$$\left[\begin{smallmatrix} x \\ \widetilde{\mu}(x) \end{smallmatrix}\right] \in \cup_{j \in \mathcal{I}_i} \mathcal{S}_j, \quad \forall\, x \in \mathcal{P}_i, \quad \forall\, i = 1, \dots, N_{\mathcal{P}}, \tag{11}$$

where $\cup_{j \in \mathcal{I}_i} \mathcal{S}_j$ are the stability tubes associated to region $\mathcal{P}_i$ through the index set $\mathcal{I}_i$. In other words, if there exists a polynomial $\widetilde{\mu}(x)$ such that $\left[\begin{smallmatrix} x \\ \widetilde{\mu}(x) \end{smallmatrix}\right]$ for all points $x \in \cup_{i=1}^{N_{\mathcal{P}}} \mathcal{P}_i$ is contained in $\mathcal{S}(V, \beta)$, then such polynomial, when applied as a state feedback, guarantees closed-loop stability and feasibility for all time for the controlled system (1).

Assumption 4.1. Through the rest of this section we assume that the set $\mathcal{S}_j$ defined over the i-th region $\mathcal{P}_i$ is convex (and thus $|\mathcal{I}_i| = 1$), and the union $\cup_{j=1}^{N_{\mathcal{S}}} \mathcal{S}_j = \mathcal{S}(V, \beta)$ is connected. □

We remark that simulations seem to indicate that most often this is fulfilled.

Under Assumption 4.1, condition (11) can be written in a matrix form as

$$S_i^{xu} \left[\begin{smallmatrix} x \\ \widetilde{\mu}(x) \end{smallmatrix}\right] \le S_i^0, \quad \forall\, x \in \mathcal{P}_i \quad \forall\, i = 1, \dots, N_{\mathcal{P}}. \tag{12}$$

When considering only a single region $\mathcal{P}_i = \{x \in \mathbb{R}^{n_x} \mid P_i^x x \le P_i^0\}$, the Positivestellensatz (Stengle, 1974) provides a sufficient condition for the existence of a suitable polynomial approximation $\widetilde{\mu}(\cdot)$ which satisfies (12):

Lemma 4.2. (Positivestellensatz). Given a polynomial $h(x) = \sum_\alpha a_\alpha x^\alpha$ and a set of r polynomials $g_r(x) = \sum_\alpha b_{\alpha,r} x^\alpha$. If there exist non-negative polynomials $s_r(x) = \sum_\alpha c_{\alpha,r} x^\alpha$ such that

$$h(x) - \sum_r s_r(x) g_r(x) \ge 0, \tag{13a}$$

$$s_r(x) \ge 0, \tag{13b}$$

then the following statement

$$g_r(x) \ge 0 \quad \Rightarrow \quad h(x) \ge 0, \tag{14}$$

holds for all $x \in \{x | g_r(x) \ge 0\}$. □

To observe the relation between the Positivestellensatz and the polynomial approximation (12), notice that one can define the polynomials $h(x)$ and $g_r(x)$ in (14) as

$$h(x) := S_i^0 - S_i^{xu} \left[\begin{smallmatrix} x \\ \widetilde{\mu}(x) \end{smallmatrix}\right], \tag{15a}$$

$$g_r(x) := [P_i^0]_r - [P_i^x]_r x, \tag{15b}$$

where $[\cdot]_r$ denotes the r-th row of the respective matrix. Therefore, in order to find $\widetilde{\mu}(x)$ we need to ensure the non-negativity of (13). This can be achieved by a sum of squares decomposition as captured by Lemma 4.3 and Lemma 4.4.

Lemma 4.3. (Parrilo, 2004) A real valued polynomial $P(x)$ is non-negative for all x if there exists a *sum of squares* (SOS) *decomposition* of the form

$$P(x) = \sum_i p_i^2(x), \quad p_i(x) \in \mathbb{R}[x]. \tag{16}$$

■

Lemma 4.4. (Sum of squares (Parrilo, 2004)). A polynomial $P(x)$ is a sum of squares if and only if it can be written as $P(x) = z'Qz$, where z is a vector of monomials (i.e. polynomials with only one term, e.g. x_i^n) in the x_i variables and Q is a symmetric, positive semi-definite matrix of suitable dimension. ■

By Lemma 4.4, the decomposition (16) can be found by solving a semi-definite program (SDP), for which efficient solvers exist (Löfberg, 2004).

We can now state the main result of this section.

Theorem 4.5. (Polynomial approximation). There exists a state feedback $\widetilde{\mu}(x)$ of the form (10) which stabilizes the PWA system (1) if the polynomials $P_i(x)$

$$P_i(x) := S_i^0 - S_i^{xu} \begin{bmatrix} x \\ \widetilde{\mu}(x) \end{bmatrix} - \sum_r s_{i,r}(x)([P_i^0]_r - [P_i^x]_r x) \tag{17}$$

and polynomials $s_{i,r}(x)$ are sum of squares $\forall i = 1, \dots, N_{\mathcal{P}}$. Moreover, the coefficients a_l, $l = 1, \dots, d$, of the polynomial $\widetilde{\mu}(x)$ in (10) can be found by an SOS decomposition (17) by solving a semi-definite program. □

Proof. By Theorem 3.3, any control law $\widetilde{\mu}(x)$ with $\left[\begin{smallmatrix} x \\ \widetilde{\mu}(x) \end{smallmatrix}\right] \in \mathcal{S}(V, \beta)$, for all $x \in \mathcal{X}_T$, stabilizes system (1) while satisfying the system constraints. The polynomial $\widetilde{\mu}(x)$ fulfills this condition for all admissible states $x \in \mathcal{X}_T$ if and only if (12) is fulfilled for all regions $\mathcal{P}_i$. According to Lemma 4.2, the satisfaction of (12) is implied by the existence of polynomials $s_r(x) \ge 0$ and the non-negativity of (13a). By substituting (15a) and (15b) into (13a) we obtain

$$S_i^0 - S_i^{xu} \begin{bmatrix} x \\ \widetilde{\mu}(x) \end{bmatrix} - \sum_i s_{i,r}(x)([P_i^0]_r - [P_i^x]_r x) \ge 0. \tag{18}$$

It follows from Lemma 4.3 that (18) will be globally non-negative if there exists a set of coefficients a_l, $l = 1, \dots, d$ of the polynomial $\widetilde{\mu}(\cdot)$ defined by (10) such that (17) is a sum of squares and, simultaneously, there exists a

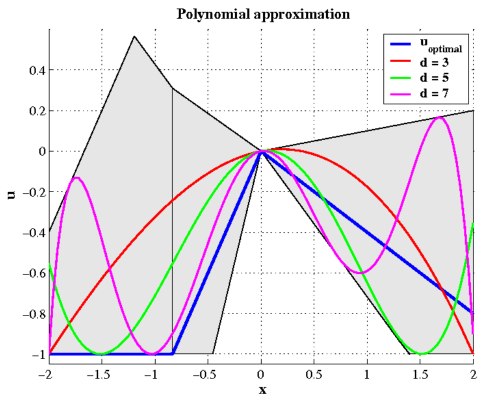

Figure 2. Stabilizing approximation of the optimal feedback law $u = \mu_{\mathrm{RH}}(\cdot)$ (blue line) by polynomials of different degrees.

SOS decomposition of the polynomials $s_{i,r}(x)$. Finally, Lemma 4.4 shows that the SOS decomposition, and hence the coefficients of the polynomial $\widetilde{\mu}(x)$, can be found using semi-definite programming techniques. ■

Remark 4.6. The conditions in Theorem 4.5 are merely sufficient for the existence of a suitable polynomial $\widetilde{\mu}(x)$. The approximation can still exist even if the SOS problem (17) is infeasible. □

Theorem 4.5 allows us to search for the coefficients of the approximation polynomial $\widetilde{\mu}(\cdot)$ by solving an SDP. We can further extend this result to give a procedure which optimizes the parameters a_l, $l = 1, \ldots, d$ such that the point-wise distance between the approximation $\widetilde{\mu}(\cdot)$ and the optimal control input $\mu_{\mathrm{RH}}(\cdot)$ is minimized, hence providing a tighter polynomial approximation of the optimal control law. This can be achieved by solving the following optimization problem:

$$\min_{a_d,\ldots,a_1} \sum_i \|\mu_{\mathrm{RH}}(x) - \widetilde{\mu}(x)\|_2 \tag{19}$$
$$\text{subj. to (17) is sum of squares,}$$
$$s_{i,r}(x) \text{ are sums of squares.}$$

SOS problems of the form (19) can be formulated and solved using higher-level optimization tools, such as YALMIP (Löfberg, 2004).

To illustrate the results of Theorem 4.5, we investigated again the Example 3.4. Three polynomials with degrees $d \in \{3, 5, 7\}$ were chosen to fit the stability tube $\mathcal{S}(J_5^*, \beta\|x\|_1)$, where $0 < \beta \ll 1$. Coefficients of the polynomials have been obtained by solving problem (19) using YALMIP. The resulting approximations are depicted in Figure 2. In all three cases the SOS problem (19) was feasible. Hence, by Theorem 4.5, each of the three polynomials guarantees closed-loop stability.

5. COMPLEXITY ANALYSIS

The aim of this section is to compare the polynomial controllers presented in Section 4 and the binary search approach of Tøndel et al. (2003) with respect to memory and CPU requirements.

In order to achieve better run-time performance as well as lower memory footprint, it is worth to replace the polynomial approximation of the form of (10) by a simpler expression, namely by

$$\widetilde{\mu}(x) := a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x. \tag{20}$$

Remark 5.1. The results of Theorem 4.5 still hold even for $\widetilde{\mu}(x)$ defined by (20), since this simpler form can be directly derived from (10) by setting to zero the coefficients which multiply the cross-products between different powers of x. This is easily done by imposing additional equality constraints when solving the approximation problem (19).

Remark 5.2. The sole purpose of the simplification (20) is to reduce the number of mathematical operations which are needed to evaluate the polynomial $\widetilde{\mu}(x)$ for a particular value of x. Since (20) has less degrees of freedom compared to the full form of (10), there could exist cases for which given stability tubes $\mathcal{S}(V, \beta)$ can be approximated by $\widetilde{\mu}(x)$ given by (10), but not with the simpler form of (20). However, in all investigated cases the approximation problem (19) was feasible even when $\widetilde{\mu}(x)$ was considered as in (20).

If the *Horner's* scheme (Eve, 1964) is used, the on-line evaluation of the polynomial (20) for a given value of x takes at most $\frac{1}{2}n_u n_x(3d + 5)$ floating point operations (FLOPS), and one needs at most $dn_u n_x$ bytes to store all its coefficients. The binary search tree, on the other hand, can be evaluated in $D(2n_x + 1) + 2n_x n_u$ FLOPS and requires as many as $D(n_x + 3) + N_u n_u(n_x + 1)$ memory elements to store all its parameters. Here D stands for the depth of the tree and N_u represents the number of unique control laws. Tøndel et al. (2003) gives $D \approx 1.7\log_2 N_{\mathcal{P}}$ as a good estimate, while from the authors' experience the number of unique control laws can be roughly estimated as $N_u \approx \frac{1}{4}N_{\mathcal{P}}$.

Therefore the complexity of the on-line implementation of the binary search tree grows logarithmically with the increasing number of controller regions $N_{\mathcal{P}}$. On the other hand, these parameters stay constant in the proposed polynomial approximation scheme. For better illustration are these correlations depicted visually in Figure 3. As can be seen from the pictures, the polynomial approximation approach easily outperforms the binary tree in terms of complexity for any reasonably complex partition, i.e. for $N_{\mathcal{P}} > 5$.

6. EXAMPLE

Consider the example from (Bemporad and Morari, 1999)

$$x(t+1) = 4/5 \begin{bmatrix} \cos\alpha(x(t)) & -\sin\alpha(x(t)) \\ \sin\alpha(x(t)) & \cos\alpha(x(t)) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ u(t) \end{bmatrix}$$
$$\alpha(x(t)) = \begin{cases} \pi/3 & \text{if } [1\ 0]x(t) \geq 0, \\ -\pi/3 & \text{if } [1\ 0]x(t) < 0, \end{cases}$$
$$x(t) \in [-10,\ 10] \times [-10,\ 10], \quad u(t) \in [-4,\ 4].$$

The CFTOC Problem (2) was solved with the parameters $Q = I_2$, $R = 1$, $T = 3$, $P = 0_2$, $p = \infty$ using the MPT toolbox (Kvasnica et al., 2004) yielding a PWA control law defined over 26 regions. Subsequently, the stability tube $\mathcal{S}$ was computed with the choice of the performance tuning parameter $\beta = 0.7143$, which corresponds to a maximum allowed performance decay of 40%. We remark that the resulting stability tubes satisfied the convexity

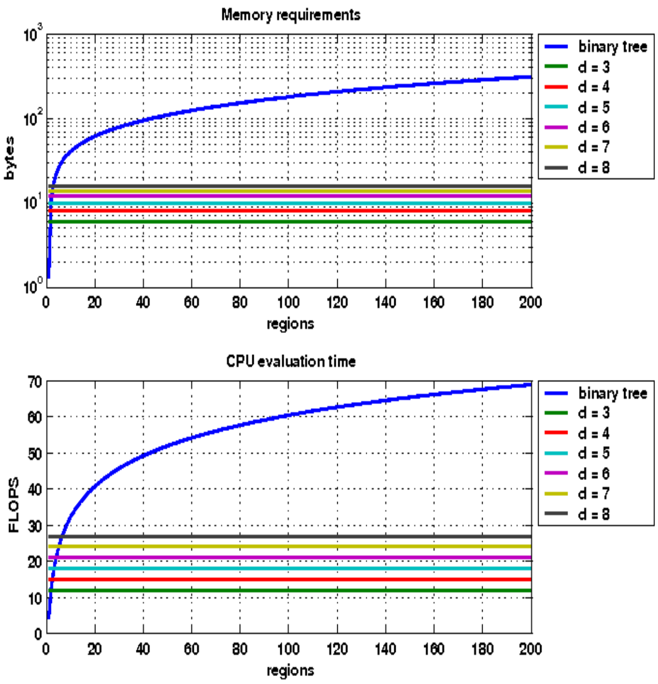

Figure 3. Memory and processing requirements needed to store and evaluate the binary search tree and the polynomial controller (20).

assumption 4.1, i.e. the union of the sets $\mathcal{S}_j$ defined over the region $\mathcal{P}_i$ was indeed convex.

The approximation problem (19) was then successfully solved for different degrees of the approximation polynomial (20) using YALMIP. Specifically, we have investigated the range $d \in \{3, 4, 5, 6\}$. Overview of the obtained results is presented in Table 1. It summarizes the time needed to solve the approximation problem (19) for a particular polynomial degree d as well as the average performance decay versus the optimal solution.

The performance drop was calculated as a ratio of the value of the cost function (2a) with $\mu_{\mathrm{RH}}(x_0)$ driven by the optimal feedback (4) versus the value of the same objective with $\widetilde{\mu}(x_0)$ given by the approximation of the form (20). The performance objective (2a) in both cases was evaluated over the closed-loop trajectory for 1000 distinct initial conditions x_0. Since a feasible solution was always recovered, the calculated polynomials, when applied as state feedback, guarantee closed-loop stability and feasibility. Note that the performance of the polynomial controller can be improved by assuming stability tubes calculated for different values of β as per Corollary 3.5.

Approximation order	3	4	5	6
Runtime [s]	70	130	220	300
Performance decay	26 %	20 %	40 %	14 %

Table 1. Numerical results for the example in Section 6.

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