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Stability Analysis of an UAV Controller using Singular Perturbation Theory ⋆

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Abstract: This paper presents the stability analysis of a hierarchical controller for an Unmanned Aerial Vehicle, using singular perturbation theory. Position and attitude control laws are successively designed by considering a time-scale separation between the translational dynamics and the orientation dynamics of a six degrees of freedom Vertical Take Off and Landing UAV model. In addition, for the design of the position controller, we consider the case where the linear velocity of the vehicle is not measured. A partial state feedback control law is proposed, based on the introduction of virtual states in the translational dynamics of the system.

1. INTRODUCTION

Miniature Unmanned Aerial Vehicles (UAV) are prone to be useful for numerous military and civil applications. Especially, thanks to features such as Vertical Take Off and Landing and hover capability, rotorcraft-based miniature UAVs are particularly well suited for missions such as video inspection of buildings for maintenance, victims localization after natural disasters, fire detection, etc. To make autonomous flight of such vehicles possible, control laws must be developed to replace the action of a human pilot.

Input-output linearization is one of the nonlinear control schemes that has been proposed for rotary wings UAVs. Since that method can only be applied to minimum phase systems, and since, generally, helicopters have unstable zero dynamics, an approximate input-output linearization has been proposed in [12]. Another solution consists in the application of backstepping techniques, by considering the model used for control design as a chain of integrators. Backstepping has been widely applied to different miniature vehicles such as conventional helicopters [6, 13], coaxial birotor helicopters [3] or four-rotor vehicles [2].
These two control strategies lead to a dynamical extension of the controller and make it difficult to use in practice, since measurements on the control inputs and their time-derivatives are not easy to obtain. In addition, measurements on the translational dynamics and on the orientation dynamics of the vehicle cannot be achieved in practice at the same sampling rate. Moreover, time-scale separation cannot be taken into account by the aforementioned control strategies.

For practical use, a more suitable approach is the hierarchical control. In that case, separate controllers can be designed to successively stabilize the translational dynamics and the orientation dynamics of the vehicle. This method, classically known in aeronautics as guidance and control, can handle time-scale separation. Considering miniature UAVs, a hierarchical control strategy has been applied, for example, to a ducted fan miniature UAV [16].
In hierarchical control, the time-scale separation between the translational dynamics (slow time-scale) and the orientation dynamics (fast-time scale) can be used to design position and orientation controllers under simplifying assumptions. Although reduced-order subsystems can hence be considered for control design, the stability must be analyzed by considering the complete closed loop system.

A theoretical background for time-scale separation approaches and stability analysis is provided by the singular perturbation theory [10, 11]. Aerospace applications of that theory can be found in [14]. In [9, 15], a time-scale separation is considered for helicopter control design, but stability issues are not considered. A theoretical stability analysis is provided in [5] using singular perturbation theory, for the altitude dynamics of a miniature VTOL UAV. As a complementary work of [4], closed loop stability is analyzed by considering a three time-scale model of a miniature helicopter mounted on a stand, incorporating collective pitch actuator dynamics. To our knowledge, this is the only work that theoretically addresses stability issues for VTOL UAVs using singular perturbation theory. However, it only focuses on the vertical motion of the vehicle, and full state measurement is assumed to be available.

In this paper, we present the stability analysis of a VTOL UAV hierarchical controller using singular perturbation

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theory. A six degrees of freedom model is considered, based on a simplified rigid body representation of miniature VTOL UAV dynamics. The kinematic representation that we use exploits the $SO(3)$ group and its manifold. For control design, we assume that no measurement of the linear velocity of the vehicle is available. This case corresponds to the practical use of an UAV equipped with a inertial measurement unit and a video camera that respectively provides measurements on attitude angles and rotation velocities, and measurement of the relative position of the vehicle with respect to the environment.

The paper is organized as follows. In the next section, we introduce notations and identities that will be used in the rest of the paper. In section 3, the UAV model and the hierarchical control strategy are presented. In section 4, a partial state feedback position controller is designed, based on previous results [1], by introducing virtual states in the translational dynamics, and without requiring an observer. The design of the attitude controller is presented in section 5, and stability of the complete closed loop system is analyzed in section 6. Concluding remarks are finally given at the end of the paper.

2. NOTATIONS

Let $SO(3)$ denote the special orthogonal group of $\mathbb{R}^{3\times 3}$ and $so(3)$ the group of antisymmetric matrices of $\mathbb{R}^{3\times 3}$.

We define by $(.)_\times$ the operator from $\mathbb{R}^3 \to so(3)$ such that

$$\forall b \in \mathbb{R}^3, b_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \quad (1)$$

where b_i denotes the i^{th} component of the vector b.

Let $V(.)$ be the inverse operator of $(.)_\times$, defined from $so(3) \to \mathbb{R}^3$, such that

$$\forall b \in \mathbb{R}^3, V(b_\times) = b \qquad \forall B \in so(3), V(B)_\times = B \quad (2)$$

For a given vector $b \in \mathbb{R}^3$ and a given matrix $M \in \mathbb{R}^{3\times 3}$, let us consider the following notations and identities:

$$P_a(M) = \frac{M - M^T}{2} \qquad P_s(M) = \frac{M + M^T}{2} \quad (3)$$

$$\text{tr}(P_a(M)P_s(M)) = 0 \quad (4)$$

$$\frac{1}{2}\text{tr}(b_\times M) = -b^T V(P_a(M)) \quad (5)$$

The following identity will also be used:

$$\forall A_a \in so(3), \frac{1}{2}\text{tr}(A_a^T A_a) = \|V(A_a)\|^2 \quad (6)$$

Denote by (γ_R, n_R) the angular-axis coordinates of a given matrix $R \in SO(3)$, and by I_d the identity matrix of $\mathbb{R}^{3\times 3}$. One has:

$$\forall R \in SO(3), \text{tr}(I_d - R) = 2(1 - \cos(\gamma_R)) \quad (7)$$

$$\forall R \in SO(3), \|V(P_a(R))\| = \cos(\frac{\gamma_r}{2})\sqrt{\text{tr}(I_d - R)} \quad (8)$$

Finally, for a given positive definite matrix $M \in \mathbb{R}^{3\times 3}$, we denote by $\lambda_i(M)$ its i^{th} eigenvalue and introduce

$$\lambda_{min}(M) = \min\{|\lambda_i(M)|, i = 1, 2, 3\} \quad (9)$$

$$\lambda_{max}(M) = \max\{|\lambda_i(M)|, i = 1, 2, 3\} \quad (10)$$

3. UAV MODEL AND CONTROL STRATEGY

3.1 VTOL UAV model

The VTOL UAV is represented by a rigid body of mass m and of tensor of inertia I. To describe the motion of the UAV, two reference frames are introduced: an inertial reference frame $(\mathcal{I})$ associated with the vector basis $(e_1,\ e_2,\ e_3)$ and a body frame $(\mathcal{B})$ attached to the UAV and associated with the vector basis $(e_1^b,\ e_2^b,\ e_3^b)$. The position and the linear velocity of the UAV in $(\mathcal{I})$ are respectively denoted $\xi = [x\ y\ z]^T$ and $v = [v_x\ v_y\ v_z]^T$. The orientation of the UAV is given by the orientation matrix $R \in SO(3)$ from $(\mathcal{I})$ to $(\mathcal{B})$, usually parameterized by Euler's pseudo angles ψ, θ, ϕ (yaw, pitch, roll). Finally, let $\Omega = [\Omega_1\ \Omega_2\ \Omega_3]^T$ be the angular velocity of the UAV defined in $(\mathcal{B})$.

We assume that a translational force F and a control torque Γ are applied to the UAV. The translational force F combines thrust, lift, drag and gravity components. For a miniature VTOL UAV in quasi-stationary flight we can reasonably assume that the aerodynamic forces are always in direction e_3^b, since the lift force predominates the other components [7]. The gravity component mge_3 can be separated from other forces and the dynamics of the VTOL UAV are written as:

$$\begin{cases} \dot{\xi} = v \\ m\dot{v} = -\mathcal{T}\, Re_3 + mge_3 \\ \epsilon\dot{R} = R\,\Omega_\times \\ \epsilon I\dot{\Omega} = -\Omega \times I\Omega + \Gamma \end{cases} \quad (11)$$

where the parameter $0 < \epsilon \ll 1$ is introduced for time-scale separation.
The control inputs that will be considered are the scalar $\mathcal{T} \in \mathbb{R}$ representing the magnitude of the external forces applied in direction e_3^b, and the control torque $\Gamma = [\Gamma_1\ \Gamma_2\ \Gamma_3]^T$ defined in $(\mathcal{B})$.

3.2 Control Strategy

Let us consider a hierarchical control strategy for stabilization of model (11). Position and attitude controllers will be successively designed, as presented below.

For the translational dynamics of (11), the full vectorial term $\mathcal{T}Re_3$ will be considered as the position control vector. We will assign its desired value [1] $(\mathcal{T}Re_3)^d = f(\xi, v)$. Assuming that actuator dynamics can be neglected before the rigid body dynamics of the UAV, the value $\mathcal{T}^d$ is considered to be instantaneously reached by $\mathcal{T}$. Therefore, we have $(\mathcal{T}Re_3)^d = \mathcal{T}R^d e_3$, where R^d is the desired orientation of the vehicle. That vector can be split into its magnitude, $\mathcal{T} = \|f(\xi, v)\|$, representing the first control input, and its direction

$$R^d e_3 = \frac{1}{\mathcal{T}} f(\xi, v) \quad (12)$$

[1] In this paper, the function f will not depend on v, since only position measurements are available for the control of the translational dynamics.

representing the desired orientation [2].

For the orientation dynamics of (11), we will assign the control torque Γ such that the orientation R of the UAV converges to the desired orientation R^d, and such that the angular velocity Ω converges to Ω^d defined by:

$$\dot{R}^d = R^d \Omega^d_\times \tag{13}$$

The computation of that desired angular velocity Ω^d is presented in Appendix A.

4. POSITION CONTROLLER

Consider the translational dynamics of (11). We assume for control design, that only measurements on the position ξ are available. Let us introduce two virtual states $q, w \in \mathbb{R}^3$ and a virtual control $\delta \in \mathbb{R}^3$ such that:

$$\begin{cases} \dot{\xi} = v \\ \dot{v} = -\dfrac{\mathcal{T}}{m} R^d e_3 + g e_3 - \dfrac{\mathcal{T}}{m}(R - R^d) e_3 \\ \dot{q} = -w \\ \dot{w} = \delta \end{cases} \tag{14}$$

We define the position control law

$$\mathcal{T} R^d e_3 = \frac{m}{k_v} \{k_x\ \xi + k_1(\xi - q) + k_1(\xi - q + w)\} + mge_3 \tag{15}$$

and the virtual control

$$\delta = -w - (\xi - q) - (\xi - q + w) \tag{16}$$

where k_x, k_v, and k_1 are strictly positive scalar gains.

Remark 1. Note that the controller (15) and the virtual input (16) do not use measurements on the linear velocity v of the vehicle.

Remark 2. The control laws (15) and (16) have been designed by considering the translational dynamics of (11) under the assumption $R = R^d$. That assumption corresponds to a time-scale separation between the translational dynamics and the orientation dynamics.

Introducing the notations

$$\alpha = \xi - q, \qquad \beta = \xi - q + w \tag{17}$$

$$u = -\frac{\mathcal{T}}{m} R e_3 + g e_3, \quad u^d = -\frac{\mathcal{T}}{m} R^d e_3 + g e_3, \quad \tilde{u} = u - u^d \tag{18}$$

the system (14) controlled by (15) and (16) can be written as

$$\begin{cases} \dot{\xi} = v \\ \dot{v} = -\dfrac{k_x}{k_v}\xi - \dfrac{k_1}{k_v}\alpha - \dfrac{k_1}{k_v}\beta + \tilde{u} \\ \dot{\alpha} = v + \beta - \alpha \\ \dot{\beta} = v - \alpha - \beta \end{cases} \tag{19}$$

Defining the vectors $X = \left[\xi^T \quad v^T \quad \alpha^T \quad \beta^T\right]^T$ and $\tilde{U} = \left[0_3^T \quad \tilde{u}^T \quad 0_3^T \quad 0_3^T\right]^T$, with $0_3 = [0 \quad 0 \quad 0]^T$, the system (19) can be represented by:

$$\dot{X} = AX + \tilde{U} \tag{20}$$

where the matrix $A \in \mathbb{R}^{12\times 12}$ is Hurwitz [3]. Therefore, the system (20) is exponentially stable for $\tilde{U} = 0$. In that case, there exist two positive definite symmetric matrices P and Q verifying the Lyapunov equation

$$\frac{1}{2}(A^T P + PA) = -Q \tag{21}$$

and such that we can define a Control Lyapunov Function

$$\mathcal{S} = \frac{1}{2} X^T P X \tag{22}$$

which verifies

$$\frac{1}{2}\lambda_{min}(P)\|X\|^2 \le \mathcal{S} \le \frac{1}{2}\lambda_{max}(P)\|X\|^2 \tag{23}$$

$$\dot{\mathcal{S}} = -X^T Q X \le -\lambda_{min}(Q)\|X\|^2 \tag{24}$$

Consider now the case $\tilde{U} \neq 0$. The time derivative of $\mathcal{S}$ along the trajectories of (20) becomes

$$\dot{\mathcal{S}} = -X^T Q X + \tilde{U}^T P X \tag{25}$$

That expression can be bounded by

$$\begin{aligned} \dot{\mathcal{S}} \le & -\lambda_{min}(Q)\|X\|^2 \\ & + \lambda_{max}(P)\|\tilde{u}\|\{\|\xi\| + \|v\| + \|\alpha\| + \|\beta\|\} \end{aligned} \tag{26}$$

To determine an upper bound on $\|\tilde{u}\|$ we compute

$$\begin{aligned} \|\tilde{u}\| &= \frac{\mathcal{T}}{m}\left\|(R - R^d)e_3\right\| = \frac{\mathcal{T}}{m}\left\|(R^d R^T - I_d) R\, e_3\right\| \\ &\le \frac{\mathcal{T}}{m}\sqrt{\mathrm{tr}((R^d R^T - I_d)^T (R^d R^T - I_d))}\ \|R\, e_3\| \\ &\le \frac{\mathcal{T}}{m}\sqrt{2\mathrm{tr}(I_d - \tilde{R})} \end{aligned} \tag{27}$$

Let $(\gamma_{\tilde{R}}, n_{\tilde{R}})$ denote the angular-axis coordinates of $\tilde{R}$. Using identity (8), we get

$$\|\tilde{u}\| \le \frac{\sqrt{2}}{m}\frac{\mathcal{T}}{\cos(\frac{\gamma_{\tilde{R}}}{2})}\left\|V(P_a(\tilde{R}))\right\| \tag{28}$$

From (26), we finally get:

$$\begin{aligned} \dot{\mathcal{S}} \le & -\lambda_{min}(Q)\left\{\|\xi\|^2 + \|v\|^2 + \|\alpha\|^2 + \|\beta\|^2\right\} \\ & + (\sqrt{2}\frac{\mathcal{T}}{m}\frac{\lambda_{max}(P)}{\cos(\frac{\gamma_{\tilde{R}}}{2})})\left\|V(P_a(\tilde{R}))\right\|\{\|\xi\| + \|v\| + \|\alpha\| + \|\beta\|\} \end{aligned} \tag{29}$$

5. ATTITUDE CONTROLLER

Let us now consider the orientation dynamics of (11) and define

$$\tilde{R} = (R^d)^T R \tag{30}$$

The orientation dynamics can be rewritten as

$$\begin{cases} \epsilon\dot{\tilde{R}} = -\epsilon\Omega^d_\times \tilde{R} + \tilde{R}\Omega_\times \\ \epsilon I\dot{\Omega} = -\Omega_\times I \Omega + \Gamma \end{cases} \tag{31}$$

We introduce

$$\tilde{\Omega} = \Omega - l_1 V(P_a(\tilde{R})^T) \tag{32}$$

where l_1 is a strictly positive scalar gain. With that notation, the kinematic relation can be transformed into

$$\dot{\tilde{R}} = -\Omega^d_\times \tilde{R} + \frac{1}{\epsilon}\tilde{R}\tilde{\Omega}_\times + \frac{l_1}{\epsilon}\tilde{R}P_a(\tilde{R})^T \tag{33}$$

[2] The desired orientation R^d can then be deduced from (12), using the pseudo Euler angle parametrization of R^d and solving for (ψ^d, θ^d, ϕ^d) for a given specified constant yaw $\psi^d(t) = \psi^d(0)$ [8].

[3] Using the fact that the gains k_x, k_v and k_1 are strictly positive, it can be easily checked that the matrix A is Hurwitz, by applying Routh's criterion on its characteristic polynomial.

Assuming that the tensor of inertia I is invertible, the time derivative of $\tilde{\Omega}$ can be expressed as

$$\dot{\tilde{\Omega}} = \frac{1}{\epsilon} I^{-1}(-\Omega_\times I\Omega) + \frac{1}{\epsilon} I^{-1}\Gamma - \frac{l_1}{2} V(\tilde{R}^T\Omega^d_\times + \Omega^d_\times \tilde{R}) + \frac{l_1}{2\epsilon} V(\Omega_\times \tilde{R}^T + \tilde{R}\Omega_\times) \quad (34)$$

By choosing the control torque

$$\Gamma = \Omega_\times I\Omega + I(-l_2\tilde{\Omega} - 2lV(P_a(\tilde{R})) - \frac{l_1}{2} V(\Omega_\times \tilde{R}^T + \tilde{R}\Omega_\times)) \quad (35)$$

with $l > 0$ and $l_2 > 0$, equation (34) becomes:

$$\dot{\tilde{\Omega}} = -\frac{l_2}{\epsilon}\tilde{\Omega} - 2\frac{l}{\epsilon} V(P_a(\tilde{R})) - \frac{l_1}{2} V(\tilde{R}^T\Omega^d_\times + \Omega^d_\times \tilde{R}) \quad (36)$$

Remark 3. The control law (35) has been designed by considering the orientation dynamics of (11) under the assumption $\Omega^d = 0$, which corresponds to a time-scale separation between the translational and the orientation dynamics.

Let $\mathcal{L}$ be a candidate Control Lyapunov Function for the orientation dynamics (31):

$$\mathcal{L} = l\,\mathrm{tr}(I_d - \tilde{R}) + \frac{1}{2}\left\|\tilde{\Omega}\right\|^2 \quad (37)$$

We use relations (33) and (34), and identities (4) and (5) to compute the time derivative of $\mathcal{L}$ along the trajectories of (31) controlled by (35). We get:

$$\dot{\mathcal{L}} = -2l\,(\Omega^d)^T V(P_a(\tilde{R})) - \frac{l\,l_1}{\epsilon}\mathrm{tr}(P_a(\tilde{R})P_a(\tilde{R})^T) - \frac{l_2}{\epsilon}\left\|\tilde{\Omega}\right\|^2 - \frac{l_1}{2}\tilde{\Omega}^T V(\tilde{R}^T\Omega^d_\times + \Omega^d_\times \tilde{R}) \quad (38)$$

By triangular inequality and applying identity (6), we obtain

$$\dot{\mathcal{L}} \leq 2l\left\|\Omega^d\right\|\left\|V(P_a(\tilde{R}))\right\| - \frac{2l\,l_1}{\epsilon}\left\|V(P_a(\tilde{R}))\right\|^2 - \frac{l_2}{\epsilon}\left\|\tilde{\Omega}\right\|^2 + \frac{l_1}{2}\left\|\tilde{\Omega}\right\|\left\|V(\tilde{R}^T\Omega^d_\times + \Omega^d_\times \tilde{R})\right\| \quad (39)$$

To get an upper bound on $\mu = \left\|V(\tilde{R}^T\Omega^d_\times + \Omega^d_\times \tilde{R})\right\|$, we compute

$$\begin{aligned}\mu^2 &\leq \frac{1}{2}\mathrm{tr}\left\{(\tilde{R}^T\Omega^d_\times + \Omega^d_\times\tilde{R})^T(\tilde{R}^T\Omega^d_\times + \Omega^d_\times\tilde{R})\right\} \\ &\leq \frac{1}{2}\mathrm{tr}((\tilde{R}^T\Omega^d_\times)^T\tilde{R}^T\Omega^d_\times) + \frac{1}{2}\mathrm{tr}((\Omega^d_\times\tilde{R})^T\Omega^d_\times\tilde{R}) \\ &\leq \mathrm{tr}((\Omega^d_\times)^T\Omega^d_\times) \leq 2\left\|\Omega^d\right\|^2\end{aligned} \quad (40)$$

It remains to find an upper bound on $\left\|\Omega^d\right\|$. In the case of stabilization, we choose $\Omega^d_3 = 0$. We get $\left\|\Omega^d\right\| = \left\|\Omega^d_\times e_3\right\|$ and can use (A.8) along with the time-derivative of (15) to obtain:

$$\left\|\Omega^d\right\| \leq \frac{m}{\mathcal{T}}\frac{1}{k_v}\left\{(k_x + 2k_1)\left\|v\right\| + 2k_1\left\|\alpha\right\|\right\} \quad (41)$$

Using (40) and (41) along with (39) leads finally to the following upper bound on the time derivative of $\mathcal{L}$:

$$\dot{\mathcal{L}} \leq -\frac{2l\,l_1}{\epsilon}\left\|V(P_a(\tilde{R}))\right\|^2 - \frac{l_2}{\epsilon}\left\|\tilde{\Omega}\right\|^2 + \frac{2m}{\mathcal{T}}\frac{l}{k_v}\left\|V(P_a(\tilde{R}))\right\|\left\{(k_x + 2k_1)\left\|v\right\| + 2k_1\left\|\alpha\right\|\right\} + \frac{\sqrt{2}}{2}\frac{m}{\mathcal{T}}\frac{l_1}{k_v}\left\|\tilde{\Omega}\right\|\left\{(k_x + 2k_1)\left\|v\right\| + 2k_1\left\|\alpha\right\|\right\} \quad (42)$$

6. STABILITY ANALYSIS

Consider now the complete system composed of the translational dynamics (14) and of the orientation dynamics (31), and define the candidate Control Lyapunov Function

$$\mathcal{V} = \mathcal{S} + \mathcal{L} \quad (43)$$

We have the following proposition :

Proposition 1. Consider the system (14)-(31) along with the control laws (15) and (35) and the virtual input (16). There exist $K_1, K_2 > 0$ and $\epsilon^* > 0$ such that,

for all initial conditions $\xi(0)$, $v(0)$, $q(0) = \xi(0)$, $w(0) = 0$, $R(0)$ and $\Omega(0)$ such that

$$\mathcal{V}(0) < \frac{K_2(g - \frac{\epsilon_g}{m})^2}{2(3\frac{K_1}{k_v})^2} \qquad (0 < \epsilon_g \ll mg) \quad (44)$$

then, for all l verifying

$$l \geq \frac{K_2(g - \frac{\epsilon_g}{m})^2}{2(3\frac{K_1}{k_v})^2(4-\eta)} \qquad (0 < \eta < 4) \quad (45)$$

and for all $\epsilon > 0$ such that $\epsilon < \epsilon^*$,
the closed loop system is exponentially stable.

Proof
First, let us consider the following assumptions that will be verified at the end of the proof:

Assumption 1. There exist two reals $\mathcal{T}_{min}$ and $\mathcal{T}_{max}$ such that

$$0 < \mathcal{T}_{min} < mg < \mathcal{T}_{max} < \infty \quad (46)$$

$$\forall t \geq 0, \quad \mathcal{T}_{min} \leq \mathcal{T}(t) \leq \mathcal{T}_{max} \quad (47)$$

Assumption 2. There exists a real $c > 0$ such that

$$\forall t \geq 0, \quad \cos(\frac{\gamma_{\tilde{R}}(t)}{2}) \geq c \quad (48)$$

Let us define the coefficients

$$s_1 = \frac{1}{2}\lambda_{max}(P)\frac{\mathcal{T}_{max}}{m}\frac{\sqrt{2}}{c}, \qquad s_2 = \frac{l}{\mathcal{T}_{min}}\frac{m}{k_v} \quad (49)$$

$$s_3 = \frac{\sqrt{2}}{4}\frac{l_1}{k_v}\frac{m}{\mathcal{T}_{min}} \quad (50)$$

With these notations and under Assumptions (1) and (2), we can use relations (29) and (42), to give the following upper bound on the time derivative of $\mathcal{V}$, computed along the trajectories of (14)-(31) controlled by (15),(35) and (16):

$$\begin{aligned}\dot{\mathcal{V}} \leq & -\lambda_{min}(Q)\left\{\|\xi\|^2+\|v\|^2+\|\alpha\|^2+\|\beta\|^2\right\} \\ & -\frac{2l\,l_1}{\epsilon}\left\|V(P_a(\tilde{R}))\right\|^2-\frac{l_2}{\epsilon}\left\|\tilde{\Omega}\right\|^2 \\ & +2s_1\|\xi\|\left\|V(P_a(\tilde{R}))\right\| \\ & +2(s_1+s_2(k_x+2k_1))\|v\|\left\|V(P_a(\tilde{R}))\right\| \\ & +2(s_1+2s_2k_1)\|\alpha\|\left\|V(P_a(\tilde{R}))\right\| \\ & +2s_1\|\beta\|\left\|V(P_a(\tilde{R}))\right\|+2s_3(k_x+2k_1)\|v\|\left\|\tilde{\Omega}\right\| \\ & +4s_3k_1\|\alpha\|\left\|\tilde{\Omega}\right\|\end{aligned} \tag{51}$$

Let us define

$$a=\lambda_{min}(Q)\ ,\quad b_1=s_1\ ,\quad b_2=s_1+s_2(k_x+2k_1) \tag{52}$$

$$b_3=s_1+2s_2k_1\ ,\quad b_4=s_3(k_x+2k_1)\ ,\quad b_5=2s_3k_1 \tag{53}$$

and introduce the state vector

$$\mathcal{X}=\begin{bmatrix}\xi^T & v^T & \alpha^T & \beta^T & V(P_a(\tilde{R}))^T & \tilde{\Omega}^T\end{bmatrix}^T \tag{54}$$

With these notations, equation (51) can be restated as

$$\dot{\mathcal{V}} \leq -\mathcal{X}^T\Sigma\,\mathcal{X} \tag{55}$$

The term $-\mathcal{X}^T\Sigma\,\mathcal{X}$ is negative definite if and only if the following matrix σ is positive definite:

$$\sigma=\begin{bmatrix} a & 0 & 0 & 0 & -b_1 & 0 \\ 0 & a & 0 & 0 & -b_2 & -b_4 \\ 0 & 0 & a & 0 & -b_3 & -b_5 \\ 0 & 0 & 0 & a & -b_1 & 0 \\ -b_1 & -b_2 & -b_3 & -b_1 & \dfrac{2l\,l_1}{\epsilon} & 0 \\ 0 & -b_4 & -b_5 & 0 & 0 & \dfrac{l_2}{\epsilon}\end{bmatrix} \tag{56}$$

Since the matrix Q is positive definite, the coefficient $a=\lambda_{min}(Q)$ is strictly positive and the four first minors of the matrix σ are strictly positive. The positivity of the minor of size five is obtained for $\epsilon<\epsilon_1^*$ with

$$\epsilon_1^*=\frac{2\lambda_{min}(Q)\,l\,l_1}{4s_1^2+s_2^2(8k_1^2+k_x^2+4k_xk_1)+2s_1s_2(4k_1+k_x)} \tag{57}$$

The strict positivity of $\det(\sigma)$ is obtained for

$$A\epsilon^2+B\epsilon+C>0 \tag{58}$$

where

$$A=a^2s_3^2s_1^2(3k_x^2+8k_xk_1+16k_1^2)>0 \tag{59}$$

$$\begin{aligned}B=-a^3(&4l_2s_1^2+8l_2s_1s_2k_1+8l_2s_2^2k_1^2+2s_3^2ll_1k_x^2 \\ &+8s_3^2ll_1k_xk_1+16s_3^2k_1^2ll_1+2l_2s_1s_2k_x \\ &+l_2s_2^2k_x^2+4l_2s_2^2k_xk_1)<0\end{aligned} \tag{60}$$

$$C=2l_2ll_1a^4>0 \tag{61}$$

With these coefficients, it can be checked that the discriminant (B^2-4AC) of (58) is strictly positive. Let us define

$$\epsilon_2^*=\frac{-B-\sqrt{B^2-4AC}}{2A} \tag{62}$$

which is strictly positive since $A>0$, $B<0$, $C>0$ and $(B^2-4AC)>0$. Hence, $\det(\sigma)$ is strictly positive for $\epsilon<\epsilon_2^*$.
Let us define

$$\epsilon^*=\min(\epsilon_1^*,\epsilon_2^*) \tag{63}$$

For all $\epsilon>0$ such that $\epsilon<\epsilon^*$, the time derivative (55) of $\mathcal{V}$ is negative definite and we can ensure [4] the exponential stability of the system (14)-(31) when (15) and (35) are used as control inputs and (16) as virtual control.

We have shown that closed loop stability is guaranteed for all $\epsilon<\epsilon^*$ under Assumptions 1 and 2. Now we have to check that both assumptions are satisfied.

Let us start with Assumption 1. Define $K_1=\max(k_x,k_1)$. Using triangular inequality with (15) yields:

$$\begin{aligned} mg-\frac{m}{k_v}K_1(\|\xi\|+\|\alpha\|+\|\beta\|) &\leq \mathcal{T} \\ \mathcal{T} \leq\ mg+\frac{m}{k_v}K_1(\|\xi\|+\|\alpha\|+\|\beta\|) &\end{aligned} \tag{64}$$

That expression can be linked to the value of the Lyapunov function $\mathcal{V}$ using (22),(23) and (43) to get for all $t\geq 0$:

$$mg-3\frac{m}{k_v}K_1\sqrt{\frac{2\mathcal{V}(t)}{K_2}}\leq\mathcal{T}(t)\leq\ mg+3\frac{m}{k_v}K_1\sqrt{\frac{2\mathcal{V}(t)}{K_2}} \tag{65}$$

with $K_2=\lambda_{min}(P)$.
The time derivative of $\mathcal{V}$ being negative for $\epsilon<\epsilon^*$, one has

$$\forall t\geq 0,\quad \mathcal{V}(t)\leq\mathcal{V}(0) \tag{66}$$

and from (65), we obtain for all $t\geq 0$:

$$mg-3\frac{m}{k_v}K_1\sqrt{\frac{2\mathcal{V}(0)}{K_2}}\leq\mathcal{T}(t)\leq\ mg+3\frac{m}{k_v}K_1\sqrt{\frac{2\mathcal{V}(0)}{K_2}} \tag{67}$$

Taking a $\epsilon_g>0$ such that $\epsilon_g\ll mg$, we can use condition (44) to finally get

$$\forall t\geq 0,\quad 0<\epsilon_g<\mathcal{T}(t)<2mg-\epsilon_g \tag{68}$$

Assumption 1 is hence verified by choosing $\mathcal{T}_{min}=\epsilon_g$ and $\mathcal{T}_{max}=2mg-\epsilon_g$.

To complete the proof, let us finally check that Assumption 2 is verified. As previously, we use the fact that $\mathcal{V}$ is decreasing, with (37) and (43), to obtain

$$\forall t\geq 0,\quad l\,\mathrm{tr}(I_d-\tilde{R}(t))\leq\mathcal{V}(t)\leq\mathcal{V}(0) \tag{69}$$

Defining a $\eta>0$ such that $\eta<4$, conditions (44) and (45) can be used successively to get:

$$\mathcal{V}(0)<(4-\eta)l \tag{70}$$

and then

$$\forall t\geq 0,\quad \mathrm{tr}(I_d-\tilde{R}(t))<4-\eta \tag{71}$$

Using (7) we obtain

$$\forall t\geq 0,\quad (1-\cos(\gamma_{\tilde{R}}(t)))<2 \tag{72}$$

Therefore, for all $t\geq 0$, we have $-\pi<\gamma_{\tilde{R}}(t)<\pi$ and there exists a $c>0$ such that

$$\cos(\frac{\gamma_{\tilde{R}}(t)}{2})\geq c>0 \tag{73}$$

Assumption 2 is hence verified, which completes the proof. ■

Remark 4. Since assumptions (46) and (47) are verified, the strict positivity of the input $\mathcal{T}$ is guaranteed. Therefore, the direction R^de_3 computed by (12) is well defined.

[4] The convergence of $\tilde{R}$ to the identity matrix I_d is guaranteed by conditions (44) and (45) from which we can show that $(1-\cos(\gamma_{\tilde{R}}))<2$ and hence $\gamma_{\tilde{R}}\to 0$. That relation will be shown in the next step of the proof.

Remark 5. Condition (44) is not restrictive. Indeed, in practice, the gains k_x, k_1 and the matrix P can be chosen to obtain respectively sufficient small and high values for K_1 and K_2, so that all initial conditions in the usual domain of flight of the vehicle will satisfy (44).

7. CONCLUSION

In this paper, we have presented both design and stability analysis of a hierarchical controller for a miniature VTOL UAV. Position and attitude controllers have been designed considering successively, and with a time-scale separation, the translational dynamics and the orientation dynamics of a six degrees of freedom VTOL UAV model. A partial state feedback controller has been proposed for position stabilization, assuming that no measurement of the linear velocity of the vehicle is available. Time-scale separation of the proposed control scheme and stability analysis have been addressed by singular perturbation theory.

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Appendix A. COMPUTATION OF THE DESIRED ANGULAR VELOCITY

A method to compute Ω^d from the control vector $\mathcal{T}R^de_3$ is presented here. From (13) we get

$$\frac{d}{dt}(R^de_3) = \dot{R}^de_3 = R^d\Omega^d_\times e_3 \tag{A.1}$$

and then

$$\Omega^d_\times e_3 = (R^d)^T\frac{d}{dt}(R^de_3) \tag{A.2}$$

To compute the time derivative of R^de_3, let us define

$$N = \mathcal{T}R^de_3 \tag{A.3}$$

so that we get

$$R^de_3 = \frac{N}{\sqrt{N^TN}} \tag{A.4}$$

The time derivative of R^de_3 is given by

$$\frac{d}{dt}(R^de_3) = \frac{\dot{N}\sqrt{N^TN} - \frac{NN^T\dot{N}}{\sqrt{N^TN}}}{N^TN} = \frac{1}{\sqrt{N^TN}}(I_d - \frac{NN^T}{N^TN})\dot{N} \tag{A.5}$$

Therefore, we have

$$\frac{d}{dt}(R^de_3) = \frac{1}{\mathcal{T}}\left\{I_d - R^de_3e_3^T(R^d)^T\right\}\frac{d}{dt}(\mathcal{T}R^de_3) \tag{A.6}$$

Defining the projector

$$\Pi_{e_3} = I_d - e_3e_3^T \tag{A.7}$$

equation (A.2) can be restated as

$$\Omega^d_\times e_3 = \begin{bmatrix}\Omega^d_2\\ -\Omega^d_1\\ 0\end{bmatrix} = \frac{1}{\mathcal{T}}\Pi_{e_3}(R^d)^T\frac{d}{dt}(\mathcal{T}R^de_3) \tag{A.8}$$

Considering the stabilization of the UAV around a fixed point, the third component Ω^d_3 of the vector Ω^d is chosen to be identically zero, and we have $\psi^d(t) = \psi^d(0)$, for a given initial yaw $\psi^d(0)$.