Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Model and Algorithm for the Vendor-Warehouse Transportation and Inventory Problem in a Three-Level Distribution System

Jianxiang Li*[1,2], Feng Chu[2], Haoxun Chen[2]

1. Department of Automatic Control, Beijing Institute of Technology, Beijing 100081, China
2. Industrial Systems Optimization Laboratory, University of Technology of Troyes, Troyes 10010, France
*(*Corresponding Author: e-mail: jianxiangli@bit.edu.cn)*

Abstract: We consider the inventory-routing problem in a three-level distribution system with a single vendor, a single warehouse and many geographically dispersed retailers. In this problem, each retailer faces a demand at a deterministic, retailer-specific rate. The demand of each retailer is replenished either from the warehouse by a small vehicle or from the vendor bypassing the warehouse by a big vehicle. Inventories are kept not only at the retailers but also at the warehouse. The objective is to find a combined inventory policy and routing pattern minimizing a long-run average system-wide cost while meeting the demand of each retailer without shortage. We present an efficient solution approach based on a fixed partition policy where the retailers are partitioned into disjoint and collectively exhaustive sets and each set of retailers is served on a separate route. Given a fixed partition, the original problem is decomposed into three subproblems. In this paper, we focus on the modelling and resolution of the vendor-warehouse transportation and inventory subproblem. We demonstrate that the subproblem can be reduced to a C/C/C/Z capacitated dynamic lot sizing problem and there exists an algorithm to solve the reduced problem to optimality in $O(T^2)$ time.

Key words: Transportation logistics; Inventory routing; Inventory; Multi-echelon; Lot sizing

1. INTRODUCTION

Nowadays, more and more companies are aware that great cost savings can be achieved by integrating inventory control and vehicle routing into a cost-effective strategy for their distribution systems, especially for Vendor Managed Inventory (VMI) systems (Campbell and Savelsbergh, 2004). Determining such a cost-effective distribution strategy is known as Inventory Routing Problem (IRP).

IRP has been successfully applied to many industrial sectors, such as retailer industries, oil and gas industries, clothing industries (Adelman, 2004; Cambelll and Savelsbergh, 2004; Gaur and Fisher, 2004). It also has been attracting the attention of academic communities. The literature on IRP distinguishes between single-period deterministic, multi-period deterministic, infinite-horizon deterministic, single-period stochastic, multi-period stochastic and infinite-horizon stochastic models according to different time horizons and demand types (Cambelll and Savelsbergh, 2004; Yu et al., 2007).

Specifically, this paper considers an infinite-horizon deterministic IRP for a three-level distribution system with a single outside vendor, a single warehouse and many geographically dispersed retailers. In this problem, each retailer faces an external demand for a single product with a deterministic, retailer specific rate. The demand of each retailer is replenished either from the vendor through the warehouse by a "small" vehicle of limited capacity or directly from the vendor bypassing the warehouse by a "big" vehicle of limited capacity. Inventories are kept not only at the retailers but also at the warehouse. The objective is to determine a combined inventory policy and routing strategy minimizing a long-run average system-wide cost while meeting each retailer's demand without shortage.

Most of existing literature focuses on the IRP for two-level distribution systems, also called one-warehouse multi-retailer distribution systems, see e.g., Anily and Bramel (2004), Anily and Federgruen (1990), Bramel and Simchi-Levi (1995), Burns et al. (1985), Chan et al. (1998), and Jung and Mathur (2007). The literature on the IRP for three-level distribution systems is rather limited. To the best of our knowledge, only Chan and Simchi-Levi (1998) and Zhao et al. (2007) are exceptional. Zhao et al. (2007) consider a simpler three-level distribution system where the inventory of the warehouse is replenished from the vendor by a single train with a large capacity. They propose a solution strategy integrating a Fixed Partition Policy (FPP) and a Power-Of-Two (POT) policy. In the strategy, firstly an FPP is implemented, i.e., the retailers are partitioned into disjoint and collectively exhaustive sets and each set of retailers is served on a separate route (Bramel and Simchi-Levi, 1995), and then a POT policy is implemented, i.e., each set of retailers and the warehouse are restricted to be visited at a replenishment interval which is power of two times a basic planning period (Roundy, 1985). The distribution system considered in Chan and Simchi-Levi (1998) is identical to the one considered in this paper, although it includes multiple warehouses. To solve the complex IRP in the three-level distribution problem, they present a solution approach that decomposes the problem into two subproblems: the warehouse-retailer transportation and inventory subproblem including a transportation decision from the warehouse to the retailers and an inventory decision at the retailer, and the vendor-warehouse transportation and inventory subproblem including a transportation decision from the vendor to the warehouse and an inventory decision at the warehouse. The former subproblem is solved based on an FPP. The latter subproblem is then solved based on the restriction to the cross-docking strategy in which the warehouse acts as a coordinator of the supply process and as a transhipment point for incoming orders from the vendor but not keep stock itself. The authors prove that the cross-docking strategy is asymptotically optimal.

As pointed out by Jung and Mathur (2007), however, for a finite number of retailers, if the inventory holding cost rate at the warehouse is relatively small to the one at the retailers, it may be profitable to keep inventory also at the warehouse. Moreover, in Chan and Simchi-Levi (1998), it is assumed that all shipments are delivered from the vendors to the retailers through the warehouse. In Li et al. (2007b), we prove that in certain conditions a strategy in which shipments are delivered from the vendor to the retailers bypassing the warehouse has a higher asymptotic optimality, and conclude that a hybrid strategy, i.e., combing the strategy bypassing the warehouse with the strategy passing the warehouse, should be used in three-level distribution systems with a limited number of retailers.

In this paper, the restriction to the cross-docking strategy is relaxed, i.e., the warehouse is allowed to keep inventories. The restriction to the strategy passing the warehouse is also relaxed, i.e., a retailer is allowed to be delivered directly from the vendor bypassing the warehouse. For simplicity, we assume that each shipment from the vendor directly to the retailers serves only one retailer. The assumption is reasonable in practice.

The remainder of this paper is organized as follows. In Section 2, we describe formally the IRP in the three-level distribution system and introduce the notation used. In Section 3, we present a solution approach for the problem, in which the problem is decomposed into three subproblems. For lack of space, in Section 4, we focus on modelling the vendor-warehouse transportation and inventory subproblem and discussing its solution algorithms. Section 5 concludes this paper.

2. THE IRP AND THE NOTATION

In the three-level distribution system considered, there are a single outside vendor, a single warehouse and N geographically dispersed retailers (see Fig. 1 for illustration). Each retailer faces an external demand for a single product with a deterministic, retailer specific rate D_i ($i \in \{1,2,...,N\}$). The vendor with an unlimited supply of the product serves the warehouse using "big" vehicles of limited capacity Q. The warehouse serves the retailers using "small" vehicles of limited capacity q. The vendor can also serve directly the retailers using big vehicles; however, it is assumed that a big vehicle departing from the vendor directly to the retailers serves only one retailer. In addition, it is assumed that split delivery is not allowed, i.e., every retailer is served by only one vehicle (Dror and Trudeau, 1989). Consequently, the demand of each retailer is replenished either from the vendor through the warehouse or directly from the vendor bypassing the warehouse. Each time a big (small) vehicle is sent out to replenish inventory to the warehouse or a set of retailers, it incurs a fixed cost C (c) plus a variable cost proportional to the total distance travelled, where the variable transportation cost per unit distance of a big (small) vehicle is U (u). A linear inventory holding cost at a constant rate h (h_0) is charged at each retailer (the warehouse) whenever stocks are kept there. The frequency in which a given retailer can be visited is bounded from above by f. The objective is to determine a combined transportation (routing) and inventory policy minimizing a long-run average system-wide cost including the transportation cost from the vendor to the warehouse, the transportation cost from the vendor to the retailers, and the transportation cost from the warehouse to the retailers, the inventory cost at the warehouse and the inventory cost at the retailers, while meeting each retailer's demand without shortage or backlogging.

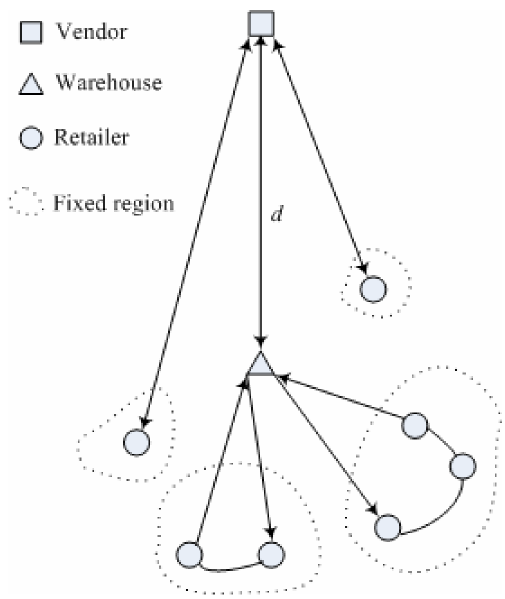

Fig. 1. The three-level distribution system

The other notation used in this paper is introduced as follows.

t	Index of time period, $t = 0,1,...,T-1$, where T is a cycle period and how to determine T will be discussed in Section 4
χ	A fixed partition of the set of the retailers under an FPP, $\chi = \{1,2,...,L\}$, where χ excludes the retailers to be replenished by direct shipping from the vendor
l	Index of fixed region, $l \in \chi$
T_l	Replenishment interval of region l,
Q_l	Replenishment quantity of region l in each replenishment interval
d	Distance from the vendor to the warehouse
$D(t)$	Total demand of the retailers (i.e., the demand of the warehouse) in period t

$Q(t)$ Delivery quantity from the vendor to the warehouse in period t

$I(t)$ Inventory level at the warehouse at the end of period t, without loss of generality, assume initial inventory level is zero

3. THE SOLUTION APPROACH FOR THE IRP

The IRP for the three-level distribution system is NP-hard since it is more complex than the NP-hard IRP for two-level distribution systems. It is therefore impossible to find an algorithm that can solve the problem with a number of retailers to optimality in an acceptable computation time. As an alternative, we develop an approach based on an FPP as in most literature. The approach combining with a metaheuristic such as tabu search or genetic algorithm can find a near-optimal solution for the problem. The metaheuristic is used for the improvement of the fixed partition. In the approach, at each iteration, a fixed partition is generated in which the retailers are partitioned into fixed regions. Under the fixed partition, the original problem is decomposed into three subproblems: the option to direct shipping subproblem in which a retailer is to be decided whether is replenished directly by the vendor bypassing the warehouse (subproblem 1), the warehouse-retailer transportation and inventory subproblem (subproblem 2), and the vendor-warehouse transportation and inventory subproblem (subproblem 3). Subproblem 1 and subproblem 2 are first solved, then subproblem 3. The fixed partition in the next iteration may be generated by exchanging two retailers between two regions or moving a retailer from one region to another region if we use local search based metaheuristic such as tabu search. If we use a genetic algorithm for fixed partition improvement, at the first iteration, multiple fixed partitions may be generated and the fixed partitions in the next iteration may be generated by crossover and/or mutation operations. The solution procedure is repeated until no further improvement is possible. The framework of the approach is illustrated in Fig. 2, where *TC* is the total cost of the three subproblems. The fixed partition policy and three subproblems are described in more detail as follows.

(1) *Fixed Partition Policy:* The proposed solution approach is based on an FPP whose task is to partition the retailers into regions to be replenished by a single vehicle. Dozens of literature are available on FPP, see e.g., Anily and Bramel (2004), Bramel and Simchi-Levi (1995), Chan et al. (1998), Chan and Simchi-Levi (1998), Jung and Mathur (2007), and Zhao et al. (2007). Moreover, the literature on Vehicle Routing Problem (VRP) can also provide useful references. Due to the complexity of the problem, it is necessary to find an efficient approach for quickly generating good fixed partitions. Some metaheuristics (e.g., genetic algorithms) or variable neighbourhood search algorithms are potential good approaches. This is one of our further research topics.

(2) *The option to direct shipping subproblem (subproblem 1)*: The task of the subproblem is to determine a retailer whether is replenished directly by the vendor bypassing the warehouse (otherwise is replenished by the vendor through the warehouse). Given a fixed partition, one can determine that a retailer is replenished directly by the vendor whenever a fixed region includes only the retailer and the demand rate of the retailer is large enough, e.g., the demand rate of the retailer is larger than the largest possible capacity of a small vehicle *qf* or is close to the largest possible capacity of a big vehicle *Qf*. Particularly if $Q \geq 2q$, one can determine whether a retailer is replenished directly by the vendor by using an explicit formula evaluating the effectiveness of direct shipping, which is derived by us in Li et al. (2007a). That is, for any retailer with a demand rate larger than *qf*, its demand must be satisfied by direct shipping from the vendor, whereas for any retailer with a demand rate less than or equal to *qf*, its demand must be replenished by the warehouse. The detail is omitted here. Once the retailers to be replenished directly by the vendor are determined, the optimal replenishment interval and replenishment quantity in each replenishment interval for each of these retailers can be easily computed, so do the transportation costs from the vendor to these retailers and the inventory costs at these retailers.

(3) *The warehouse-retailer transportation and inventory subproblem (subproblem 2)*: Under a given fixed partition, the subproblem is to determine the optimal replenishment interval and replenishment quantity in each replenishment interval for each region to be replenished from the warehouse by a single vehicle. This subproblem is easy to solve based on the solution to its corresponding Travelling Salesman Problem (TSP), so the transportation costs from the warehouse to the retailers and the inventory holding costs at the retailers can be computed easily.

(4) *The vendor-warehouse transportation and inventory subproblem (subproblem 3)*: The subproblem is to determine simultaneously the transportation decision between the vendor and the warehouse and the inventory decision at the warehouse, i.e., to determine the delivery quantity from the vendor to the warehouse and the inventory level at the warehouse in every period, with the objective to minimize long-run average total transportation and inventory costs while meeting the retailers' demands without shortage. The results of subproblem 2 are input parameters of this subproblem. That is, when the replenishment interval and replenishment quantity in each replenishment interval for each region is known, the demand of the warehouse in each period is determined. To solve this subproblem, Chan and Simchi-Levi (1998) and Zhao et al. (2007) consider a specific strategy, cross-docking and power-of-two polices respectively, and thus obtain only a suboptimal solution of the subproblem. In this paper, we solve the subproblem to optimality based on the following important finding: the subproblem can be reduced to a C/C/C/Z capacitated dynamic lot size problem in a certain condition. The notation $\alpha / \beta / \gamma / \delta$ introduced by Bitran and Yanassee (1982) represents a specific family of dynamic lot size problems, where $\alpha, \beta, \gamma, \delta$ specify respectively the time structure of the setup costs, holding costs, production costs, and production capacities, and may be taken the following letters: *G*, *C*, *ND*, *NI*, *Z* to indicate arbitrary pattern, constant, nondescreasing, nonincreasing, and zero. Once the subproblem is solved, the corresponding transportation costs from the vendor to the

warehouse and the corresponding inventory holding costs at the warehouse can be known.

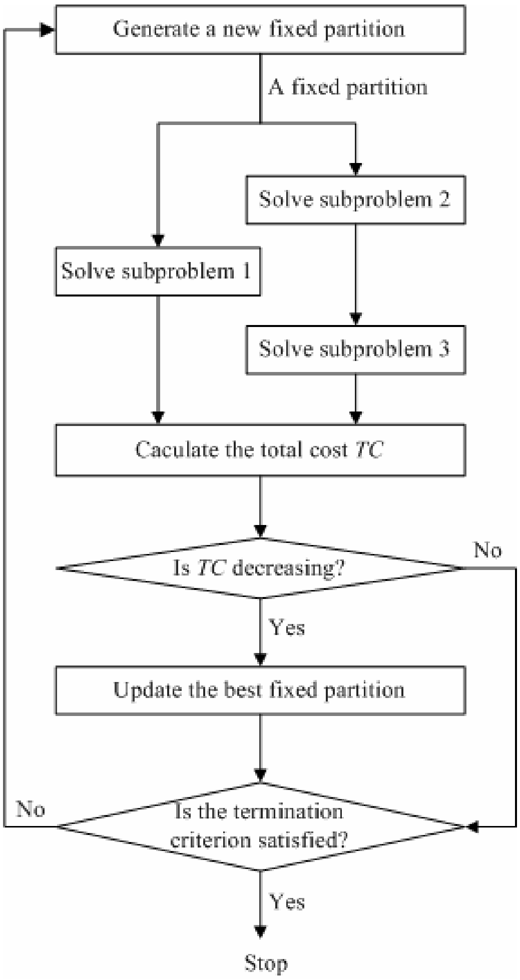

Fig. 2. Framework of the proposed solution approach

4. THE VENDOR-WAREHOUSE TRANSPORTATION AND INVENTORY SUBPROBELM

T is a cycle period, i.e., $D(t+T)=D(t)$, $Q(t+T)=Q(t)$, $I(t+T)=I(t)$. Without loss of generality, only the situations during T are needed to be considered for the vendor-warehouse transportation and inventory subproblem with an infinite-horizon.

4.1 Model

Given a fixed partition $\chi=\{1,2,...,L\}$, the vendor-warehouse transportation and inventory subproblem can be formulated as follows.

$$\text{P}: \qquad \min[\sum_{t=0}^{T-1}\lceil Q(t)/Q\rceil(C+2dU)+\sum_{t=0}^{T-1}I(t)h_0]/T\,, \qquad (1)$$

S.t.

$$I(t)=\sum_{\tau=0}^{t}Q(\tau)-\sum_{l=1}^{L}(1+\lfloor t/T_l\rfloor)Q_l\,, \quad t=0,1,...,T-1\,, \quad (2)$$

$$Q(t), I(t)\ge 0\,, \qquad t=0,1,...,T-1\,, \quad (3)$$

The objective function is to minimize the sum of the transportation cost from the vendor to the warehouse and the inventory cost of the warehouse. Constraints (2) ensure the flow balance of the warehouse, i.e., the inventory level equals the cumulative delivery quantities minus the cumulative demands. Constraints (3) are variable domain constraints.

In what follows we discuss properties of optimal solutions of the model and prove that the problem (P) is a C/C/C/C capacitated dynamic lot size problem. Without loss of generality we assume that deliveries are used to satisfy demand in a first-in-first-out basis. $D(t)$ can be computed by Equations (4).

$$D(t)=\sum_{l=1}^{L}(1+\lfloor t/T_l\rfloor)Q_l-\sum_{l=1}^{L}(1+\lfloor (t-1)/T_l\rfloor)Q_l\,, \qquad t=0,1,...,T-1\,. \quad (4)$$

Without loss of generality, let $D(t)=k_tQ+q_t$, $k_t\ge 0$ and integer, $0\le q_t<Q$, $\forall t$.

4.2 Properties of Optimal Solutions

Proposition 1: $I(t-1)*(Q(t)\,\%\,Q)=0$, $\forall t$.

Proof: by contradiction. Assume there is an optimal solution: $\exists\tau$, $I(\tau-1)>0$ and $Q(\tau)=kQ+x$, where $k\ge 0$ and integer, $0<x<Q$. It is clear that a part of the demand after period τ (including period τ), $I(\tau-1)$, is replenished before period τ. Let $y=\min\{I(\tau-1),Q-x\}>0$. We can construct a new solution where $S'(\tau-1)=S(\tau-1)-y$, which means that y inside the part $I(\tau-1)$ is no longer delivered at the corresponding period(s) before period τ, instead delivered in period τ, i.e., $Q'(\tau)=Q(\tau)+y\le(k+1)Q$. This alternation does not incur additional transportation cost (possibly decrease transportation cost) and does save the inventory cost at least yh_0. This implies that the original solution is not an optimal solution. □

Proposition 2: $0\le I(t)<Q$, $\forall t$.

Proof: by contradiction. Assume there is an optimal solution: $\exists\tau$, $I(\tau)=kQ+x$, where $k\ge 1$ and integer, $0\le x<Q$. We can construct a new solution: $S'(\tau)=S(\tau)-kQ$ and thus $I'(\tau)=I(\tau)-kQ=x$, i.e., $0\le I'(\tau)<Q$; $Q'(\eta)=Q'(\eta)+kQ$ and thus $I'(\eta)=I(\eta)$, where η ($>\tau$) is the earliest (or first) period when the delivery quantity of the vendor is larger than zero. This alternation does not incur additional transportation cost and does save the inventory cost $(\eta-\tau)kQh_0$. This implies that the original solution is not an optimal solution. □

Proposition 3: $k_tQ\le Q(t)\le(k_t+1)Q$, $\forall t$.

Proof:

i). Firstly, we prove $Q(t) \ge k_t Q$.

Case 1: $k_t = 0$. The conclusion is clear.

Case 2: $k_t \ge 1$. By contradiction. Assume there is an optimal solution: $\exists \tau$, $Q(\tau) < k_\tau Q$. Without loss of generality, let $k_\tau Q - Q(\tau) = nQ + x > 0$, $n \ge 0$ and integer, $n \le k_\tau - 1$, $0 \le x < Q$. This implies that the demand in period τ has at least $nQ + x$ to be satisfied before period τ , i.e., $I(\tau - 1) \ge nQ + x$. If $n \ge 1$, we can construct a new solution: $S'(\tau - 1) = S(\tau - 1) - nQ$; $Q'(\tau) = Q(\tau) + nQ$. This alternation does not incur additional transportation cost and does save the inventory cost at least nQh_0 . This implies that the original solution is not an optimal solution. If $n = 0$, i.e., $Q(\tau) = k_\tau Q + x$ and $0 < x < Q$, i.e., $Q(t)\,\%\,Q = x > 0$. At the same time we have $I(\tau - 1) \ge x > 0$. According to Proposition 1, it is impossible.

ii). Now we prove $Q(t) \le (k_t + 1)Q$.

The proof of i) indicates that we need to consider only q_t for $D(t)$, $\forall t$. That is, the original proposition turns to prove $Q(t) \le Q$ whenever $D(t) = q_t$, $0 \le q_t < Q$, $\forall t$. By contradiction. Assume there is an optimal solution: $\exists \tau$, $Q(\tau) > Q$. It is clear that that a part of the demand after period τ is satisfied in period τ . Therefore, $q'_\tau + q^+_\tau = nQ + x = Q(\tau)$, $n \ge 1, 0 \le x < Q$, where q'_τ is the whole or part of demand in period τ to be satisfied in period τ , and $q'_\tau \le q_\tau$ because a part of the demand in period τ maybe satisfied before period τ ; q^+_τ is the total demand after period τ that is satisfied in period τ . Observe $q_t + q_{t'} < 2Q$ for $\forall t, t'$. This implies that the demands of at least *n*-1 periods after period τ are satisfied in period τ . Without loss of generality, let η ($> \tau$) is the earliest (or first) period in these periods. If $n = 1$, $q'_\tau + q^+_\tau = Q + x$ and $0 < x < Q$, and then $q^+_\tau > x$ since $q'_\tau \le q_\tau < Q$. We can construct a new solution: $Q'(\tau) = Q(\tau) - x = Q$ (decreasing a vehicle) ; $Q'(\eta) = Q(\eta) + x$ (increasing at most a vehicle). This alternation does not incur additional transportation cost (possibly decrease transportation cost) and does save the inventory cost at least $(\eta - \tau)xh_0$. If $n \ge 2$, $Q(\tau) = q'_\tau + q^+_\tau = nQ + x$ and $0 \le x < Q$, and then $q^+_\tau > (n-1)Q + x$ since $q'_\tau \le q_\tau < Q$. We can construct a new solution: $Q'(\tau) = Q(\tau) - (n-1)Q$, $Q'(\eta) = Q(\eta) + (n-1)Q$. This alternation does not incur additional transportation cost and does save the inventory cost at least $(n-1)(\eta - \tau)Qh_0$. This implies that the original solution is not an optimal solution. □

Proposition 4: $Q(t) = k_t Q$ **whenever** $D(t) = k_t Q$ **,** $k_t \ge 0$ **and integer.**

Proof: by contradiction. Assume there is an optimal solution: $\exists \tau$, $Q(\tau) = k_\tau Q + x$ **,** where $k_\tau \ge 0$ and integer , $0 < x \le Q$. Note that $k_\tau Q \le Q(\tau) \le (k_\tau + 1)Q$ according to Proposition 3. Similar to the proof of Proposition 3, we can construct a new solution: $Q'(\tau) = Q(\tau) - x = k_\tau Q$ (decreasing a vehicle), $Q'(\eta) = Q(\eta) + x$ (increasing at most a vehicle). This alternation does not incur additional transportation cost (possibly decrease transportation cost) and does save the inventory cost at least $(\eta - \tau)xh_0$. □

It is important to note that, according to Proposition 4, $Q(t) = 0$ whenever $D(t) = 0$ **,** $\forall t$ **.**

4.3 The reduced problem

P1 is a special case of P where $D(t) = q_t$, $0 \le q_t < Q$, $\forall t$.

Theorem 1: P can be reduced to P1.

Proof: According to Proposition 3 and 4, the portion $k_t Q$ of $D(t)$ is surely replenished by k_t vehicles in period *t*. Therefore, only the portion q_t of $D(t)$ needs to be decided how to be replenished. □

Theorem 2: Given a planning horizon *T*, P1 is a *C/C/C/Z* capacitated dynamic lot sizing problem.

Proof: In P1, if $Q(t)$ is viewed as the production (or order) amount to be decided, Q corresponds to the production capacity bound, $C + 2dU$ corresponds to the setup cost, h_0 is the inventory holding cost per unit item per unit time, and the production cost per unit item is zero and hence is omitted. P1 is therefore a *C/C/C/Z* capacitated dynamic lot sizing problem**.** □

4.4 Solution algorithms of the reduced problem

According to Theorem 2, given a planning horizon, P1 is a *C/C/C/Z* capacitated dynamic lot sizing problem. Consequently, all algorithms for the *C/C/C/Z* capacitated dynamic lot sizing problem without backlog can be also used to solve P1.

Theorem 3: Given a planning horizon *T*, there is an algorithm to solve P1 in $O(T^2)$ time.

Proof: In Chung and Lin (1988), the authors designed a dynamic programming algorithm to solve the *NI/G/NI/ ND* capacitated dynamic lot sizing problem in $O(T^2)$ time. It is clear that the *C/ C/ C/Z* problem is a special case of the *NI/G/NI/ ND*, therefore the algorithm can be also used to solve the *C/C/C/Z* problem. □

4.5 Determining T

Lastly, the remaining problem is to determine the planning horizon *T*. It is clear that *T* is a cycle period ensuring $D(t+T)=D(t)$, $Q(t+T)=Q(t)$, $I(t+T)=I(t)$. We assume that $D(t)$, $Q(t)$, $\forall t$ and Q are rationales. Hence, without loss of generality, $D(t)$, $Q(t)$, $\forall t$ and Q are assumed to be integers, see also Chan et al. (1998). Let *M* be the smallest common multiple of $T_1, T_2, ..., T_L$, i.e., *M* is a cycle period satisfying $D(t+M)=D(t)$.

Theorem 4: $M \le T \le QM$ **.**

Proof: According to Proposition 2, $0 \le I(t) < Q$, $\forall t$. This implies that $I(t)$ has at most *Q* kind of possible values since $I(t)$ is an integer. Therefore, within the planning horizon $Q(M + 1) + 1$, for any τ ($0 \le \tau \le M-1$), there exists two periods in τ, $M+\tau$, $2M+\tau$, …, $QM+\tau$, without loss of generality, let the two periods be $iM+\tau$ and $jM+\tau$, satisfying $I(iM+\tau)=I(jM+\tau)$. In addition, $D(iM+\tau+1)$. As a consequence, the transportation and inventory sub-policy during $[\, iM+\tau+1, jM+\tau \,]$ is not an optimal sub-policy. Hence, $I(iM+\tau+1)=I(jM+\tau+1)$. This implies that $(j-i)M=T$ is a cycle period satisfying $D(t+T)$ $=D(t)$, $Q(t+T)=Q(t)$, $I(t+T)=I(t)$. As a consequence, $M \le T=(j-i)M \le QM$. □

How to obtain an accurate value of *T*? A feasible method is described as follows. Let *T* = *M*, 2*M*, …, *QM* respectively, solve P1 and obtain respective optimal solution, and then select the best one from them.

5. CONCLUSIONS

For the inventory-routing problem in the three-level distribution system, we present a solution approach that decomposes the problem into three subproblems. For lack of space, we emphasize on modelling and solving the vendor-warehouse transportation and inventory subproblem in this paper. We have demonstrated that the vendor-warehouse transportation and inventory problem can be reduced to a *C/C/C/Z* capacitated dynamic lot sizing problem and there is an algorithm to solve the reduced problem to optimality in $O(T^2)$ time.

ACKNOWLEDGEMENT

The work of the first author is partially supported by a post-doctoral fellowship granted by the Conseil Regional de Champagne-Ardenne, France and Natural Science Foundation of China (Grant no. 70702007). The authors are also grateful to the reviewers for their valuable comments.

REFERENCES

Adelman, D. (2004). A price-directed approach to stochastic inventory/routing. *Operations Research*, **52**(4), 499-514.

Anily, S. and J. Bramel. (2004). An asymptotic 98.5%-effective lower bound on fixed partition policies for the inventory-routing problem. *Discrete Applied Mathematics*, **145**(1), 22-39.

Anily, S. and A. Federgruen. (1990). One warehouse multiple retailers systems with vehicle routing costs. *Management Science*, **36**(1), 92-113.

Bitran, G. and H. Yanasse. (1982). Computational complexity of the capacitated lot size problem. *Management Science*, **28**(10), 1174-1186.

Bramel, J. and D. Simchi-Levi. (1995). A location based heuristic for general routing problems. *Operations Research*, **43**(4), 649-660.

Burns, L.D., R.W. Hall, D.E. Blumenfeld and C.F. Daganzo. (1985). Distribution strategies that minimize transportation and inventory costs. *Operations Research*, **33**(3), 469-490.

Campbell, A. and M. Savelsbergh. (2004). A decomposition approach for the inventory-routing problem. *Transportation Science*, **38**(4), 488-502.

Chan, L.M.A., A. Federgruen and D. Simchi-Levi. (1998). Probabilistic analyses and practical algorithms for inventory-routing problems. *Operations Research*, **46**(1), 96-106.

Chan, L.M.A. and D. Simchi-Levi. (1998). Probabilistic analyses and algorithms for three-level distribution systems. *Management Science*, **44**(11), 1562-1576.

Chung, C.-S. and C.M. Lin. (1988). An $O(T^2)$ algorithm for the *NI/G/NI/ND* capacitated lot size problem. *Management Science*, **34**(3), 420-426.

Dror, M. and P. Trudeau. (1989). Savings by split delivery routing. *Transportation Science*, 23, 141-145.

Gaur, V. and M.L. Fisher. (2004). A periodic inventory routing problem at a supermarket chain. *Operations Research*, **52**(6), 813-822.

Jung, J. and K. Mathur. (2007). An efficient heuristic algorithm for a two-echelon joint inventory and routing problem. *Transportation Science*, **41**(1), 55-73.

Li, J., H. Chen and F. Chu. (2007a). On the effectiveness of fixed partition policies for one-warehouse multi-retailer distribution systems. Working paper, Industrial Systems Optimization Laboratory, University of Technology of Troyes, France.

Li, J., F. Chu and H. Chen. (2007b). Probabilistic analysis on three-level distribution systems. In *the proceedings of The 2007 IEEE International Conference on Systems, Man, and Cybernetics (SMC 2007), Montreal, Canada,* 1729-1734.

Roundy, R.O. (1985). 98% Effective integer-ratio lot sizing for one-warehouse multi-retailer systems. *Management Science*, **31**(11), 1416-1430.

Yu, Y., H. Chen and F. Chu. (2007). A new model and hybrid approach for large scale inventory routing problems. *European Journal of Operational Research*, doi:10.1016/j.ejor.2007.02.061

Zhao, Q.-H., S. Chen and C.-X. Zang. (2007). Model and algorithm for inventory/routing decision in a three-echelon logistics system. *European Journal of Operational Research*, doi:10.1016/j.ejor.2006.12.056