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Delay-Dependent Robust H_∞ Control for Uncertain Stochastic Systems ⋆

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Abstract: This note deals with the problems of robust H_∞ control for uncertain stochastic systems with a time-varying delay in the state. Based on the Lyapunov stability theory and the stochastic analysis tools, delay-dependent sufficient condition is established in terms of weak coupling linear matrix inequality (LMI) equations. The equations are derived by constructing a more efficient Lyapunov function candidate and combining LMI approach with free-weighting matrix technique. Properties of conservatism are only appeared with free-weighting matrices in a equation, which is coupled with another equation weakly. So the new criteria is of low conservatism with large time-delay, large time-varying rate and small disturbance attenuation. Numerical examples are given to demonstrate the benefits of the proposed criteria.

Keywords: Stochastic systems; Uncertain systems; Time-delay systems; Lyapunov functional; Linear matrix inequality; Stability.

1. INTRODUCTION

In the past decades, stochastic systems have attracted much attention due to the extensive applications of stochastic systems in mechanical systems, economics, systems with human operators, and other areas, see Wonham [1968]. Recently, many fundamental results of robust control for deterministic systems have been extended to stochastic systems, see Wonham [1968],Yaz [1993]. The robust stochastic stability problem for uncertain parameters and time-delay was studied by Liao et al. [2000], Mao et al. [1998], and Xie et al. [2000] respectively. Very recently, the stochastic version of bounded real lemma was derived in Hinrichsen et al. [1998]; based on this, necessary and sufficient conditions for the existence of H_∞ controllers were proposed in Ghaoui. [1995]. The corresponding results for discrete-time systems were studied in Bouhtouri et al. [1999]. Furthermore, the problems of robust H_∞ control for the systems with uncertain parameters and time-varying delays appearing simultaneously were discussed in Xu et al. [2002], Xu et al. [2004]. However, those conditions are of considerable conservatism and delay-independence. To the best of the authors' knowledge, corresponding delay-dependent condition has not been presented.

On the other hand, some free-weighting matrix methods were proposed to reduce the conservatism. The results in Lee et al. [2004] and Jing et al. [2004] are included or equivalent to those in He et al. [2004] and Wu et al. [2004]. The augmented Lyapunov functional presented by Wu et al. [2004] and He et al. [2005] is only applicable for neutral systems with time-invariant delay. And it was extended to time-varying delay case in He et al. [2007] via free-weighting matrices technique. To the best of the authors' knowledge, the free-weighting matrices technique for stochastic systems with state delay is still open.

In this note, a Lyapunov functional candidate is proposed. By incorporating additional terms in the candidate, we are able to reduce the conservatism. And the candidate is used to analyze the robust control problem for uncertain stochastic systems with state delay via combining linear matrix inequality (LMI) approach with free-weighting matrices technique. Furthermore, we gain the delay-dependent sufficient conditions for such systems in terms of weak coupling LMI equations. The properties of conservatism, upper time-delay, time varying rate and disturbance attenuation, are only appeared in the equation which involves free-weighting matrices. And the equation is coupled with another equation which involves parameters of systems weakly. So, we gain low conservative conditions with large time-delay, large time-varying rate and small disturbance attenuation. Numerical examples are given to demonstrate the considerable merits of the proposed criteria.

2. PROBLEM FORMULATION

Consider the following stochastic system with state-delay and parameter uncertainties:

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$$(\Sigma): dx(t) = [(A+\Delta A(t))x(t) + (A_d+\Delta A_d(t))x(t-\tau(t)) + (B+\Delta B(t))u(t) + B_\nu \nu(t)]dt + [(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t)) + E_\nu \nu(t)]d\omega(t), \quad (1)$$

$$z(t) = Cx(t) + Du(t), \quad (2)$$

$$x(t) = \phi(t), \forall t \in [-h, 0]. \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state,$u(t) \in \mathbb{R}^m$ is the control input,$\nu(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $L_2[0,\infty)$,$z(t) \in \mathbb{R}^q$ is the controlled output, and $\omega(t)$ is a one-dimensional (1-D) Brownian motion satisfying

$$\mathscr{E}\{d\omega(t)\} = 0, \mathscr{E}\{d\omega(t)^2\} = dt.$$

and $\tau(t)$ is the time-varying delay satisfying

$$0 < \tau(t) \le h < \infty, 0 < \dot{\tau}(t) \le \mu < 1 \quad (4)$$

where h, μ are real constant scalars;$\phi(t)$ is initial condition, $A, A_d, B, B_\nu, C, D, E, E_d$ and E_ν are known real constant matrices,$\Delta A(t), \Delta A_d(t), \Delta B(t), \Delta E(t)$ and $\Delta E_d(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix}\Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta E(t) & \Delta E_d(t)\end{bmatrix} = MF(t)\begin{bmatrix}N_a & N_{ad} & N_b & N_e & N_{ed}\end{bmatrix} \quad (5)$$

where M, N_a, N_{ad}, N_b, N_e and N_{ed} are known real constant matrices and $F(\cdot): \mathbb{R} \to \mathbb{R}^{k\times l}$ is an unknown time-varying matrix function satisfying

$$F(t)^T F(t) \le I, \forall t. \quad (6)$$

It is assumed that all the elements of $F(t)$ are Lebesgue measurable.$\Delta A(t), \Delta A_d(t), \Delta B(t), \Delta E(t)$ and $\Delta E_d(t)$ are said to be admissible if both (5) and (6) hold.

3. MAIN RESULTS

Well-used lyapunov functional candidate was adopted to solve the robust stochastic stabilization problem in the previous works such as Xu et al. [2002] and Xu et al. [2004], which is similar to the following form:

$$V(x(t),t) = x(t)^T Px(t) + \int_{t-\tau(t)}^{t} x(s)^T Qx(s)ds. \quad (7)$$

Some important terms were ignored when estimating the upper bound of the derivative of Lyapunov functional for systems in the candidate, such as $-\int_{t-h}^{t} \dot{x}(s)^T Z_1 \dot{x}(s)ds$.

To handle this term, we improve the Lyapunov functional candidate for stochastic time-delay systems as follows:

$$V(x(t),t) = x(t)^T Px(t) + \int_{t-\tau(t)}^{t} x(s)^T Qx(s)ds + \int_{t-h}^{t} x(s)^T \cdot Rx(s)ds + \int_{-h}^{0}\int_{t+\theta}^{0} \dot{x}(s)^T (Z_1+Z_2)\dot{x}(s)dsd\theta \quad (8)$$

where $P = P^T > 0, Q = Q^T \ge 0, R = R^T \ge 0$ and $Z_i = Z_i^T > 0, i = 1,2$ are to be determined.

We use this improved Lyapunov functional candidate to deal with the robust H_∞ control for uncertain stochastic systems with state delay.

Before proceeding further, we give the following lemma which will be used in the proof of our main results.

Lemma 1. Wang et al. [1992] Let A, D, S, W and F be real matrices of appropriate dimensions such that $W > 0$ and $F^T F \le I$. Then, we have the following:

1)For scalar $\varepsilon > 0$ and vectors $x, y \in \mathbb{R}^n$

$$2x^T DFSy \le \varepsilon^{-1} x^T DD^T x + \varepsilon y^T S^T Sy.$$

2)For any scalar $\varepsilon > 0$ such that $W - \varepsilon DD^T > 0$

$$(A+DFS)^T W^{-1} (A+DFS) \le A^T(W-\varepsilon DD^T)^{-1}A + \varepsilon^{-1} S^T S.$$

3.1 Robust Stochastic Stabilization

In this section, we propose a sufficient condition for the stochastic asymptotically mean-square stabilization result. The main result is given in the following theorem.

Theorem 2. Consider the uncertain stochastic delay system (1) and (3) with $\nu(t) = 0$. Given scalars $h > 0$ and μ,this system is robustly stochastically stabilizable if there exist scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$ and matrices

$$X > 0, Q = Q^T \ge 0, R = R^T \ge 0, G = G^T \ge 0,$$
$$H = H^T \ge 0, Z_i = Z_i^T > 0, i = 1,2,$$

$$L = \begin{bmatrix}L_1\\L_2\\L_3\end{bmatrix}, S = \begin{bmatrix}S_1\\S_2\\S_3\end{bmatrix}, J = \begin{bmatrix}J_1\\J_2\\J_3\end{bmatrix}$$

and Y such that the following LMIs hold.

$$\begin{bmatrix}\Omega_{11} & A_d X & \Omega_{13} & XN_e^T & XE^T\\ * & -G & XN_{ad}^T & XN_{ed}^T & XE_d^T\\ * & * & -\varepsilon_1 I & 0 & 0\\ * & * & * & -\varepsilon_2 I & 0\\ * & * & * & * & \varepsilon_2 MM^T - X\end{bmatrix} < 0, \quad (9)$$

$$\begin{bmatrix}\Phi & hL & hS & hJ\\ * & -hZ_1 & 0 & 0\\ * & * & -hZ_1 & 0\\ * & * & * & -hZ_2\end{bmatrix} < 0, \quad (10)$$

where

$$\Omega_{11} = AX + XA^T + BY + Y^T B^T + \varepsilon_1 MM^T + G + H,$$
$$\Omega_{13} = XN_a^T + Y^T N_b^T,$$
$$\Phi = \Phi_1 + \Phi_2 + \Phi_2^T,$$
$$\Phi_1 = \begin{bmatrix}0 & 0 & 0\\ * & \mu Q & 0\\ * & * & -R\end{bmatrix},$$
$$\Phi_2 = \begin{bmatrix}L+J & S-L & -S-J\end{bmatrix}.$$

In this case, an appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), \qquad K = YX^{-1}. \quad (11)$$

Proof. Applying the controller (11) to (1) with $\nu(t) = 0$, we obtain the resulting closed-loop system as

$$dx(t) = [(A_{cK} + \Delta A_{cK}(t))x(t) + (A_d + \Delta A_d(t))x(t-\tau(t))]dt + [(E+\Delta E(t))x(t) + (E_d + \Delta E_d(t))x(t-\tau(t))]d\omega(t) \quad (12)$$

where

$$A_{cK} = A + BK, \Delta A_{cK}(t) = MF(t)N_{cK}, N_{cK} = N_a + N_b K. \quad (13)$$

Let

$$P = X^{-1}, Q = X^{-1}GX^{-1}, R = X^{-1}HX^{-1}, \quad (14)$$

then, from (9), it is easy to see that

$$P^{-1} - \varepsilon_2 MM^T > 0. \quad (15)$$

Now, use the Lyapunov function candidate as (8) for the system in (12) and by *Itô's* formula, we obtain:

$$dV(x(t),t) = \mathscr{L}V(x(t),t)dt + 2x(t)^T P[(E+\Delta E(t))x(t) + (E_d + \Delta E_d(t))x(t-\tau(t))]d\omega(t) \quad (16)$$

where

$$\begin{aligned}\mathscr{L}\, V(x(t),t)dt &= x(t)^T(Q+R)x(t) - (1-\dot{\tau}(t))x(t-\tau(t))^T Q \\ &\cdot x(t-\tau(t)) - \int_{t-h}^{t} \dot{x}(s)^T(Z_1+Z_2)\dot{x}(s)ds - x(t-h)^T Rx(t-h) \\ &+ 2x(t)^T P[(A_{cK}+\Delta A_{cK}(t))x(t) + (A_d+\Delta A_d(t))x(t-\tau(t))] \\ &+ [(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t))]^T P \\ &\cdot[(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t))],\end{aligned}$$

it follows from (4) that

$$\begin{aligned}\mathscr{L}\, V(x(t),t)dt &\le x(t)^T(Q+R)x(t) - (1-\mu)x(t-\tau(t))^T Q \\ &\cdot x(t-\tau(t)) - x(t-h)^T Rx(t-h) - \int_{t-h}^{t} \dot{x}(s)^T(Z_1+Z_2)\dot{x}(s)ds \\ &+ 2x(t)^T P[(A_{cK}+\Delta A_{cK}(t))x(t) + (A_d+\Delta A_d(t))x(t-\tau(t))] \\ &+ [(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t))]^T P \\ &\cdot[(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t))]. \quad (17)\end{aligned}$$

Denote

$$\begin{aligned}\mathscr{L}\, V_1(x(t),t) &= x(t)^T(Q+R)x(t) - x(t-\tau(t))^T Qx(t-\tau(t)) \\ &+ 2x(t)^T P[(A_{cK}+\Delta A_{cK}(t))x(t) + (A_d+\Delta A_d(t)) \\ &\cdot x(t-\tau(t))] + [(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t))]^T \\ &\cdot P[(E+\Delta E(t))x(t) + (E_d+\Delta E_d(t))x(t-\tau(t))], \quad (18)\end{aligned}$$

$$\begin{aligned}\mathscr{L}\, V_2(x(t),t) &= \mu x(t-\tau(t))^T Qx(t-\tau(t)) \\ &- x(t-h)^T Rx(t-h) - \int_{t-h}^{t} \dot{x}(s)^T(Z_1+Z_2)\dot{x}(s)ds. \quad (19)\end{aligned}$$

Noting (15), using Lemma 1 in (18), we have

$$\begin{aligned}&2x(t)^T P[\Delta A_{cK}(t)x(t) + \Delta A_d(t)x(t-\tau(t))] \\ &= 2x(t)^T PMF(t)[N_{cK}x(t) + N_{ad}x((t-\tau(t))] \\ &\le \varepsilon_1 x(t)^T PMM^T Px(t) + \varepsilon_1^{-1}[N_{cK}x(t) \\ &+ N_{ad}x((t-\tau(t))]^T[N_{cK}x(t) + N_{ad}x((t-\tau(t))] \quad (20)\end{aligned}$$

and

$$\begin{aligned}&[\bar{E} + M\,F\,(t)\bar{N}]^T P[\bar{E} + MF(t)\bar{N}] \le \\ &\bar{E}^T(P^{-1} - \varepsilon_2 MM^T)^{-1}\bar{E} + \varepsilon_2^{-1}\bar{N}^T\bar{N} \quad (21)\end{aligned}$$

where $\bar{E} = [E \quad E_d \quad 0], \bar{N} = [N_e \quad N_{ed} \quad 0]$. Therefore, it follows from (18) and (20)-(21) that

$$\mathscr{L}V_1(x(t),t) \le \xi(t)^T\Theta_1\xi(t) \quad (22)$$

where

$$\xi(t) = \begin{bmatrix} x(t)^T & x(t-\tau(t))^T & x(t-h)^T \end{bmatrix}^T,$$

$$\begin{aligned}\Theta_1 &= \begin{bmatrix} \bar{\Omega}_{11} & PA_d & 0 \\ * & -Q & 0 \\ * & * & 0 \end{bmatrix} + \varepsilon_1^{-1}\begin{bmatrix} N_{cK}^T \\ N_{ad}^T \\ 0 \end{bmatrix}\begin{bmatrix} N_{cK} & N_{ad} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} E^T \\ E_d^T \\ 0 \end{bmatrix}(P^{-1} - \varepsilon_2 MM^T)^{-1}\begin{bmatrix} E & E_d & 0 \end{bmatrix} \\ &+ \varepsilon_2^{-1}\begin{bmatrix} N_e^T \\ N_{ed}^T \\ 0 \end{bmatrix}\begin{bmatrix} N_e & N_{ed} & 0 \end{bmatrix}\end{aligned}$$

with $\bar{\Omega}_{11} = PA_{cK} + A_{cK}^T P^T + \varepsilon_1 PMM^T P + Q + R$. On the other hand, pre- and post-multiplying (9) by $diag(P,P,I,I,I)$ results in

$$\begin{bmatrix} \bar{\Omega}_{11} & PA_d & N_{cK}^T & N_e^T & E^T \\ * & -Q & N_{ad}^T & N_{ed}^T & E_d^T \\ * & * & -\varepsilon_1 I & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & \varepsilon_2 MM^T - P^{-1} \end{bmatrix} < 0, \quad (23)$$

which, by Schur complement, implies that $\Theta_1 < 0$.This together with (22) implies that for all $\xi(t)^T \neq 0$ we have

$$\mathscr{L}V_1(x(t),t) < 0. \quad (24)$$

Now, we observe the $\mathscr{L}V_2(x(t),t)$ in (19). From the Leibniz-Newton formula, the following equations are true for any of the matrices L, S and J with appropriate dimensions:

$$2\xi(t)^T L[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds] = 0,$$

$$2\xi(t)^T S[x(t-\tau(t)) - x(t-h) - \int_{t-h}^{t-\tau(t)} \dot{x}(s)ds] = 0,$$

$$2\xi(t)^T J[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s)ds] = 0.$$

Add them to (19), then

$$\begin{aligned}\mathscr{L}V_2(x(t\ \ ),t) &\le \xi(t)^T[\Phi + hLZ_1^{-1}L^T + hSZ_1^{-1}S^T + hJZ_2^{-1}J^T]\xi(t) \\ &- \int_{t-\tau(t)}^{t} [\dot{x}(s)^T Z_1 + \xi(t)^T L]Z_1^{-1}[L^T\xi(t) + Z_1^T\dot{x}(s)]ds \\ &- \int_{t-h}^{t-\tau(t)} [\dot{x}(s)^T Z_1 + \xi(t)^T S]Z_1^{-1}[S^T\xi(t) + Z_1^T\dot{x}(s)]ds \\ &- \int_{t-h}^{t} [\dot{x}(s)^T Z_2 + \xi(t)^T J]Z_2^{-1}[J^T\xi(t) + Z_2^T\dot{x}(s)]ds \\ &\le \xi(t)^T[\Phi + hLZ_1^{-1}L^T + hSZ_1^{-1}S^T + hJZ_2^{-1}J^T]\xi(t).\end{aligned}$$

We can write it in the following form:

$$\mathscr{L}V_2(x(t),t) \le \xi(t)^T\Theta_2\xi(t)$$

where $\Theta_2 = \Phi + hLZ_1^{-1}L^T + hSZ_1^{-1}S^T + hJZ_2^{-1}J^T$. Which, by Schur complement, (10) implies that $\Theta_2 < 0$,we have

$$\mathscr{L}V_2(x(t),t) < 0. \quad (25)$$

So, from (17)-(19),(24) and (25), we can obtain

$$\mathscr{L}V(x(t),t) \le \mathscr{L}V_1(x(t),t) + \mathscr{L}V_2(x(t),t) < 0. \quad (26)$$

Then, by Xu et al. [2002] Definition 1 and Kolmanovskii et al. [1992], we know that the closed-loop system in (12) is robustly stable. The proof of Theorem 2 is complete.

Remark 1. If $R = 0, Z_1 = Z_2 = 0$, Theorem 2 provides a complementary method to the result in Xu et al. [2002], Th.1.

When there are no parameter uncertainties in the system in (1) and (3), Theorem 2 is specialized as follows.

Corollary 3. Consider the stochastic delay system in (1) and (3) with $v(t) = 0, \Delta A(t) = 0, \Delta A_d(t) = 0, \Delta B(t) = 0, \Delta E(t) = 0$ and $\Delta E_d(t) = 0$. Then, this system is stochastically stabilizable if there exist matrices as in Theorem 2 such that the following LMI and (10) hold.

$$\begin{bmatrix} \tilde{\Omega}_{11} & A_d X & XE^T \\ * & 0 & XE_d^T \\ * & * & -X \end{bmatrix} < 0.$$

where $\tilde{\Omega}_{11} = AX + XA^T + BY + Y^T B^T + G + H$. In this case, the controller can be chosen as Theorem 2.

3.2 Robust H_∞ Control

In this section, we propose a sufficient condition for the solvability of robust H_∞ control problem for uncertain stochastic delay systems. The main result is given in the following theorem.

Theorem 4. Consider the uncertain stochastic delay system (Σ).Given scalars $h > 0$, $\gamma > 0$ and μ, then this system is robustly stochastically stabilizable with disturbance attenuation γ if there exist scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$ and matrices

$$X > 0, Q = Q^T \geq 0, R = R^T \geq 0, G = G^T \geq 0,$$
$$H = H^T \geq 0, Z_i = Z_i^T > 0, i = 1, 2,$$

$$\bar{L} = \begin{bmatrix} L \\ 0 \end{bmatrix}, \bar{S} = \begin{bmatrix} S \\ 0 \end{bmatrix}, \bar{J} = \begin{bmatrix} J \\ 0 \end{bmatrix}$$

and Y such that the following LMIs hold.

$$\begin{bmatrix} \Omega_{11} & A_d X & \Omega_{13} & XN_e^T & XE^T & XC^T + Y^T D^T \\ * & -G & XN_{ad}^T & XN_{ed}^T & XE_d^T & 0 \\ * & * & -\varepsilon_1 I & 0 & E_\nu^T & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & \varepsilon_2 MM^T - X & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{27}$$

$$\begin{bmatrix} \bar{\Phi} & h\bar{L} & h\bar{S} & h\bar{J} \\ * & -hZ_1 & 0 & 0 \\ * & * & -hZ_1 & 0 \\ * & * & * & -hZ_2 \end{bmatrix} < 0, \tag{28}$$

where $\Omega_{11}, \Omega_{13}, L, S, J$ are given in Theorem 2,

$$\bar{\Phi} = \bar{\Phi}_1 + \bar{\Phi}_2 + \bar{\Phi}_2^T,$$

$$\bar{\Phi}_1 = \begin{bmatrix} 0 & 0 & 0 & PB_\nu \\ * & \mu Q & 0 & 0 \\ * & * & -R & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\bar{\Phi}_2 = \begin{bmatrix} \bar{L} + \bar{J} & \bar{S} - \bar{L} & -\bar{S} - \bar{J} & 0 \end{bmatrix}.$$

In this case, an appropriate state feedback controller can be chosen by

$$u(t) = Kx(t), \qquad K = YX^{-1}. \tag{29}$$

Proof. By the state feedback in (29), the system (Σ) becomes

$$\begin{aligned} (\Sigma_c) : dx(t) &= [(A_{cK} + \Delta A_{cK}(t))x(t) + (A_d + \Delta A_d(t))x(t - \tau(t)) \\ &\quad + B_\nu \nu(t)]dt + [(E + \Delta E(t))x(t) + (E_d + \Delta E_d(t)) \\ &\quad \cdot x(t - \tau(t)) + E_\nu \nu(t)]d\omega(t), \end{aligned} \tag{30}$$

$$z(t) = C_{cK} x(t) \tag{31}$$

where A_{cK} and $\Delta A_{cK}(t)$ are given in (13), and

$$C_{cK} = C + DK.$$

It is easy to see that (27-28) implies the LMI in (9-10), so that the closed-loop system (Σ_c) is robustly stochastically stable. Next, according to Xu et al. [2002] Definition 2, we shall show that system (Σ_c) satisfies

$$\|z(t)\|_{E_2} < \gamma \|\nu(t)\|_2 \tag{32}$$

for all nonzero $\nu(t) \in L_2[0, \infty)$,where

$$\|z(t)\|_{E_2} = (\mathscr{E}\{\int_0^\infty |z(t)|^2 dt\})^{1/2}.$$

To this end, we assume zero initial condition, that is, $x(t) = 0$ for $t \in [-h, 0]$. Thus, by $It\hat{o}'s$ formula, we can derive

$$\mathscr{E}\{V(x(t), t)\} = \mathscr{E}\{\int_0^t \mathscr{L}V(x(s), s)ds\} \tag{33}$$

where the Lyapunov function candidate $V(x(t), t)$ is given in (8),and

$$\begin{aligned} dV(x(t), t) &= \mathscr{L}V(x(t), t)dt + 2x(t)^T P[(E + \Delta E(t))x(t) \\ &\quad + (E_d + \Delta E_d(t))x(t - \tau(t)) + E_\nu \nu(t)]d\omega(t) \end{aligned} \tag{34}$$

where

$$\begin{aligned} &\mathscr{L}V(x(t), t)dt \\ &\leq x(t)^T (Q + R)x(t) - (1 - \mu)x(t - \tau(t))^T Qx(t - \tau(t)) \\ &\quad - \int_{t-h}^t \dot{x}(s)^T (Z_1 + Z_2)\dot{x}(s)ds - x(t - h)^T Rx(t - h) \\ &\quad + 2x(t)^T P[(A_{cK} + \Delta A_{cK}(t))x(t) + (A_d + \Delta A_d(t)) \\ &\quad \cdot x(t - \tau(t)) + B_\nu \nu(t)] + [(E + \Delta E(t))x(t) \\ &\quad + (E_d + \Delta E_d(t))x(t - \tau(t)) + E_\nu \nu(t)]^T P \\ &\quad \cdot [(E + \Delta E(t))x(t) + (E_d + \Delta E_d(t))x(t - \tau(t)) \\ &\quad + E_\nu \nu(t)] \end{aligned} \tag{35}$$

and $Q > 0, R > 0$ are defined in (14). Now,set

$$\mathscr{J}(t) = \mathscr{E}\{\int_0^t [z(s)^T z(s) - \gamma^2 \nu(s)^T \nu(s)]ds\} \tag{36}$$

where $t > 0$. From (33) to (36), it is easy to show that

$$\begin{aligned} \mathscr{J}(t) &= \mathscr{E}\{\int_0^t [z(s)^T z(s) - \gamma^2 \nu(s)^T \nu(s) + \mathscr{L}(Vx(s), s)]ds\} \\ &\quad - \mathscr{E}\{V(x(t), t)\} \\ &\leq \mathscr{E}\{\int_0^t [z(s)^T z(s) - \gamma^2 \nu(s)^T \nu(s) + \mathscr{L}(Vx(s), s)]ds\}. \end{aligned} \tag{37}$$

Denote

$$\begin{aligned} \mathscr{L}V_1(x(t), t) &= x(t)^T (Q + R)x(t) - x(t - \tau(t))^T Qx(t - \tau(t)) \\ &\quad + 2x(t)^T P[(A_{cK} + \Delta A_{cK}(t))x(t) + (A_d + \Delta A_d(t)) \\ &\quad \cdot x(t - \tau(t))] + [(E + \Delta E(t))x(t) \\ &\quad + (E_d + \Delta E_d(t))x(t - \tau(t)) + E_\nu \nu(t)]^T P[(E + \Delta E(t))x(t) \\ &\quad + (E_d + \Delta E_d(t))x(t - \tau(t)) + E_\nu \nu(t)], \end{aligned} \tag{38}$$

$$\begin{aligned} \mathscr{L}V_2(x(t), t) &= -x(t - h)^T Rx(t - h) + \mu x(t - \tau(t))^T Qx(t - \tau(t)) \\ &\quad + 2x(t)^T PB_\nu \nu(t) - \int_{t-h}^t \dot{x}(s)^T (Z_1 + Z_2)\dot{x}(s)ds. \end{aligned} \tag{39}$$

By Lemma 1 , it can be shown that for $\varepsilon_1 > 0$

$$\begin{bmatrix} P\Delta A_{cK}(t) + \Delta A_{cK}(t)^T P & P\Delta A_d(t) & 0 & 0 \\ \Delta A_d(t)^T P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} PM \\ 0 \\ 0 \\ 0 \end{bmatrix} F(t) \begin{bmatrix} N_{cK} & N_{ad} & 0 & 0 \end{bmatrix} + \begin{bmatrix} N_{cK}^T \\ N_{ad}^T \\ 0 \\ 0 \end{bmatrix} F(t)^T \begin{bmatrix} M^T P & 0 & 0 & 0 \end{bmatrix}. \quad (40)$$

Considering $P = X^{-1}$, it then follows from (27) that (15) is satisfied, therefore, by Lemma 1 again, we have

$$[\bar{E}_1 + M F(t)\bar{N}_1]^T P[\bar{E}_1 + MF(t)\bar{N}_1] \leq \bar{E}_1^T (P^{-1} - \varepsilon_2 MM^T)^{-1}\bar{E}_1 + \varepsilon_2^{-1}\bar{N}_1^T \bar{N}_1 \quad (41)$$

where $\bar{E}_1 = [E \quad E_d \quad 0 \quad E_\nu], \bar{N} = [N_e \quad N_{ed} \quad 0 \quad 0]$. Therefore,

$$\mathscr{L}V_1(x(t),t) \leq \bar{\xi}(t)^T \Xi \bar{\xi}(t) \quad (42)$$

where

$$\bar{\xi}(t) = \begin{bmatrix} x(t)^T & x(t-\tau(t))^T & x(t-h)^T & \nu(t)^T \end{bmatrix}^T,$$

$$\Xi = \begin{bmatrix} \bar{\Omega}_{11} & PA_d & 0 & 0 \\ * & -Q & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} + \varepsilon_1^{-1} \begin{bmatrix} N_{cK}^T \\ N_{ad}^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} N_{cK} & N_{ad} & 0 & 0 \end{bmatrix} + [\bar{E}_1^T (P^{-1} - \varepsilon_2 MM^T)^{-1}\bar{E}_1 + \varepsilon_2^{-1}\bar{N}_1^T \bar{N}_1].$$

By the same line as pervious subsection, we have

$$\mathscr{L}\, V_2(x(t),t) \leq \bar{\xi}(t)^T [\tilde{\Phi} + h\bar{L}Z_1^{-1}\bar{L}^T + h\bar{S}Z_1^{-1}\bar{S}^T + h\bar{J}Z_2^{-1}\bar{J}^T]\bar{\xi}(t)$$

where

$$\tilde{\Phi} = \tilde{\Phi}_1 + \bar{\Phi}_2 + \bar{\Phi}_2^T,$$

$$\tilde{\Phi}_1 = \begin{bmatrix} 0 & 0 & 0 & PB_\nu \\ & \mu Q & 0 & 0 \\ & * & -R & 0 \\ & * & * & 0 \end{bmatrix}.$$

Observe

$$\begin{aligned} & z(s)^T z(s) - \gamma^2 \nu(s)^T \nu(s) + \mathscr{L}V(x(s),s) \\ &= x(s)^T C_{cK}^T C_{cK} x(s) - \gamma^2 \nu(s)^T \nu(s) + \mathscr{L}V(x(s),s) \\ &= \bar{\xi}(s)^T \begin{bmatrix} C_{cK}^T C_{cK} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} \bar{\xi}(s) + \mathscr{L}V(x(s),s) \\ &\leq \bar{\xi}(s)^T \{ \begin{bmatrix} \bar{\Omega}_{11} + C_{cK}^T C_{cK} & PA_d & 0 & 0 \\ * & -Q & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \\ &\quad + \varepsilon_1^{-1} \begin{bmatrix} N_{cK}^T \\ N_{ad}^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} N_{cK} & N_{ad} & 0 & 0 \end{bmatrix} \\ &\quad + [\bar{E}_1^T (P^{-1} - \varepsilon_2 MM^T)^{-1}\bar{E}_1 + \varepsilon_2^{-1}\bar{N}_1^T \bar{N}_1]\}\bar{\xi}(s) \\ &\quad + \bar{\xi}(s)^T [\tilde{\Phi} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} + h\bar{L}Z_1^{-1}\bar{L}^T + h\bar{S}Z_1^{-1}\bar{S}^T \\ &\quad + h\bar{J}Z_2^{-1}\bar{J}^T]\bar{\xi}(s) \\ &= \bar{\xi}(s)^T \Gamma \bar{\xi}(s) + \xi(s)^T [\bar{\Phi} + hLZ_1^{-1}L^T + hSZ_1^{-1}S^T \\ &\quad + hJZ_2^{-1}J^T]\xi(s). \end{aligned} \quad (43)$$

Now, pre- and post-multiplying (27) by $diag(P,P,I,I,I,I)$ result in

$$\begin{bmatrix} \bar{\Omega}_{11} & PA_d & N_{cK}^T & N_e^T & E^T & C_{cK}^T \\ * & -Q & N_{ad}^T & N_{ed}^T & E_d^T & 0 \\ * & * & -\varepsilon_1 I & 0 & E_\nu^T & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & \varepsilon_2 MM^T - P^{-1} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \quad (44)$$

which, by Schur complement, (44) implies that

$$\Gamma < 0. \quad (45)$$

On the other hand, (28) implies that

$$\bar{\Phi} + h\bar{L}Z_1^{-1}\bar{L}^T + h\bar{S}Z_1^{-1}\bar{S}^T + h\bar{J}Z_2^{-1}\bar{J}^T < 0. \quad (46)$$

From (43,45-46), we obtain

$$z(s)^T z(s) - \gamma^2 \nu(s)^T \nu(s) + \mathscr{L}V(x(t),t) < 0.$$

So, from (37)

$$\mathscr{J}(t) \leq \mathscr{E}\{ \int_0^t [z(s)^T z(s) - \gamma^2 \nu(s)^T \nu(s) + \mathscr{L}(Vx(s),s)]ds\} < 0 \quad (47)$$

for all $t > 0$.Then, (32) follows immediately from (47) and (36). The proof of Theorem 4 is complete.

In the case when there are no parameter uncertainties in the system (Σ), Theorem 4 is specialized as follows.

Corollary 5. Consider the stochastic delay system (Σ) with $\Delta A(t) = 0, \Delta A_d(t) = 0, \Delta B(t) = 0, \Delta E(t) = 0$ and $\Delta E_d(t) = 0$. Then, this system is stochastically stabilizable with disturbance attenuation γ if there exist matrices as in Theorem 4 such that the following LMI and (28) hold.

$$\begin{bmatrix} \tilde{\Omega}_{11} & A_d X & XC^T + Y^T D^T & XE^T \\ * & -G & 0 & XE_d^T \\ * & * & -I & E_\nu^T \\ * & * & * & -X \end{bmatrix} < 0$$

where $\tilde{\Omega}_{11}$ is given in Corollary 3. In this case, the controller can be chosen as Theorem 4.

4. NUMERICAL EXAMPLE

In this section, we shall give numerical examples to show the benefit of our results.

Example: Xu et al. [2002] Consider the uncertain stochastic delay system (Σ) with parameters as follows:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0.2 & 0.5 & -1 \\ 0.3 & 0.3 & -1.2 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.1 & -0.8 & 0.2 \\ 1 & -0.3 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_\nu = \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.5 & 0.1 & 0.2 \\ 0 & 1 & -0.1 \\ -0.2 & 0.3 & 0 \end{bmatrix}, \quad E_d = \begin{bmatrix} -0.5 & 0 & 0.2 \\ 0.1 & 0.4 & 0.1 \\ 0.2 & -0.1 & 0.5 \end{bmatrix},$$

$$E_\nu = \begin{bmatrix} 0 \\ 0.2 \\ -0.3 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5 & 0.1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & -0.3 \end{bmatrix},$$

$$N_a = \begin{bmatrix} 0.1 & 0 & 0.1 \end{bmatrix}, \quad N_{ad} = \begin{bmatrix} 0 & 0.1 & 0.1 \end{bmatrix},$$

$$N_e = \begin{bmatrix} 0.1 & 0.2 & 0 \end{bmatrix}, \quad N_{ed} = \begin{bmatrix} 0 & 0.1 & 0.2 \end{bmatrix}, \quad N_b = \begin{bmatrix} 0.2 & 0 \end{bmatrix}.$$

Using Matlab LMI Control Toolbox to solve the LMIs (27-28), we obtain the solution as follows:

$$X = \begin{bmatrix} 5.3142 & -0.6633 & 3.9000 \\ -0.6633 & 4.4041 & 3.1194 \\ 3.9000 & 3.1194 & 7.6861 \end{bmatrix},$$

$$Y = \begin{bmatrix} -54.2389 & -25.2927 & -57.8588 \\ -9.1690 & -18.8504 & -11.5039 \end{bmatrix}.$$

Therefore, by Theorem 4, it follows that the robust H_∞ control problem is solvable, and the desired state feedback control law can be chosen as

$$u(t) = \begin{bmatrix} -15.1525 & -11.4229 & 4.7969 \\ -7.8046 & -10.1056 & 6.5648 \end{bmatrix} x(t).$$

Next, we shall show the large upper bounds h and small minimum disturbance attenuations γ on many time-varying rates μ. h, μ and γ are the properties of conservatism for systems.

Table 1 lists the upper bounds on the time delay h for many μ by Theorem 4 in this note.

Table 1. Maximum h for many μ and $\gamma = 1.8$

μ	0.3	0.5	0.9
Theorem 4	5.0661×10^9	1.0413×10^9	2.5232×10^7

Table 2 lists the minimum disturbance attenuations γ for many μ and $h = 1$ by Theorem 4 in this note.

Table 2. Minimum γ for many μ and $h = 1$

μ	0.3	0.5	0.9
Theorem 4	0.0615	0.0790	0.2351

It is clear that the upper bound time-delay h and time-varying rate μ are large enough , while the disturbance attenuation γ is small. Hence, the results guarantee the low conservatism for systems. Unfortunately, the corresponding h and γ are always worse in most delay-dependent results for various systems, such as He et al. [2004], Wu et al. [2004] and He et al. [2007].

The key why we gained these excellent results is we proposed the conditions in terms of weak coupling LMI equations. The equation in (27) is deduced via constructing a more efficient Lyapunov function candidate and adopting LMI approach, which guarantee the low conservatism of systems. (28) is deduced by adopting free-weighting matrix technique to deal with some terms in Lyapunov candidate. (28) involves free-weighting matrices ,which can be valued freely, and the properties of conservatism mainly. And (27) is coupled with (28) just only by Q. So, the weak coupling equations in Theorem 4 is low conservatism, delay-dependence with large time-delays, large time-varying rate and small disturbance attenuation.

5. CONCLUSION

An improved Lyapunov functional candidate has been proposed in this note, where we incorporate the more terms to reduce the conservatism, which is the base of low conservatism. The LMI approach and free-weighting matrix technique have been adopted, which is the possibility of low conservatism. The weak coupling equations have been presented, which is the key of gaining low conservatism with large time-delays, large time-varying rate and small disturbance attenuation. Some numerical examples have been provided to demonstrate the usefulness of the proposed criteria.

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